Theories with NIP. List 5.

We work in a monster model \( \mathfrak{C} \) of a complete theory \( T \). Recall that our convention is that \( \varphi(x, a) \in L(A) \) means that \( \varphi(x, y) \in L \) and \( a \) is a tuple from \( A \).

**Problem 1.** Let \( \varphi(x, y, z) \in L(\mathfrak{C}) \). Show that \( \text{alt}(R_{\varphi(x, y, c)}(x, y)) \) computed in \( \text{Th}(M^{Sh}) \) is less than or equal to \( \text{alt}(\varphi(x, y, c)) \) computed in \( T \).

**Problem 2.** Show that if \( M \) is the random graph, then \( M^{Sh} \) does not have quantifier elimination.

**Problem 3.** Let \( M \prec \mathfrak{C} \) and \( \varphi(x, y, b) \in L(\mathfrak{C}) \). Assume that \( \varphi(M, b) \) is the graph of a function. Prove that there is \( \psi(x, y, d) \in L(\mathfrak{C}) \) such that \( \psi(M, b) = \varphi(M, b) \) and \( \mathfrak{C} \models (\forall x)(\exists \leq 1 y)\psi(x, y, d) \).

**Problem 4.** Prove that \( T \) has NIP if and only if for every finite tuple \( b \) and indiscernible sequence \( I \) of cofinality at least \( |T|^+ \), some finite segment of \( I \) is indiscernible over \( b \).

**Problem 5.** Let \( I \) be a linear order, and let \( J \) be its completion. Let \( \sim \) be a convex equivalence relation on \( I \).

(i) Assume that \( \sim \) is finite. Prove that there is a finite tuple \( \bar{c} \subseteq J \) such that \( \sim_{\bar{c}}|_{\bar{c}} \subseteq \sim \) and \( (\forall i, j \in I \setminus \bar{c})(i \sim j \iff i \sim_{\bar{c}} j) \).

(ii) Assume that \( \sim \) is essentially of size \( \kappa \). Prove that there is a tuple \( \bar{c} \subseteq J \) of length at most \( \kappa \) satisfying the same conditions as in (i).

**Problem 6.** Assume \( T \) has NIP. Let \( I = (a_i)_{i \in I} \) be an indiscernible sequence, and \( \varphi(x_1, \ldots, x_n, b) \in L(\mathfrak{C}) \). Prove that there exists a coarsest finite convex equivalence relation on \( I \) such that for every \( \bar{i}, \bar{j} \in I^n \) we have
\[
\bar{i} \sim \bar{j} \Rightarrow \models (\varphi(a_i, b) \iff \varphi(a_j, b)).
\]

*Comment.* From the lecture we know that there is a finite convex equivalence relation on \( I \) satisfying the above equivalence. In this problem, the only thing to do is to deduce that there exists a coarsest such relation.

**Problem 7.** Assume \( T \) has NIP. Let \( I = (a_i)_{i \in I} \) be an indiscernible sequence, where \( I \) is a saturated model of DLO. Let \( \text{Aut}(I) \) be the group of elementary permutations of \( I \). For every \( n \), \( \text{Aut}(I) \) acts naturally on \( S_n(I) \). Prove that the number of orbits under this action is at most \( \kappa \).