Theories with NIP. List 15.

We work in a monster model $\mathfrak{C}$ of a complete theory T; $G = G(\mathfrak{C})$ is a $\emptyset$-definable group.

Problem 1. Let $M \prec N \prec \mathfrak{C}$, where N is $|M|^+$-saturated. Assume that $p \in S_G(M)$ satisfies that gp does not fork over M for all $g \in G(N)$. Prove that p extends to a global type which is (strongly) f-generic over M.

Problem 2. Suppose G is definably amenable in all reducts of $\mathfrak{C}$ to countable sublanguages in which G remains to be $\emptyset$-definable. Prove that G is definably amenable.

Problem 3. Let $H \lhd G$ be $\emptyset$-definable.
(i) Prove that if G is definably amenable, then so is G/H.
(ii) Assume NIP. Prove that if H and G/H are both definably amenable, then so is G.

Problem 4. Assume NIP. Prove that a definably amenable group G admits a global Keisler measure which is both left and right invariant.
Hint. First, observe that there is a left invariant and invariant under $\mathrm{Aut}(\mathfrak{C}/M)$ *Keisler measure* μ *on* G. *Define* $\mu^{-1}(X) := \mu(X^{-1})$. *Check that* μ^{-1} *is a right invariant and invariant under* $\mathrm{Aut}(C/M)$ *Keisler measure on* G. *Put* $\nu(\varphi(x)) := \mu \otimes \mu^{-1}(\varphi(x \cdot y))$ *and show that it is as required.*

Problem 5. Let $M := (\mathbb{R}, +, \cdot)$ and $G(M) := (\mathbb{R}, +) \rtimes \{-1, 1\}$ with $(x_0, \epsilon_0)(x_1, \epsilon_1) := (x_0 + \epsilon_0 x_1, \epsilon_0 \epsilon_1)$. Find a left invariant Keisler measure on $G = G(\mathfrak{C})$ which is not right invariant.

Problem 6. (i) Let $G(\mathbb{Q}) := (\mathbb{Q}^2, +)$ seen as a group definable in $(\mathbb{Q}, +, \leq)$. Prove that $G = G(\mathfrak{C})$ has $2^{\aleph_0}$ global left invariant types.
(ii) Let $G(\mathbb{R}) := (\mathbb{R}^2, +)$ seen as a group definable in RCF. Prove that $G = G(\mathfrak{C})$ has unboundedly many global left invariant types.

Question Assume $p \in S_G(\mathfrak{C})$ is a strong f-generic. Is it true that for sufficently large countable sublanguages L_0 of L, the type $p|_{L_0}$ is strongly f-generic (working in the reduct to L_0)?