ON FIRST ORDER AMENABILITY

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Abstract. We introduce the notion of *first order [extreme] amenability*, as a property of a first order theory T: every complete type over $\emptyset$, in possibly infinitely many variables, extends to an automorphism-invariant global Keisler measure [type] in the same variables. [Extreme] amenability of T will follow from [extreme] amenability of the (topological) group $\mathrm{Aut}(M)$ for all sufficiently large $\aleph_0$-homogeneous countable models M of T (assuming T to be countable), but is radically less restrictive. First, we study basic properties of amenable theories, giving many equivalent conditions. Then, applying a version of the stabilizer theorem from [5], we prove that if T is amenable, then T is G-compact, namely Lascar strong types and Kim-Pillay strong types over $\emptyset$ coincide. This extends and essentially generalizes a similar result proved via different methods for ω-categorical theories in [14]. In the special case when amenability is witnessed by $\emptyset$-definable global Keisler measures (which is for example the case for amenable ω-categorical theories), we also give a different proof, based on stability in continuous logic.

0. Introduction

We introduce the notions of *amenable* and *extremely amenable* first order theory. This is part of our attempt to extract the model-theoretic content of the circle of ideas around [extreme] amenability of automorphism groups of countable structures, which we discuss further below. We say that T is *amenable* if for every $p \in S_{\bar{x}}(\emptyset)$, in any (possibly infinite) tuple of variables $\bar{x}$, there exists an $\mathrm{Aut}(\mathfrak{C})$-invariant, Borel probability measure on $S_p(\mathfrak{C}) := \{q \in S_{\bar{x}}(\mathfrak{C}) : p \subseteq q\}$, where $\mathfrak{C}$ is a monster model of T. *Extreme amenability* of T means that the invariant measure above can be chosen to be a *Dirac*, namely: every $p \in S_{\bar{x}}(\emptyset)$ extends to a global $\mathrm{Aut}(\mathfrak{C})$-invariant complete type. We study properties of [extreme] amenability, showing for example that they are indeed properties of the theory (i.e. do not depend on $\mathfrak{C}$) and providing several equivalent definitions. We will discuss here amenability, leaving the extreme version to further paragraphs. One of the equivalent definitions of amenability of T is that $\mathrm{Aut}(\mathfrak{C})$ is *relatively definably amenable*

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(i.e. there is an $\mathrm{Aut}(\mathfrak{C})$-invariant, finitely additive, probability measure on the Boolean algebra of relatively definable subsets of $\mathrm{Aut}(\mathfrak{C})$ treated as a subset of $\mathfrak{C}^{\mathfrak{C}}$). Relative definable amenability of $\mathrm{Aut}(\mathfrak{C})$ (or, more generally, of the group of automorphisms of any model) is a natural counterpart of definable amenability of a definable group. The above observations work for any $\aleph_0$-saturated and strongly $\aleph_0$-homogeneous model M in place of $\mathfrak{C}$. For such an M, if $\mathrm{Aut}(M)$ is amenable as a topological group (with the pointwise convergence topology), then T is amenable. We point out in a similar fashion that (for countable T) if $\mathrm{Aut}(M)$ is amenable for all sufficiently large $\aleph_0$-homogeneous countable models, then T is amenable. In the NIP context, we get a full characterization of amenability of T in various terms, e.g. by saying that $\emptyset$ is an extension base, which also yields a class of examples of amenable theories, e.g. all stable or o-minimal or c-minimal theories are amenable. Also, the theories of measurable structures in the sense of Elwes and Macpherson (e.g. pseudo-finite fields) [3] are amenable.

This paper is concerned with the implications of [extreme] amenability of a first order theory T for the Galois group $\mathrm{Gal}_L(T)$. So let us discuss briefly those Galois groups as well as the notions of G-compactness and G-triviality and why they should be considered important. Formal definitions will be given in Section 1, but we give a rather more relaxed description now. See also the introduction to [14]. At the centre are the key notions of *strong types*. Two tuples $\bar{a}$ and $\bar{b}$ from the monster model $\mathfrak{C}$, of the same (bounded) length, have the same *Lascar strong type* if $E(\bar{a},\bar{b})$ whenever E is an $\mathrm{Aut}(\mathfrak{C})$-invariant equivalence relation with boundedly many classes. If we instead consider only bounded equivalence relations E which are *type-definable* over $\emptyset$, we obtain the notion of having the same *Kim-Pillay strong type* (in short, KP-strong type). The group of permutations of all Lascar strong types induced by $\mathrm{Aut}(\mathfrak{C})$ is called the *Lascar Galois group* $\mathrm{Gal}_L(T)$; $\mathrm{Gal}_{KP}(T)$ is defined analogously. When Lascar strong types coincide with KP-strong types, $\mathrm{Gal}_L(T)$ has naturally the structure of a compact Hausdorff group, and T is said to be *G-compact*. When Lascar strong types coincide with types (over $\emptyset$), then $\mathrm{Gal}_L(T)$ is trivial, and T is said to be *G-trivial*. Lascar strong types present *obstructions* to various kinds of type amalgamation. Also in [16], where the Lascar Galois group was first defined, they present obstacles to recovering an ω-categorical theory T from its category of models. As KP-strong types types are much easier to handle than Lascar strong types, G-compactness is a desirable property. In any case, $\mathrm{Gal}_L(T)$ and $\mathrm{Gal}_{KP}(T)$ are important invariants of an arbitrary complete first order theory T and play important roles in model theory.

The main result of this paper (proved in Section 4) is the following

Theorem 0.1. *Every amenable theory is G-compact.*

This result essentially generalizes Theorem 0.7 from [14] which says that whenever M is a countable, ω-categorical structure and $\mathrm{Aut}(M)$ is amenable as a topological group, then $\mathrm{Th}(M)$ is G-compact. Theorem 0.7 of [14] was deduced (by a non-trivial argument which is interesting in its own right) from [14, Theorem 0.5], more precisely, from the fact that amenability of a topological group implies equality of certain model-theoretic/topological connected components. In [14], this last fact was proved for groups possessing a basis of open neighborhoods of the identity consisting of open subgroups, which was sufficient in the proof of [14, Theorem 0.7], because $\mathrm{Aut}(M)$ has this property; later, this fact was proved in full generality in [5, Corollary 2.37]. As to our very general Theorem 0.1, we do not have an argument showing that it follows from [5, Corollary 2.37]; instead we give a direct proof working with *relatively definable subsets* of the group of automorphisms of the monster model and using a version from [5] of Massicot-Wagner stabilizer theorem [18]. In Section 3, we give a simpler proof of Theorem 0.1, but under the stronger assumption of the existence of $\emptyset$-definable Keisler measures on all $\emptyset$-definable sets and using stability theory in continuous logic. This also includes the ω-categorical context from [14, Theorem 0.7], yielding yet another proof of [14, Theorem 0.7].

Extreme amenability of automorphism groups of (arbitrary) countable structures M was studied in detail by Kechris, Pestov, and Todorćević. Their paper [10] inspired a whole school, connecting to structural Ramsey combinatorics and dynamics. When $\mathrm{Th}(M)$ is ω-categorical, then extreme amenability of $\mathrm{Aut}(M)$ is a property of this first order theory, so is a model-theoretic notion (in the sense of model theory being the study of first order theories rather than arbitrary structures). Some of this extends to homogeneous models of arbitrary theories and to continuous logic (thanks to Todor Tsankov for a conversation about this with one of the authors).

Let us comment on the relation between extreme amenability of the automorphism group of an ω-categorical, countable structure M as considered in [10] (which we call KPT-extreme amenability) and extreme amenability of $\mathrm{Th}(M)$ in our sense. KPT-extreme amenability concerns *all* flows of the topological group $\mathrm{Aut}(M)$ and says that the universal flow (or rather ambit) has a fixed point. Our first order extreme amenability (of $\mathrm{Th}(M)$) can also be read off from flows of $\mathrm{Aut}(M)$ and says that a *particular flow* $S_{\bar{m}}(M)$ has a fixed point (where $\bar{m}$ is an enumeration of M and $S_{\bar{m}}(M)$ here denotes the space of complete extensions of $\mathrm{tp}(\bar{m})$ over M). The class of KPT-extremely amenable, ω-categorical theories T is not at present explicitly classified, but appears to be very special (analogous to monadic stability in the stable world). It follows from their definition that whenever $\mathcal{L}'$ is a language extending the language $\mathcal{L}$ of T and T' is a universal $\mathcal{L}'$-theory consistent with T, then the countable model M of T has an expansion to a model of T' where the new symbols in $\mathcal{L}'$ are interpreted as certain $\emptyset$-definable sets in M. Note in particular that KPT-amenability of an ω-categorical structure M implies the existence of a $\emptyset$-definable linear ordering on M. By contrast, our first-order extreme amenability is a quite common property; in particular, all Fraïssé classes with free (or, more generally, canonical) amalgamation enjoy it; so does T expanded by constants for a model, or, when T is stable, for an algebraically closed set in T^{eq}, and often also when T is NIP. Although not explicitly named or identified, this property has also been useful in various situations, such as for elimination of imaginaries.

Keisler measures play a big role in this paper (especially in the notion of first order amenability) and we generally assume that the reader is familiar with them. A Keisler measure on a sort (or definable set) X over a model M is simply a finitely additive (probability) measure on the Boolean algebra of definable (over M) subsets of X. As such it is a natural generalization of a complete type over M containing the formula defining X. As pointed out at the beginning of Section 4 of [7], a Keisler measure on X over M is the "same thing" as a regular Borel probability measure on the space $S_X(M)$ of complete types over M containing the formula defining X. Keisler measures are completely natural in model theory, but it took some time for them to be studied systematically. They were introduced in Keisler's seminal paper [11] mainly in a stable environment, and later played an important role in [6] in the solution of some conjectures relating o-minimal groups to compact Lie groups.

This paper contains the material in Section 4 of our preprint "Amenability and definability". Following the advice of editors and referees we have divided that preprint into two papers, the current paper being the second.

1. Preliminaries on G-compactness

We only recall a few basic definitions and facts about Lascar strong types and Galois groups. For more details the reader is referred to [17], [2] or [21].

As usual, by a monster model of a given complete theory we mean a κ-saturated and strongly κ-homogeneous model for a sufficiently large cardinal κ (typically, $\kappa > |T|$ is a strong limit cardinal). Where recall that the (standard) expression "strongly κ-homogeneous" means that any partial elementary map between subsets of the model of cardinality $< \kappa$ extends to an automorphism of the model. A set [tuple] is said to be small [short] if it is of bounded cardinality (i.e. $< \kappa$).

Let $\mathfrak{C}$ be a monster model of a complete theory T.

Definition 1.1.

i) *The group of Lascar strong automorphisms*, which is denoted by $\mathrm{Autf}_L(\mathfrak{C})$, is the subgroup of $\mathrm{Aut}(\mathfrak{C})$ which is generated by all automorphisms fixing a small submodel of $\mathfrak{C}$ pointwise, i.e. $\mathrm{Autf}_L(\mathfrak{C}) = \langle \sigma : \sigma \in \mathrm{Aut}(\mathfrak{C}/M)$ for a small $M \prec \mathfrak{C}\rangle$.

ii) *The Lascar Galois group of* T, which is denoted by $\mathrm{Gal}_L(T)$, is the quotient group $\mathrm{Aut}(\mathfrak{C})/\mathrm{Autf}_L(\mathfrak{C})$ (which makes sense, as $\mathrm{Autf}_L(\mathfrak{C})$ is a normal subgroup of $\mathrm{Aut}(\mathfrak{C})$). It turns out that $\mathrm{Gal}_L(T)$ does not depend on the choice of $\mathfrak{C}$.

The orbit equivalence relation of $\mathrm{Autf}_L(\mathfrak{C})$ acting on any given product S of boundedly (i.e. less than the degree of saturation of $\mathfrak{C}$) many sorts of $\mathfrak{C}$ is usually denoted by E_L. It turns out that this is the finest bounded (i.e. with boundedly many classes), invariant equivalence relation on S; and the same is true after the restriction to the set of realizations of any type in $S(\emptyset)$. The classes of E_L are called *Lascar strong types*. It turns out that $\mathrm{Autf}_L(\mathfrak{C})$ coincides with the the group of all automorphisms fixing setwise all E_L-classes on all (possibly infinite) products of sorts.

For any small $M \prec \mathfrak{C}$ enumerated as $\bar{m}$, we have a natural surjection from $S_{\bar{m}}(M) := \{p \in S(M) : \mathrm{tp}(\bar{m}/M) \subseteq p\}$ to $\mathrm{Gal}_L(T)$ given by $\mathrm{tp}(\sigma(\bar{m})/M) \mapsto \sigma/\mathrm{Autf}_L(\mathfrak{C})$ for $\sigma \in \mathrm{Aut}(\mathfrak{C})$. We can equip $\mathrm{Gal}_L(T)$ with the quotient topology induced by this surjection, and it is easy to check that this topology does not depend on the choice of M. In this way, $\mathrm{Gal}_L(T)$ becomes a topological (but not necessarily Hausdorff) group (see [21] for a detailed exposition).

Definition 1.2.

i) By $\mathrm{Gal}_0(T)$ we denote the closure of the identity in $\mathrm{Gal}_L(T)$.
ii) *The group of Kim-Pillay strong automorphisms*, which is denoted by $\mathrm{Autf}_{KP}(\mathfrak{C})$, is the preimage of $\mathrm{Gal}_0(T)$ under the quotient homomorphism $\mathrm{Aut}(\mathfrak{C}) \to \mathrm{Gal}_L(T)$.
iii) *The Kim-Pillay Galois group of* T, which is denoted by $\mathrm{Gal}_{KP}(T)$, is the quotient group $\mathrm{Gal}_L(T)/\mathrm{Gal}_0(T) \cong \mathrm{Aut}(\mathfrak{C})/\mathrm{Autf}_{KP}(\mathfrak{C})$ equipped with the quotient topology. It is a compact, Hausdorff topological group.

The orbit equivalence relation of $\mathrm{Autf}_{KP}(\mathfrak{C})$ acting on any given product S of (boundedly many) sorts of $\mathfrak{C}$ is usually denoted by E_{KP}. It turns out that this is the finest bounded (i.e. with boundedly many classes), type-definable over $\emptyset$ equivalence relation on S; and the same is true after the restriction to the set of realizations of any type in $S(\emptyset)$. The classes of E_{KP} are called *Kim-Pillay strong types*. It turns out that $\mathrm{Autf}_{KP}(\mathfrak{C})$ coincides with the the group of all automorphisms fixing setwise all E_{KP}-classes on all (possibly infinite) products of sorts.

The theory T is said to be *G-compact* if the following *equivalent* conditions hold.

(1) $\mathrm{Autf}_L(\mathfrak{C}) = \mathrm{Autf}_{KP}(\mathfrak{C})$.
(2) $\mathrm{Gal}_L(T)$ is Hausdorff.
(3) Lascar strong types coincide with Kim-Pillay strong types on any (possibly infinite) products of sorts.

By the definition of E_L, we see that $\bar{\alpha}\, E_L\, \bar{\beta}$ if and only if there are $\bar{\alpha}_0 = \bar{\alpha}, \bar{\alpha}_1, \dots, \bar{\alpha}_n = \bar{\beta}$ and models $M_0, \dots, M_{n-1}$ such that

$$\bar{\alpha}_0 \equiv_{M_0} \bar{\alpha}_1 \equiv_{M_1} \dots \bar{\alpha}_{n-1} \equiv_{M_{n-1}} \bar{\alpha}_n.$$

In this paper, by the *Lascar distance* from $\bar{\alpha}$ to $\bar{\beta}$ (denoted by $d_L(\bar{\alpha}, \bar{\beta})$) we mean the smallest natural number n as above. By the *Lascar diameter* of a Lascar strong type $[\bar{\alpha}]_{E_L}$ we mean the supremum of $d_L(\bar{\alpha}, \bar{\beta})$ with $\bar{\beta}$ ranging over $[\bar{\alpha}]_{E_L}$. It is well known (proved in [19]) that $[\bar{\alpha}]_{E_L} = [\bar{\alpha}]_{E_{KP}}$ if and only if the Lascar diameter of $[\bar{\alpha}]_{E_L}$ is finite.

2. Amenable theories: definitions and basic results

As usual, $\mathfrak{C}$ is a monster model of an arbitrary complete theory T. Let $\bar{c}$ be an enumeration of $\mathfrak{C}$ and let $S_{\bar{c}}(\mathfrak{C}) = \{\mathrm{tp}(\bar{a}/\mathfrak{C}) \in S(\mathfrak{C}) : \bar{a} \equiv \bar{c}\}$. More generally, for a partial type $\pi(\bar{x})$ over $\emptyset$, put $S_\pi(\mathfrak{C}) = \{q(\bar{x}) \in S(\mathfrak{C}) : \pi \subseteq q\}$. If $p(\bar{x}) \in S(\emptyset)$ and $\bar{\alpha} \models p$, then $S_{\bar{\alpha}}(\mathfrak{C}) := S_p(\mathfrak{C}) = \{q(\bar{x}) \in S(\mathfrak{C}) : p \subseteq q\}$. (Note that we allow here tuples $\bar{x}$ of unbounded length (i.e. greater than the degree of saturation of $\mathfrak{C}$). Each $S_\pi(\mathfrak{C})$ is naturally an $\mathrm{Aut}(\mathfrak{C})$-flow.

Let us start from the local version of amenability.

Definition 2.1. A partial type $\pi(\bar{x})$ over $\emptyset$ is *amenable* if there is an $\mathrm{Aut}(\mathfrak{C})$-invariant, Borel (regular) probability measure on $S_\pi(\mathfrak{C})$.

Remark 2.2. The following conditions are equivalent for a type $\pi(\bar{x})$ over $\emptyset$.

1. $\pi(\bar{x})$ is amenable.
2. There is an $\mathrm{Aut}(\mathfrak{C})$-invariant, Borel (regular) probability measure μ on $S_{\bar{x}}(\mathfrak{C})$ concentrated on $S_\pi(\mathfrak{C})$, i.e. for any formula $\varphi(\bar{x}, \bar{a})$ inconsistent with $\pi(\bar{x})$, $\mu([\varphi(\bar{x}, \bar{a})]) = 0$ (where $[\varphi(\bar{x}, \bar{a})]$ is the subset of $S_{\bar{x}}(\mathfrak{C})$ consisting of all types containing $\varphi(\bar{x}, \bar{a})$).
3. There is an $\mathrm{Aut}(\mathfrak{C})$-invariant, finitely additive probability measure on relatively $\mathfrak{C}$-definable subsets of $\pi(\bar{x})$.
4. There is an $\mathrm{Aut}(\mathfrak{C})$-invariant, finitely additive probability measure on $\mathfrak{C}$-definable sets in variables $\bar{x}$, concentrated on $\pi(\bar{x})$ (i.e. for any formula $\varphi(\bar{x}, \bar{a})$ inconsistent with $\pi(\bar{x})$, $\mu(\varphi(\bar{x}, \bar{a})) = 0$).

Proof. Follows easily using the fact (see [20, Chapter 7.1]) that whenever G acts by homeomorphisms on a compact, Hausdorff, 0-dimensional space X, then each G-invariant, finitely additive probability measure on the Boolean algebra of clopen subsets of X extends to a G-invariant, Borel (regular) probability measure on X. □

Thus, by a *global* $\mathrm{Aut}(\mathfrak{C})$*-invariant Keisler measure extending* $\pi(\bar{x})$ we mean a measure from any of the items of Remark 2.2. And similarly working over any model M in place of $\mathfrak{C}$.

Proposition 2.3. *Amenability of a given type $\pi(\bar{x})$ (over $\emptyset$) is absolute in the sense that it does not depend on the choice of the monster model $\mathfrak{C}$. It is also equivalent to the amenability of $\pi(\bar{x})$ computed with respect to an $\aleph_0$-saturated and strongly $\aleph_0$-homogeneous model M in place of $\mathfrak{C}$.*

Proof. Let M and M' be two $\aleph_0$-saturated and strongly $\aleph_0$-homogeneous models. Assume that there is an $\mathrm{Aut}(M)$-invariant, Borel (regular) probability measure μ on $S_\pi(M)$. We want to find such $\mathrm{Aut}(M')$-invariant measure μ' on $S_\pi(M')$.

Consider any formula $\varphi(\bar{x},\bar{a}')$ with $\bar{a}'\in M'$. Choose (using the $\aleph_0$-saturation of M) any $\bar{a}\in M$ such that $\bar{a}'\equiv\bar{a}$, and define

$$\mu'([\varphi(\bar{x},\bar{a}')]\cap S_\pi(M')) := \mu([\varphi(\bar{x},\bar{a})]\cap S_\pi(M)).$$

By the strong $\aleph_0$-homogeneity of M and $\mathrm{Aut}(M)$-invariance of μ, we see that μ' is well-defined and $\mathrm{Aut}(M')$-invariant. It is also clear that $\mu'(S_\pi(M'))=1$. It remains to check μ' is finitely additive on clopen subsets (as then μ' extends to the desired Borel measure). Take $\varphi(\bar{x},\bar{a}')$ and $\psi(\bar{x},\bar{a}')$ such that $[\varphi(\bar{x},\bar{a}')]\cap S_\pi(M')$ is disjoint from $[\psi(\bar{x},\bar{a}')]\cap S_\pi(M')$. This just means that $\varphi(\bar{x},\bar{a}')\wedge\psi(\bar{x},\bar{a}')$ is inconsistent with $\pi(\bar{x})$. Take $\bar{a}\in M$ such that $\bar{a}\equiv\bar{a}'$. Then $\varphi(\bar{x},\bar{a})\wedge\psi(\bar{x},\bar{a})$ is still inconsistent with $\pi(\bar{x})$, so $\mu'(([\varphi(\bar{x},\bar{a}')]\cap S_\pi(M'))\cup([\psi(\bar{x},\bar{a}')]\cap S_\pi(M')))=\mu(([\varphi(\bar{x},\bar{a})]\cap S_\pi(M))\cup([\psi(\bar{x},\bar{a})]\cap S_\pi(M)))=\mu([\varphi(\bar{x},\bar{a})]\cap S_\pi(M))+\mu([\psi(\bar{x},\bar{a})]\cap S_\pi(M))=\mu'([\varphi(\bar{x},\bar{a}')]\cap S_\pi(M'))+\mu'([\varphi(\bar{x},\bar{a}')]\cap S_\pi(M'))$. □

Remark 2.4. Assume T to be countable, and let $\pi(\bar{x})$ be a partial type. Then $\pi(\bar{x})$ is amenable if and only if for all [sufficiently large] countable, ($\aleph_0$-)homogeneous models M, $\pi(\bar{x})$ has an extension to a Keisler measure $\mu(\bar{x})$ over M which is $\mathrm{Aut}(M)$-invariant. If T is uncountable, the same is true but with "countable, $\aleph_0$-homogeneous models" replaced by "strongly $\aleph_0$-homogeneous models of cardinality at most $|T|$".

Proof. For each [sufficiently large] countable homogeneous model $M\prec\mathfrak{C}$, let $\mu_M(\bar{x})$ be an $\mathrm{Aut}(M)$-invariant Keisler measure over M extending π, and let $\bar{\mu}_M$ be an arbitrary global Keisler measure extending μ_M. Working in the compact space of global Keisler measures, there is a subnet of the net $\{\bar{\mu}_M\}_M$, which converges to some $\bar{\mu}$. But then $\bar{\mu}$ is $\mathrm{Aut}(\mathfrak{C})$-invariant: For otherwise, for some formula $\phi(\bar{x},\bar{y})$ and tuples $\bar{a},\bar{b}$ in $\mathfrak{C}$ with the same type, we have $\bar{\mu}(\phi(\bar{x},\bar{a}))=r$ and $\bar{\mu}(\phi(\bar{x},\bar{b}))=s$ for some $r<s$. But then we can find some countable homogeneous model M containing $\bar{a},\bar{b}$ and such that $\bar{\mu}_M(\phi(\bar{x},\bar{a}))<\bar{\mu}_M(\phi(\bar{x},\bar{b}))$, contradicting the $\mathrm{Aut}(M)$-invariance of $\bar{\mu}_M$. □

Lemma 2.5. *A type $\pi(\bar{x})$ (over $\emptyset$) is amenable if and only if each formula $\varphi(\bar{x})$ (without parameters) implied by $\pi(\bar{x})$ is amenable.*

Proof. The implication ($\rightarrow$) is obvious, as $S_\pi(\mathfrak{C})\subseteq S_\varphi(\mathfrak{C})$, and so for any formula $\psi(\bar{x},\bar{a})$ we can define $\mu'([\psi(\bar{x},\bar{a})]\cap S_\varphi(\mathfrak{C})) := \mu([\psi(\bar{x},\bar{a})]\cap S_\pi(\mathfrak{C}))$, where μ is an $\mathrm{Aut}(\mathfrak{C})$-invariant, Borel probability measure on $S_\pi(\mathfrak{C})$.

($\leftarrow$). On the set of formulas (without parameters) implied by $\pi(\bar{x})$, consider an ultrafilter $\mathcal{U}$ containing for every $\varphi(\bar{x})\dashv\pi(\bar{x})$ the set $\{\psi(\bar{x}):\pi(\bar{x})\vdash\psi(\bar{x})\vdash\varphi(\bar{x})\}$.

By assumption and Remark 2.2, for any $\varphi(\bar{x})\dashv\pi(\bar{x})$ we have an $\mathrm{Aut}(\mathfrak{C})$-invariant, finitely additive probability measure μ_φ on $\mathfrak{C}$-definable subsets of $\mathfrak{C}^{\bar{x}}$ which is concentrated on $\varphi(\bar{x})$. Put $\mathfrak{C}' := \prod_{\varphi(\bar{x})\dashv\pi(\bar{x})}\mathfrak{C}/\mathcal{U}$ and define

$$\mu' := \mathrm{st}\left(\prod_{\varphi(\bar{x})\dashv\pi(\bar{x})}\mu_\varphi/\mathcal{U}\right),$$

where st stands for the standard part map. It is clear that μ' is is a finitely additive probability measure on definable subsets of $\mathfrak{C}'^{\bar{x}}$. By the choice of $\mathcal{U}$, μ' is concentrated on $\pi(\bar{x})$. By the $\mathrm{Aut}(\mathfrak{C})$-invariance of all μ_φ, for any finite $\bar{a} \equiv \bar{b}$ from $\mathfrak{C}$, for any $\varphi(\bar{x}) \dashv \pi(\bar{x})$ and any $\psi(\bar{x}, \bar{y})$, we have $\mu_\varphi(\psi(\bar{x}, \bar{a})) = \mu_\varphi(\psi(\bar{x}, \bar{b}))$. Therefore, $\mu'(\psi(\bar{x}, \bar{a})) = \mu'(\psi(\bar{x}, \bar{b}))$ (treating $\bar{a}$ and $\bar{b}$ as tuples from $\mathfrak{C}'$). Finally, let μ be the restriction of μ' to the algebra of $\mathfrak{C}$-definable sets. We conclude that μ is an $\mathrm{Aut}(\mathfrak{C})$-invariant, finitely additive probability measure on definable subsets of $\mathfrak{C}^{\bar{x}}$ which is concentrated on $\pi(\bar{x})$, which is enough by Remark 2.2. □

Lemma 2.6. *All types in $S(\emptyset)$ (possibly in unboundedly many variables) are amenable if and only if all finitary types in $S(\emptyset)$ are amenable.*

Proof. The implication ($\rightarrow$) is trivial. For the other implication, take $p(\bar{x}) \in S_{\bar{x}}(\emptyset)$. Consider the compact space $X := [0, 1]^{\{\varphi(\bar{x},\bar{a}) :\ \varphi(\bar{x},\bar{y}) \text{ a formula, } \bar{a}\in\mathfrak{C}\}}$ with the pointwise convergence topology (where $\bar{x}$ is the fixed tuple of variables). Then the $\mathrm{Aut}(\mathfrak{C})$-invariant, finitely additive probability measures on $\mathfrak{C}$-definable sets in variables $\bar{x}$ concentrated on $p(\bar{x})$ form a closed subset $\mathcal{M}$ of X. We can present $\mathcal{M}$ as the intersection of a directed family of closed subsets of X each of which witnessing a finite portion of information of being in $\mathcal{M}$. But each such finite portion of information involves only finitely many variables, so the corresponding closed set is nonempty by the assumption that all finitary types are amenable and Remark 2.2. By the compactness of X, we conclude that $\mathcal{M}$ is nonempty. □

Corollary 2.7. *The following conditions are equivalent.*

1. *All partial types (possibly in unboundedly many variables) over $\emptyset$ are amenable.*
2. *All complete types (possibly in unboundedly many variables) over $\emptyset$ are amenable.*
3. *All finitary complete types over $\emptyset$ are amenable.*
4. *All consistent formulas (in finitely many variables $\bar{x}$) over $\emptyset$ are amenable.*
5. $\mathrm{tp}(\bar{c}/\emptyset)$ *is amenable.*
6. $\mathrm{tp}(\bar{m}/\emptyset)$ *is amenable for some tuple $\bar{m}$ enumerating a model.*

Proof. The equivalence (1) $\leftrightarrow$ (2) is obvious (for (2) $\rightarrow$ (1) use the argument as in the proof of ($\rightarrow$) in Lemma 2.5). The equivalence (2) $\leftrightarrow$ (3) is Lemma 2.6. The equivalence (3) $\leftrightarrow$ (4) follows from Lemma 2.5. The implications (1) $\rightarrow$ (5) $\rightarrow$ (6) are trivial. Finally, (6) $\rightarrow$ (4) also follows from Lemma 2.5, because taking all possible finite subtuples $\bar{m}'$ of $\bar{m}$ and $\varphi(\bar{x}') \in \mathrm{tp}(\bar{m}'/\emptyset)$, we will get all consistent formulas over $\emptyset$. □

Definition 2.8. The theory T is *amenable* if the equivalent conditions of Corollary 2.7 hold.

By Proposition 2.3, we see that amenability of T is really a property of T, i.e. it does not depend on the choice of $\mathfrak{C}$.

Analogously, one can define the stronger notion of an extremely amenable theory.

Definition 2.9. A type $\pi(\bar{x})$ over $\emptyset$ is *extremely amenable* if there is an $\mathrm{Aut}(\mathfrak{C})$-invariant type in $S_\pi(\mathfrak{C})$. The theory T is *extremely amenable* if every type (in any number of variables) in $S(\emptyset)$ is extremely amenable.

As in the case of amenability, compactness arguments easily show that these notions are absolute (i.e. do not depend on the choice of $\mathfrak{C}$), and, in fact, they can be tested on any $\aleph_0$-saturated and strongly $\aleph_0$-homogeneous model in place of $\mathfrak{C}$; moreover, T is extremely amenable if and only if all finitary types in $S(\emptyset)$ are extremely amenable. Note that Remark 2.4 specializes to extremely amenable partial types, too. So note that for countable theories, both amenability and extreme amenability can be seen at the level of countable models. It is also easy to see that in a stable theory, a type in $S(\emptyset)$ is extremely amenable if and only if it is stationary.

Yet another equivalent approach to amenability of T is via $\mathrm{Aut}(\mathfrak{C})$-invariant, finitely additive probability measures on the algebra of so-called relatively definable subsets of $\mathfrak{C}$. This will be the exact analogue of the definition of definable amenability of definable groups (via the existence of an invariant Keisler measure). We will use this approach in Section 4.

The idea of identifying $\mathrm{Aut}(\mathfrak{C})$ with the subset $\{\sigma(\bar{c}) : \sigma \in \mathrm{Aut}(\mathfrak{C})\}$ of $\mathfrak{C}^{\bar{c}}$ and considering relatively definable subsets of $\mathrm{Aut}(\mathfrak{C})$, i.e. subsets of the form $\{\sigma \in \mathrm{Aut}(\mathfrak{C}) : \mathfrak{C} \models \varphi(\sigma(\bar{c}), \bar{c})\}$ for a formula $\varphi(\bar{x}, \bar{c})$, already appeared in [15, Appendix A]. Here, we extend this notion of relative definability to the local context and introduce an associated notion of amenability which is easily seen to be equivalent to the amenability of T [or of a certain type in the extended local version].

Let M be any model of T and let $\bar{m}$ be its enumeration.

Definition 2.10. i) By a *relatively definable subset* of $\mathrm{Aut}(M)$ we mean a subset of the form $\{\sigma \in \mathrm{Aut}(M) : M \models \varphi(\sigma(\bar{m}), \bar{m})\}$, where $\varphi(\bar{x}, \bar{y})$ is a formula without parameters.
ii) If $\bar{\alpha}$ is a tuple of some elements of M, by *relatively $\bar{\alpha}$-definable subset* of $\mathrm{Aut}(M)$ we mean a subset of the form $\{\sigma \in \mathrm{Aut}(M) : M \models \varphi(\sigma(\bar{\alpha}), \bar{m})\}$, where $\varphi(\bar{x}, \bar{y})$ is a formula without parameters.

The above definition differs from the standard terminology in which "A-definable" means "definable over A"; here, "relatively $\bar{\alpha}$-definable" has nothing to do with the parameters over which the set is relatively definable. One should keep this in mind from now on.

For a formula $\varphi(\bar{x}, \bar{y})$ and tuples $\bar{a}$, $\bar{b}$ from M corresponding to $\bar{x}$ and $\bar{y}$, respectively, we will use the following notation

$$A_{\varphi,\bar{a},\bar{b}} = \{\sigma \in \mathrm{Aut}(M) : M \models \varphi(\sigma(\bar{a}), \bar{b})\}.$$

When $\bar{x}$ and $\bar{y}$ are of the same length (by which we mean that they are also of the same sorts) and $\bar{a} = \bar{b}$, then this set will be denoted by $A_{\varphi,\bar{a}}$.

Definition 2.11. i) The group $\mathrm{Aut}(M)$ is said to be *relatively definably amenable* if there exists a left $\mathrm{Aut}(M)$-invariant, finitely additive probability measure on the Boolean algebra of relatively definable subsets of $\mathrm{Aut}(M)$.
ii) If $\bar{\alpha}$ is a tuple of some elements of M, the group $\mathrm{Aut}(M)$ is said to be *relatively $\bar{\alpha}$-definably amenable* if there exists a left $\mathrm{Aut}(M)$-invariant, finitely additive probability measure on the Boolean algebra of relatively $\bar{\alpha}$-definable subsets of $\mathrm{Aut}(M)$.

In particular, $\mathrm{Aut}(M)$ being relatively definably amenable means exactly that it is relatively $\bar{m}$-definably amenable.

We will mostly focus on the case when $M = \mathfrak{C}$ is a monster model. But often one can work in the more general context when M is $\aleph_0$-saturated and strongly $\aleph_0$-homogeneous, including the case of the unique countable model of an ω-categorical theory.

Remark 2.12. Let M be $\aleph_0$-saturated and strongly $\aleph_0$-homogeneous enumerated as $\bar{m}$. Let $\bar{\alpha}$ be a tuple of some elements of M. Then $\mathrm{Aut}(M)$ is relatively $\bar{\alpha}$-definably amenable if and only if there is an $\mathrm{Aut}(M)$-invariant, (regular) Borel probability measure on $S_{\bar{\alpha}}(M)$ (equivalently, $\mathrm{tp}(\bar{\alpha}/\emptyset)$ is amenable). In particular, $\mathrm{Aut}(M)$ is relatively definable if and only if there is an $\mathrm{Aut}(M)$-invariant, (regular) Borel probability measure on $S_{\bar{m}}(M)$ (equivalently, T is amenable).

Proof. Suppose first that $\mathrm{Aut}(M)$ is relatively $\bar{\alpha}$-definably amenable, witnessed by a measure μ. For a formula $\varphi(\bar{x}, \bar{m})$ let $[\varphi(\bar{x}, \bar{m})]$ be the basic clopen set in $S_{\bar{\alpha}}(M)$ given by this formula. Define $\tilde{\mu}([\varphi(\bar{x}, \bar{m})]) := \mu(A_{\varphi,\bar{m}})$. It is clear that $\tilde{\mu}$ is an $\mathrm{Aut}(M)$-invariant, finitely additive probability measure on the algebra of clopen subsets of $S_{\bar{\alpha}}(M)$. This $\tilde{\mu}$ extends (by [20, Chapter 7.1]) to an $\mathrm{Aut}(M)$-invariant, (regular) Borel probability measure on $S_{\bar{\alpha}}(M)$.

Conversely, assume that $\tilde{\mu}$ is an $\mathrm{Aut}(M)$-invariant, Borel probability measure on $S_{\bar{\alpha}}(M)$. For any relatively $\bar{\alpha}$-definable subset $A_{\varphi,\bar{m}}$ define $\mu(A_{\varphi,\bar{m}}) := \tilde{\mu}([\varphi(\bar{x}, \bar{m})])$. By the $\aleph_0$-saturation and strong $\aleph_0$-homogeneity of M, we easily get that μ is a well-defined, $\mathrm{Aut}(M)$-invariant, finitely additive probability measure on relatively $\bar{\alpha}$-definable subsets of $\mathrm{Aut}(M)$.

The fact that the existence of an $\mathrm{Aut}(M)$-invariant, (regular) Borel probability measure on $S_{\bar{\alpha}}(M)$ is equivalent to amenability of $\mathrm{tp}(\bar{\alpha}/\emptyset)$ follows from Proposition 2.3. And then, the fact that the existence an $\mathrm{Aut}(M)$-invariant, (regular) Borel probability measure on $S_{\bar{m}}(M)$ is equivalent to amenability of T follows from Corollary 2.7. □

So the terminologies "$\mathrm{Aut}(M)$ is relatively [$\bar{\alpha}$-]definably amenable" and "T [resp. $\mathrm{tp}(\bar{\alpha}/\emptyset)$] is amenable" will be used interchangeably.

Corollary 2.13. *[For a given tuple $\bar{\alpha}$] relative [$\bar{\alpha}$-]definable amenability of an $\aleph_0$-saturated and strongly $\aleph_0$-homogeneous model M [containing $\bar{\alpha}$] does not depend on the choice of M.*

Recall that a G-*flow* (for a topological group G) is a pair (G, X), where X is a compact, Hausdorff space on which G acts continuously; a G-*ambit* is a G-flow (G, X, x_0) with a distinguished point $x_0 \in X$ with dense G-orbit. The topological group G is said to be *[extremely] amenable* if each G-flow (equivalently, the universal G-ambit) has an invariant, Borel probability measure [respectively, a fixed point].

Corollary 2.14. *Let M be $\aleph_0$-saturated and strongly $\aleph_0$-homogeneous. Then, if* $\mathrm{Aut}(M)$ *is amenable as a topological group (with the pointwise convergence topology), then it is relatively definably amenable, which in turn implies that it is relatively $\bar{\alpha}$-definably amenable for any tuple $\bar{\alpha}$ of elements M.*

Similarly, extreme amenability of $\mathrm{Aut}(M)$ *as a topological group implies extreme amenability of T.*

Proof. Amenability of $\mathrm{Aut}(M)$ implies that there is an $\mathrm{Aut}(M)$-invariant, Borel probability measure on $S_{\bar{m}}(M)$. By Remark 2.12, this implies relative definable amenability of $\mathrm{Aut}(M)$. Furthermore, since there is an obvious flow homomorphism from $S_{\bar{m}}(M)$ to $S_{\bar{\alpha}}(M)$, a measure on $S_{\bar{m}}(M)$ induces a measure on $S_{\bar{\alpha}}(M)$, and this is enough by Remark 2.12. □

As in the introduction, we will call a countable $\aleph_0$-categorical theory KPT-[extremely] amenable if the automorphism group of its unique countable model is [extremely] amenable as a topological group.

So, by Corollary 2.14, KPT-[extreme] amenability of a (countable $\aleph_0$-categorical) theory T implies [extreme] amenability of T in the new sense of this paper. In fact most, if not all, of the examples of not only KPT-extremely amenable theories (such as dense linear orderings) but also KPT-amenable (not necessarily KPT-extremely amenable) theories (such as the random graph) come from Fraïssé classes with *canonical amalgamation*, hence are extremely amenable in our sense. Only canonical amalgamation over $\emptyset$ is needed here (see the next paragraph for a justification) which says that there is a map $\otimes$ taking pairs of finite structures (A, B) from the Fraïssé class to an amalgam $A \otimes B$ (also in the Fraïssé class) which is compatible with embeddings, i.e. if $f : B \to C$ is an embedding of finite structure structures from the Fraïssé class, then there exists an embedding from $A \otimes B$ to $A \otimes C$ which commutes with f and with the embeddings: $A \to A \otimes B$, $B \to A \otimes B$, $A \to A \otimes C$, and $C \to A \otimes C$. A typical example is a Fraïssé class with "free amalgamation", namely adding no new relations.

Let us briefly explain why canonical amalgamation of a Fraïssé class of finite structures in a relational language [or, more generally, finitely generated structures in any language] whose Fraïssé limit M is ω-categorical implies extreme amenability. First, note that canonical amalgamation implies that for any finite tuples $\bar{d}, \bar{a}_1, \bar{b}_1, \ldots, \bar{a}_n, \bar{b}_n$ from M, if the structures $\bar{a}_i$ and $\bar{b}_i$ are isomorphic (i.e. have the same quantifier-free type), then we can amalgamate structures $\bar{d}$ and

$(\bar{a}_i, \bar{b}_i : i \leq n)$ into a structure $\bar{d}', \bar{a}'_1, \bar{b}'_1, \dots, \bar{a}'_n, \bar{b}'_n$ in such a way that $\bar{a}'_i$ is isomorphic with $\bar{b}'_i$ over $\bar{d}'$ for all $i \leq n$. Therefore, using ω-categoricity and quantifier elimination, one concludes by compactness that any finitary type in $S(\emptyset)$ extends to an $\mathrm{Aut}(M)$-invariant type in $S(M)$, so T is extremely amenable (since M is ω-categorical).

In [14], we proved that both KPT-amenability and KPT-extreme amenability are preserved by adding finitely many parameters. This is not the case for our notion of first order [extreme] amenability. For example, if T is the theory of two equivalence relations E_1, E_2, where E_1 has infinitely many classes, all infinite, and each E_1-class is divided into two E_2-classes, both infinite, then T is extremely amenable, but adding an (imaginary) parameter for an E_1-class destroys extreme amenability. Similar examples can be built by putting uniformly in each E_1-class some non amenable theory.

Let us state one more corollary of Remark 2.12, which will be useful in Section 4.

Corollary 2.15. *If* $\mathrm{Aut}(\mathfrak{C})$ *is relatively* $\bar{\alpha}$*-definably amenable, where* $\bar{\alpha}$ *is a tuple in* $\mathfrak{C}$ *(e.g.* $\bar{\alpha} = \bar{c}$*), then there exists an* $\mathrm{Aut}(\mathfrak{C})$*-invariant, finitely additive probability measure on the Boolean algebra generated by relatively* $\bar{\alpha}$*-type-definable sets, i.e. sets of the form* $\{\sigma \in \mathrm{Aut}(\mathfrak{C}) : \mathfrak{C} \models \pi(\sigma(\bar{a}), \bar{b}))\}$ *for some partial type* $\pi(\bar{x}, \bar{y})$*, where* $\bar{x}$ *and* $\bar{y}$ *are short tuples of variables, and* $\bar{a}$, $\bar{b}$ *are tuples from* $\mathfrak{C}$ *corresponding to* $\bar{x}$ *and* $\bar{y}$*, respectively, such that* $\bar{a}$ *is a subtuple of* $\bar{\alpha}$*.*

In particular, if $\mathrm{Aut}(\mathfrak{C})$ *is relatively definably amenable, then there exists an* $\mathrm{Aut}(\mathfrak{C})$*-invariant, finitely additive probability measure on the Boolean algebra generated by relatively type-definable sets (i.e., relatively* $\bar{c}$*-type-definable sets).*

Proof. A set X belongs to the Boolean algebra in question if and only if it is of the form $\{\sigma \in \mathrm{Aut}(\mathfrak{C}) : \mathrm{tp}(\sigma(\bar{a})/A)) \in \mathcal{P}\}$, where $A \subseteq \mathfrak{C}$ is a (small) set, $\bar{a}$ is a short subtuple of $\bar{\alpha}$, and $\mathcal{P}$ is a finite Boolean combination of closed subsets of $S_{\bar{a}}(A)$. By Remark 2.12, there is an $\mathrm{Aut}(\mathfrak{C})$-invariant, (regular) Borel probability measure $\tilde{\mu}$ on $S_{\bar{\alpha}}(\mathfrak{C})$. Then define $\mu(X) := \tilde{\mu}(\pi^{-1}[\mathcal{P}])$, where $\pi \colon S_{\bar{\alpha}}(\mathfrak{C}) \to S_{\bar{a}}(A)$ is the restriction map. It is easy to check that it is a well-defined measure as required. □

Recall that in an NIP theory, for any global type p the following conditions are equivalent (see [7, Proposition 2.11]).

(1) p does not fork over $\emptyset$.
(2) The $\mathrm{Aut}(\mathfrak{C})$-orbit of p is bounded.
(3) p is Kim-Pillay invariant (i.e. invariant under $\mathrm{Autf}_{KP}(\mathfrak{C})$).
(4) p is Lascar invariant.

More importantly, Proposition 4.7 of [7] tells us that in a NIP theory, a type $p \in S(\emptyset)$ is amenable if and only if it does not fork over $\emptyset$ (equivalently, it has a global non-forking extension).

Corollary 2.16. *Assume T has NIP. Then, T is amenable if and only if $\emptyset$ is an extension base (i.e. any type over $\emptyset$ does not fork over $\emptyset$). In particular, stable, o-minimal, and c-minimal theories are all amenable (even after adding constants).*

Note that, in fact, in an arbitrary amenable theory, $\emptyset$ is an extensions base. To see this, consider an arbitrary type $p(\bar{x}) \in S_{\bar{x}}(\emptyset)$. Let μ be a global, invariant Keisler measure extending $p(\bar{x})$. Choose a μ-wide type $q(\bar{x}) \in S_{\bar{x}}(\mathfrak{C})$ (i.e. any formula in $q(\bar{x})$ is of positive measure). Then $q(\bar{x})$ is a non-forking extension of $p(\bar{x})$, so we are done. Thus, amenability of T is a strong form of saying that $\emptyset$ is an extension base; and amenability after adding any constants is a strong form of saying that every set is an extensions base.

By [7, Corollary 2.10], the characterization from Corollary 2.16 gives us

Corollary 2.17. *Assume T has NIP. Then amenability of T implies G-compactness.*

Theorem 0.1 is a generalization of the last corollary to arbitrary amenable theories, but it requires completely different methods compared with the NIP case.

It is worth mentioning that Theorem 7.7 of [13] yields several other conditions equivalent (under NIP) to the existence of $p \in S_{\bar{c}}(\mathfrak{C})$ with bounded $\mathrm{Aut}(\mathfrak{C})$-orbit (and so to amenability of T), for example: some (equivalently, every) minimal left ideal of the Ellis semigroup of the $\mathrm{Aut}(\mathfrak{C})$-flow $S_{\bar{c}}(\mathfrak{C})$ is of bounded size. In particular, a variant of Newelski's conjecture proved in [13, Theorem 0.7] can be stated as follows: if T is an amenable theory with NIP, then a certain natural epimorphism from the Ellis group of T to $\mathrm{Gal}_{KP}(T)$ is an isomorphism. This also implies G-compactness of amenable, NIP theories.

Let us finally mention in this section some relations between our notions of amenability and extreme amenability of a theory T and the notion of a strongly determined over $\emptyset$ theory from [8] (originating in work of Ivanov and Macpherson [9]). Decoding the definition in [8], T is strongly determined over $\emptyset$ if any complete type $p(\bar{x})$ over $\emptyset$ has an extension to a complete type $p'(\bar{x})$ over $\mathfrak{C}$ which is $\mathrm{acl}^{eq}(\emptyset)$-invariant. So clearly T extremely amenable implies T is strongly determined over $\emptyset$. Moreover, by Corollary 2.16, assuming NIP, T strongly determined over $\emptyset$ implies amenability of T. In fact, if T is NIP and KP-strong types agree with usual strong types (over $\emptyset$), then T is strongly determined over $\emptyset$ iff T is amenable.

3. Amenability implies G-compactness: the case of definable measures

Theorem 0.1 will be proved in full generality in Section 4. However, some special cases have a relatively easy proof. One such is the NIP case above. Another case is when T is extremely amenable, where the proof of Remark 4.21 of [14] shows that in fact T is G-trivial (the Lascar group is trivial). This is made explicit in Proposition 4.2 below. Ivanov's observation in [8] that if T is strongly determined over $\emptyset$, then Lascar strong types coincide with (Shelah) strong types follows from

Proposition 4.2 by working over $\mathrm{acl}^{eq}(\emptyset)$. However, deducing G-compactness of T from amenability of T in general is more complicated, and the proof in Section 4 uses a version of the stabilizer theorem (i.e. Corollary 2.12 of [5]) and requires adaptations of some ideas from Section 2 of [5] involving various computations concerning relatively definable subsets of $\mathrm{Aut}(\mathfrak{C})$. This section is devoted to a proof of the main result in the special case when amenability of T is witnessed by $\emptyset$-definable global Keisler measures, rather than just $\emptyset$-invariant Keisler measures. We will make use of continuous logic stability as in Section 3 of [5]. But this time we will also make explicit use of results from [1].

Recall the standard notion of a definable function from a model to a compact, Hausdorff space. A function $f\colon M^n \to C$ (where M is a model and C is a compact, Hausdorff space) is called *definable* if the preimages under f of any two disjoint closed subsets of C can be separated by a definable subset of M^n; equivalently, f is induced by a (unique) continuous map from $S_n(M)$ to C. This is equivalent to the condition that f has a (unique) extension to an M*-definable* function $\hat{f}\colon \mathfrak{C}^n \to C$ (where $\mathfrak{C}$ is a monster model), meaning that the preimages under $\hat{f}$ of all closed subsets of C are type-definable over M. A function from $\mathfrak{C}^n$ to C is said to be A-definable, if the preimages of all closed subsets are type-definable over A. In particular, a Keisler measure $\mu(\bar{x})$ is said to be $\emptyset$*-definable* if for every formula $\varphi(\bar{x}, \bar{y})$, the function $\mu(\phi(\bar{x}, \bar{y})) : \mathfrak{C}^{|\bar{y}|} \to [0,1]$ is $\emptyset$-definable.

We first discuss the relationship between our formalism from Section 3 of [5] and that of [1]. Start with our (classical) complete first order theory T, which we assume for convenience to be 1-sorted. This is a theory in continuous logic in the sense of [1], but where the metric is discrete and all relation symbols are $\{0,1\}$ valued, where 0 is treated as "true" and 1 as "false". The type spaces $S_n(T)$ are of course Stone spaces. Recall from Definition 3.4 of [5] that by a *continuous logic (CL) formula over* A we mean a continuous function $\phi\colon S_n(A) \to \mathbb{R}$. If ϕ is such a CL-formula, then for any $\bar{b} \in M^n$ (where $M \models T$) by $\phi(\bar{b})$ we mean $\phi(\mathrm{tp}(\bar{b}/A))$. So CL-formulas over A can be thought of as A-definable maps from $\mathfrak{C}^n$ to compact subsets of $\mathbb{R}$ (note that the range of every CL-formula is compact). What are called *definable predicates*, in finitely many variable and without parameters, in [1] are precisely CL-formulas over $\emptyset$ in our sense, but where the range is contained in $[0,1]$. Namely, a definable predicate in n variables is given by a continuous function from $S_n(T)$ to $[0,1]$. The CL-generalization of Morleyizing T consists of adding all such definable predicates as new predicate symbols in the sense of continuous logic. So if M is a model of T, $\phi(\bar{x})$ is such a new predicate symbol, and M is a model of T, then the interpretation $\phi(M)$ of ϕ in M is the function taking an n-tuple $\bar{a}$ from M to $\phi(\mathrm{tp}(\bar{a}))$. Let us call this new theory T_{CL} (a theory of continuous logic with quantifier elimination), to which we can apply the results of [1]. As just remarked, any model M of T expands uniquely to a model of T_{CL}, but we will still call it M.

To understand *imaginaries* as in Section 5 of [1], we have to also consider definable predicates, without parameters, but in possibly infinitely (yet countably) many variables. As in Proposition 3.10 of [1], such a definable predicate in infinitely many variables can be identified with a continuous function from $S_\omega(T)$ to $[0,1]$, where $S_\omega(T)$ is the space of complete types of T in a fixed countable sequence of variables. We feel free to call such a function (and the corresponding function on ω-tuples in models of T to $[0,1]$) a CL-formula in infinitely many variables. Let us now fix a definable predicate $\phi(\bar{x},\bar{y})$, where $\bar{x}$ is a finite tuple of variables, and $\bar{y}$ is a possibly infinite (but countable) sequence of variables. A "code" for the CL-formula (with parameters $\bar{a}$ and finite tuple $\bar{x}$ of free variables) $\phi(\bar{x},\bar{a})$ will then be a CL-imaginary in the sense of [1], and all CL-imaginaries will arise in this way. The precise formalism (involving new sorts with their own distance relation) is not so important, but the point is that the code will be something fixed by precisely those automorphisms (of a saturated model) which fix the formula $\phi(\bar{x},\bar{a})$. In other words, the code will be the equivalence class of $\bar{a}$ with respect to the obvious equivalence relation $E_\phi(\bar{y},\bar{z})$, on tuples of the appropriate length. If $\bar{y}$ is a finite tuple of variables, then we will call a corresponding imaginary (i.e. code for $\phi(\bar{x},\bar{a})$) a *finitary CL-imaginary.* We will work in the saturated model $\bar{M} = \mathfrak{C}$ of T which will also be a saturated model of T_{CL}. When we speak about interdefinability of various objects, we mean a priori in the sense of automorphisms of $\bar{M}$.

The notion of hyperimaginary is well-established in (usual, classical) model theory [17]. A *hyperimaginary* is by definition $\bar{a}/E$, where $\bar{a}$ is a possibly infinite (but small compared with the saturation) tuple and E a type-definable over $\emptyset$ equivalence relation on tuples of the relevant size. It is known that up to interdefinability we may restrict to tuples of length at most ω, which we henceforth do. When the length of $\bar{a}$ is finite, we call $\bar{a}/E$ a *finitary hyperimaginary.* The following is routine, but we sketch the proof.

Lemma 3.1. *(i) Any [finitary] CL-imaginary is interdefinable with a [finitary] hyperimaginary.*
(ii) If E is a bounded, type-definable over $\emptyset$ equivalence relation on finite tuples, then each class of E is interdefinable with a sequence of finitary CL-imaginaries.

Proof. (i) If $\phi(\bar{x},\bar{y})$ is a CL-formula where $\bar{y}$ is a possibly countably infinite tuple, then the equivalence relation $E(\bar{y},\bar{z})$ which says of $(\bar{a},\bar{b})$ that the functions $\phi(\bar{x},\bar{a})$ and $\phi(\bar{x},\bar{b})$ are the same is a type-definable over $\emptyset$ equivalence relation in T.
(ii) It is well-known that E is equivalent to a conjunction of equivalence relations each of which is defined by a countable collection of formulas over $\emptyset$ and is also bounded. So we may assume that E is defined by a countable collection of formulas. Then $\mathfrak{C}/E$ is a compact space, metrizable via an $\mathrm{Aut}(\mathfrak{C})$-invariant metric d (see [12, Section 3, p. 237]). Define $\psi(\bar{x},\bar{y}) := d(\bar{x}/E,\bar{y}/E)$. This is clearly a CL-formula, and we see that each $\bar{a}/E$ is interdefinable with the code of $\psi(\bar{x},\bar{a})$. $\square$

Let $\mathrm{acl}^{eq}_{CL}(\emptyset)$ denote the collection of CL-imaginaries which have a bounded number of conjugates under $\mathrm{Aut}(\bar{M})$. Likewise $\mathrm{bdd}^{heq}(\emptyset)$ is the collection of hyperimaginaries with a bounded number of conjugates under $\mathrm{Aut}(\bar{M})$. Now, Theorem 4.15 of [17] says that any bounded hyperimaginary is interdefinable with a sequence of finitary bounded hyperimaginaries. Therefore, by Lemma 3.1, we get

Corollary 3.2. *(i) Up to interdefinability,* $\mathrm{acl}^{eq}_{CL}(\emptyset)$ *coincides with* $\mathrm{bdd}^{heq}(\emptyset)$*.*
(ii) Moreover, $\mathrm{acl}^{eq}_{CL}(\emptyset)$ *is interdefinable with the collection of finitary CL-imaginaries with a bounded number of conjugates under* $\mathrm{Aut}(\bar{M})$*.*

We now appeal to the local stability results in [1] (which go somewhat beyond what we deduced purely from Grothendieck in Section 3 of [5]). Fix a finite tuple $\bar{x}$ of variables and consider $\Delta(\bar{x})$, the collection of all stable formulas (without parameters) $\phi(\bar{x},\bar{y})$ of T_{CL}, where $\bar{y}$ varies and where stability of $\phi(x,y)$ means that for all $\epsilon > 0$ there do not exist $\bar{a}_i, \bar{b}_i$ for $i < \omega$ (in the monster model) such that for all $i < j$, $|\phi(\bar{a}_i,\bar{b}_j) - \phi(\bar{a}_j,\bar{b}_i)| \geq \epsilon$. For an n-tuple $\bar{b}$ and set A of parameters (including possibly CL-imaginaries), $\mathrm{tp}_\Delta(\bar{b}/A)$ is the function taking the formula $\phi(\bar{x},\bar{a})$ to $\phi(\bar{b},\bar{a})$, where $\phi(\bar{x},\bar{y}) \in \Delta$ and $\phi(\bar{x},\bar{a})$ is over A (i.e invariant under $\mathrm{Aut}(\bar{M}/A)$). By definition, a *complete* Δ*-type over* A is something of the form $\mathrm{tp}_\Delta(\bar{b}/A)$ (and $\bar{b}$ is a realization of it).

Remark 3.3. For any $\bar{b}$, $\mathrm{tp}(\bar{b}/\mathrm{bdd}^{heq}(\emptyset))$ (in the classical case) coincides with $\mathrm{tp}_\Delta(\bar{b}/\mathrm{acl}^{eq}_{CL}(\emptyset))$ in the continuous framework, meaning that $\mathrm{tp}(\bar{b}/\mathrm{bdd}^{heq}(\emptyset)) = \mathrm{tp}(\bar{b}'/\mathrm{bdd}^{heq}(\emptyset))$ if and only if $\mathrm{tp}_\Delta(\bar{b}/\mathrm{acl}^{eq}_{CL}(\emptyset)) = \mathrm{tp}_\Delta(\bar{b}'/\mathrm{acl}^{eq}_{CL}(\emptyset))$.

Proof. Using Corollary 3.2, the left hand side always implies the right hand side. For the other direction, since $\bar{x} \equiv_{\mathrm{bdd}^{heq}(\emptyset)} \bar{y}$ is a bounded, type-definable over $\emptyset$ equivalence relation (in fact, it is exactly E_{KP}), it is enough to show that for any bounded, type-definable over $\emptyset$ equivalence relation E, whenever $\mathrm{tp}_\Delta(\bar{b}/\mathrm{acl}^{eq}_{CL}(\emptyset)) = \mathrm{tp}_\Delta(\bar{b}'/\mathrm{acl}^{eq}_{CL}(\emptyset))$, then $E(\bar{b},\bar{b}')$. Let $\psi(\bar{x},\bar{y})$ be the CL-formula from the proof of Lemma 3.1(ii). As E is bounded, $\psi(\bar{x},\bar{y})$ is stable. The code of $\psi(\bar{x},\bar{b})$ is interdefinable with $\bar{b}/E$, hence it is in $\mathrm{acl}^{eq}_{CL}(\emptyset)$, and so $\psi(\bar{x},\bar{b})$ is over $\mathrm{acl}^{eq}_{CL}(\emptyset)$. Since clearly $\psi(\bar{b},\bar{b}) = 0$, we conclude that $\psi(\bar{b}',\bar{b}) = 0$ which means that $E(\bar{b},\bar{b}')$. □

If M is a model, then $p = \mathrm{tp}_\Delta(\bar{b}/M)$ can be identified with the collection of functions $f_\phi \colon M^n \to \mathbb{R}$ taking $\bar{a} \in M^n$ to $\phi(\bar{b},\bar{a})$, for $\phi(\bar{x},\bar{y}) \in \Delta$. The type $\mathrm{tp}_\Delta(\bar{b}/M)$ is said to be *definable* (over M) if the functions f_ϕ are induced by CL-formulas over M; it is *definable over* A if the f_ϕ's are induced by CL-formulas over A. A $\varphi(\bar{x},\bar{y})$*-definition of* p is a CL-formula $\chi(\bar{y})$ such that $\varphi(\bar{b},\bar{a}) = \chi(\bar{a})$ for all $\bar{a}$ from M.

The following is a consequence of the local theory developed in Section 7 of [1] and the discussion around glueing in Section 8 of the same paper. We restrict ourselves to the case needed, i.e. over $\emptyset$.

Fact 3.4. *Let $p(\bar{x})$ be a complete Δ-type over $\operatorname{acl}^{eq}_{CL}(\emptyset)$. Then for any model M (which note contains $\operatorname{acl}^{eq}_{CL}(\emptyset)$) there is a unique complete Δ-type $q(\bar{x})$ over M such that $q(\bar{x})$ extends $p(\bar{x})$ and q is definable over $\operatorname{acl}^{eq}_{CL}(\emptyset)$. We say $q = p|M$. In particular, if $M \prec N$, then $p|M$ is precisely the restriction of $p|N$ to M.*

Definition 3.5. We say that $\bar{b}$ is stably independent from B (or that $\bar{b}$ and B are stably independent) if $\operatorname{tp}_\Delta(\bar{b}/B)$ equals the restriction of $p|M$ to B, where M is some model containing B and $p = \operatorname{tp}_\Delta(\bar{b}/\operatorname{acl}^{eq}_{CL}(\emptyset))$.

The usual Erdös-Rado arguments, together with Fact 3.4 give:

Corollary 3.6. *Let $p(\bar{x})$ be a complete Δ-type over $\operatorname{acl}^{eq}_{CL}(\emptyset)$. Then there is an infinite sequence $(\bar{b}_i : i < \omega)$ of realizations of p which is indiscernible and such that $\bar{b}_i$ is stably independent from $\{\bar{b}_j : j < i\}$ for all i.*

The following consequence of Fact 3.4 will also be important for us:

Corollary 3.7. *Suppose we have finite tuples $\bar{b}$ and $\bar{c}$ from the (classical) model $\mathfrak{C}$. Suppose that $\bar{b}$ is stably independent from $\bar{c}$. Then for any stable CL-formula $\psi(\bar{x}, \bar{y})$ (over $\emptyset$), the value of $\psi(\bar{b}, \bar{c})$ depends only on $\operatorname{tp}(\bar{b}/\operatorname{bdd}^{heq}(\emptyset))$ and $\operatorname{tp}(\bar{c}/\operatorname{bdd}^{heq}(\emptyset))$ (in the sense of the classical structure $\mathfrak{C}$).*

Proof. Let $p(\bar{x}) = \operatorname{tp}(\bar{b}/\operatorname{bdd}^{heq}(\emptyset))$, which by Remark 3.3 coincides with $\operatorname{tp}_\Delta(\bar{b}/\operatorname{acl}^{eq}_{CL}(\emptyset))$. The $\psi(\bar{x}, \bar{y})$-type of $p|\mathfrak{C}$ is by Fact 3.4 definable by a CL-formula $\chi(\bar{y})$ over $\operatorname{acl}^{eq}_{CL}(\emptyset) = \operatorname{bdd}^{heq}(\emptyset)$. So assuming the stable independence of $\bar{b}$ and $\bar{c}$, by definition and Fact 3.4, the value of $\psi(\bar{b}, \bar{c})$ is equal to $\chi(\bar{c})$, which by Remark 3.3 depends only on $\operatorname{tp}(\bar{c}/\operatorname{bdd}^{heq}(\emptyset))$. If $\bar{b}$ is replaced by another realization $\bar{b}'$ of p which is stably independent from another realization $\bar{c}'$ of $\operatorname{tp}(\bar{c}/\operatorname{bdd}^{heq}(\emptyset))$, then the above shows that $\psi(\bar{b}', \bar{c}') = \chi(\bar{c}') = \chi(\bar{c}) = \psi(\bar{b}, \bar{c})$. □

Proposition 3.8. *Let $\mu(\bar{x})$ be a global, $\emptyset$-definable Keisler measure. Let $\bar{a}$ and $\bar{b}$ be tuples of the same length from $\mathfrak{C}$, with the same type over $\operatorname{bdd}^{heq}(\emptyset)$, and stably independent. Let $p(\bar{x}, \bar{a})$ be a complete type over $\bar{a}$ which is "μ-wide" in the sense that every formula in $p(\bar{x}, \bar{a})$ gets μ-measure > 0. Then the partial type $p(\bar{x}, \bar{a}) \cup p(\bar{x}, \bar{b})$ is also μ-wide (again in the sense that every formula implied by it has μ-measure > 0).*

Proof. By definition, we have to show that if $\phi(\bar{x}, \bar{a})$ is a formula with μ-measure > 0, then $\phi(\bar{x}, \bar{a}) \wedge \phi(\bar{x}, \bar{b})$ has μ-measure > 0. By $\emptyset$-definability of μ, the function $\psi(\bar{y}, \bar{z})$ defined to be $\mu(\phi(\bar{x}, \bar{y}) \wedge \phi(\bar{x}, \bar{z}))$ is definable over $\emptyset$, i.e. is a CL-formula without parameters. Moreover, by Proposition 2.25 of [4], $\psi(\bar{y}, \bar{z})$ is stable. Bearing in mind Remark 3.3, let, by Corollary 3.6, $(\bar{a}_i : i < \omega)$ be an indiscernible sequence of realizations of $\operatorname{tp}(\bar{a}/\operatorname{bdd}^{heq}(\emptyset))$ such that $\bar{a}_j$ and $\bar{a}_i$ are stably independent for all $i < j$ (equivalently for some $i < j$). Since μ is $\operatorname{Aut}(\mathfrak{C})$-invariant, we see that $\mu(\phi(\bar{x}, \bar{a}_i))$ is positive and constant for all i, and $\mu(\phi(\bar{x}, \bar{a}_i) \wedge \phi(\bar{x}, \bar{a}_j))$ is positive (and constant) for $i \neq j$. In particular, $\psi(\bar{a}_0, \bar{a}_1) > 0$. By Corollary 3.7, $\psi(\bar{a}, \bar{b}) > 0$, which is what we had to prove. □

Proposition 3.9. *Suppose that amenability of (the classical, first order theory) T is witnessed by $\emptyset$-definable Keisler measures. Namely, for every formula $\phi(\bar{x})$ over $\emptyset$ there is a global $\emptyset$-definable Keisler measure $\mu(\bar{x})$ concentrating on $\phi(\bar{x})$. Then T is G-compact.*

Proof. We have to show that if $\bar{b}$, $\bar{c}$ are tuples of the same (but possibly infinite) length and with the same type over $\mathrm{bdd}^{heq}(\emptyset)$, then they have the same Lascar strong type.

Assume first that $\bar{b}$ and $\bar{c}$ are stably independent in the sense of Definition 3.5. (If the length of these tuples is infinite, we mean that any two finite corresponding subtuples of $\bar{a}$ and $\bar{b}$ are stably independent.) Fix a model M_0 and enumerate it. We will find a copy M of M_0 such that $\mathrm{tp}(\bar{b}/M) = \mathrm{tp}(\bar{c}/M)$ (which immediately yields that $\bar{b}$ and $\bar{c}$ have the same Lascar strong type). By compactness, given a consistent formula $\phi(\bar{y})$ in finitely many variables, it suffices to find some realization $\bar{m}$ of $\phi(\bar{y})$ such that $\mathrm{tp}(\bar{b}/\bar{m}) = \mathrm{tp}(\bar{c}/\bar{m})$. Again by compactness, we may assume that $\bar{b}$, $\bar{c}$ are finite tuples. By assumption, let $\mu(\bar{y})$ be a $\emptyset$-definable, global Keisler measure concentrating on $\phi(\bar{y})$. Let $p(\bar{y}, \bar{b})$ be a complete type over $\bar{b}$ which is μ-wide. By Proposition 3.8, $p(\bar{y}, \bar{b}) \cup p(\bar{y}, \bar{c})$ is also μ-wide, in particular consistent. So let $\bar{m}$ realize it.

In general, given (possibly infinite) tuples $\bar{b}$, $\bar{c}$ with the same type over $\mathrm{bdd}^{heq}(\emptyset)$, let $\bar{d}$ have the same type over $\mathrm{bdd}^{heq}(\emptyset)$ and be stably independent from $\{\bar{b}, \bar{c}\}$ (by Fact 3.4, uniqueness, and compactness). By what we have just shown, $\bar{b}$ and $\bar{d}$ have the same Lascar strong type, and $\bar{c}$ and $\bar{d}$ have the same Lascar strong type. So $\bar{b}$ and $\bar{c}$ do, too. □

4. Amenability implies G-compactness: the general case

Let T be an arbitrary theory, $\mathfrak{C} \models T$ a monster model, and $\bar{c}$ an enumeration of $\mathfrak{C}$. The goal of this section is to prove Theorem 0.1; in fact, we will get more precise information:

Theorem 4.1. *If T is amenable, then T is G-compact. In fact, the diameter of each Lascar strong type (over $\emptyset$) is bounded by 4.*

Before we start our analysis towards the proof of Theorem 4.1, let us first note the analogous statement for extreme amenability, which is much easier to prove.

Proposition 4.2. *If $p(\bar{x}) \in S(\emptyset)$ is extremely amenable, then $p(\bar{x})$ is a single Lascar strong type. Moreover, the Lascar diameter of $p(\bar{x})$ is at most 2.*

In particular, if T is extremely amenable, then the Lascar strong types coincide with complete types (over $\emptyset$), i.e. the Lascar Galois group $\mathrm{Gal}_L(T)$ is trivial.

Proof. Choose $\mathfrak{C}$ so that $\bar{x}$ is short in $\mathfrak{C}$. Let $q \in S_p(\mathfrak{C})$ be invariant under $\mathrm{Aut}(\mathfrak{C})$. Fix $\bar{\alpha} \models q$ (in a bigger model). Take a small $M \prec \mathfrak{C}$ and choose $\bar{\beta} \in \mathfrak{C}$ such that $\bar{\beta} \models q|_M$. Then $\bar{\alpha} \, E_L \, \bar{\beta}$. But also, for any $\sigma \in \mathrm{Aut}(\mathfrak{C})$, $\sigma(\bar{\beta}) \models \sigma(q)|_{\sigma[M]} = q|_{\sigma[M]}$, and so $\sigma(\bar{\beta})\, E_L\, \bar{\alpha}$. Therefore, $\sigma(\bar{\beta})\, E_L\, \bar{\beta}$ for any $\sigma \in \mathrm{Aut}(\mathfrak{C})$, which shows that $p(\bar{x})$ is a single Lascar strong type.

For the "moreover part" notice that, in the above argument, both $d_L(\bar{\alpha}, \bar{\beta})$ and $d_L(\sigma(\bar{\beta}), \bar{\alpha})$ are bounded by 1. □

Recall from Corollary 2.15 that by a *relatively type-definable subset* of $\mathrm{Aut}(\mathfrak{C})$ we mean a subset of the form $A_{\pi,\bar{a},\bar{b}} := \{\sigma \in \mathrm{Aut}(\mathfrak{C}) : \mathfrak{C} \models \pi(\sigma(\bar{a}), \bar{b}))\}$ for some partial type $\pi(\bar{x}, \bar{y})$ (without parameters), where $\bar{x}$ and $\bar{y}$ are short tuples of variables and $\bar{a}$, $\bar{b}$ are from $\mathfrak{C}$. (Note that although here we allow repetitions in the tuple $\bar{a}$, whereas in Corollary 2.15 $\bar{a}$ was a subtuple of $\bar{c}$, both versions yield the same class of relatively type-definable sets.) Without loss $\bar{x}$ is of the same length as $\bar{y}$ and $\bar{a} = \bar{b}$, and then we write $A_{\bar{\pi},\bar{a}}$. In fact, the following remark is very easy.

Remark 4.3. For any partial types $\pi_1(\bar{x}_1, \bar{y}_1)$ and $\pi_2(\bar{x}_2, \bar{y}_2)$ and tuples $\bar{a}_1, \bar{a}_2, \bar{b}_1, \bar{b}_2$ in $\mathfrak{C}$ corresponding to $\bar{x}_1, \bar{x}_2, \bar{y}_1, \bar{y}_2$, one can find partial types $\pi_1'(\bar{x}, \bar{y})$ and $\pi_2'(\bar{x}, \bar{y})$ with $\bar{x}$ of the same length (by which we also mean of the same sorts) as $\bar{y}$ and a tuple $\bar{a}$ in $\mathfrak{C}$ corresponding to $\bar{x}$ such that $A_{\pi_1,\bar{a}_1,\bar{b}_1} = A_{\pi_1',\bar{a}}$ and $A_{\pi_2,\bar{a}_2,\bar{b}_2} = A_{\pi_2',\bar{a}}$.

For a short tuple $\bar{\alpha}$ and a short tuple of parameters $\bar{b}$, a subset of $\mathrm{Aut}(\mathfrak{C})$ is called *relatively $\bar{\alpha}$-type-definable over $\bar{b}$* if it is of the form $A_{\pi,\bar{\alpha},\bar{b}}$ for some partial type $\pi(\bar{x}, \bar{y})$.

The next fact was observed in [13].

Fact 4.4 (Proposition 5.2 of [13]). *If G is a closed, bounded index subgroup of* $\mathrm{Aut}(\mathfrak{C})$ *(with* $\mathrm{Aut}(\mathfrak{C})$ *equipped with the pointwise convergence topology), then* $\mathrm{Autf}_L(\mathfrak{C}) \leq G$.

Using an argument similar to the proof of Fact 4.4, we will first show

Proposition 4.5. *If G is a relatively type-definable, bounded index subgroup of* $\mathrm{Aut}(\mathfrak{C})$*, then* $\mathrm{Autf}_{KP}(\mathfrak{C}) \leq G$.

Proof. Let σ_i, $i < \lambda$, be a set of representatives of the left cosets of G in $\mathrm{Aut}(\mathfrak{C})$ (so λ is bounded). Then

$$G' := \bigcap_{\sigma \in \mathrm{Aut}(\mathfrak{C})} G^{\sigma} = \bigcap_{i<\lambda} G^{\sigma_i}$$

is a normal, bounded index subgroup of $\mathrm{Aut}(\mathfrak{C})$ (where $G^{\sigma} := \sigma G \sigma^{-1}$).

Let us show now that G' is relatively type-definable. We have $G = A_{\pi,\bar{a}} = \{\sigma \in \mathrm{Aut}(\mathfrak{C}) : \mathfrak{C} \models \pi(\sigma(\bar{a}), \bar{a})\}$ for some type $\pi(\bar{x}, \bar{y})$ (with short $\bar{x}$, $\bar{y}$) and tuple $\bar{a}$ in $\mathfrak{C}$. Then $G^{\sigma_i} = \{\sigma \in \mathrm{Aut}(\mathfrak{C}) : \mathfrak{C} \models \pi(\sigma(\sigma_i(\bar{a})), \sigma_i(\bar{a}))\}$, so putting $\bar{a}' = \langle \sigma_i(\bar{a}) \rangle_{i<\lambda}$, $\bar{x}' = \langle \bar{x}_i \rangle_{i<\lambda}$, $\bar{y}' = \langle \bar{y}_i \rangle_{i<\lambda}$ (where $\bar{x}_i$ and $\bar{y}_i$ are copies of $\bar{x}$ and $\bar{y}$, respectively) and $\pi'(\bar{x}', \bar{y}') = \bigcup_{i<\lambda} \pi(\bar{x}_i, \bar{y}_i)$ (as a set of formulas), we see that

$$G' = A_{\pi',\bar{a}'} = \{\sigma \in \mathrm{Aut}(\mathfrak{C}) : \mathfrak{C} \models \pi'(\sigma(\bar{a}'), \bar{a}')\},$$

which is clearly relatively type-definable.

The orbit equivalence relation E of the action of G' on the set of realizations of $\mathrm{tp}(\bar{a}'/\emptyset)$ is a bounded equivalence relation. This relation is type-definable, because

$$\bar{\alpha}\, E\, \bar{\beta} \iff (\exists g \in G')(g(\bar{\alpha}) = \bar{\beta}) \iff (\exists \bar{b}')(\pi'(\bar{b}', \bar{a}') \wedge \bar{a}'\bar{\alpha} \equiv \bar{b}'\bar{\beta}).$$

But E is also invariant (as G' is a normal subgroup of $\mathrm{Aut}(\mathfrak{C})$), so E is type-definable over $\emptyset$. Therefore, E is refined by E_{KP}.

Now, take any $\sigma \in \mathrm{Autf}_{KP}(\mathfrak{C})$. By the last conclusion, there is $\tau \in G'$ such that $\sigma(\bar{a}') = \tau(\bar{a}')$. Then $\tau^{-1}\sigma(\bar{a}') = \bar{a}'$ and $\sigma = \tau(\tau^{-1}\sigma)$. Since the above formula for G' shows that $G' \cdot \mathrm{Fix}(\bar{a}') = G'$, we get $\sigma \in G'$. Thus, $\mathrm{Autf}_{KP}(\mathfrak{C}) \leq G' \leq G$. □

Recall that a subset C of a group is called *(left) generic* if finitely many left translates of it covers the whole group; C is called *symmetric* if it contains the neutral element and $C^{-1} = C$.

Corollary 4.6. *If $\{C_i : i \in \omega\}$ is a family of relatively definable, generic, symmetric subsets of* $\mathrm{Aut}(\mathfrak{C})$ *such that $C_{i+1}^2 \subseteq C_i$ for all $i \in \omega$, then $\bigcap_{i\in\omega} C_i$ is a subgroup of* $\mathrm{Aut}(\mathfrak{C})$ *containing* $\mathrm{Autf}_{KP}(\mathfrak{C})$.

Proof. It is clear that $\bigcap_{i\in\omega} C_i$ is a subgroup of $\mathrm{Aut}(\mathfrak{C})$, and it is easy to show that it has bounded index (at most $2^{\aleph_0}$). Moreover, it is clearly relatively type-definable. Thus, the fact that it contains $\mathrm{Autf}_{KP}(\mathfrak{C})$ follows from Proposition 4.5. □

Lemma 4.7. *i) Let $\pi(\bar{x}, \bar{y})$ be a partial type (over $\emptyset$) and $\bar{a}$, $\bar{b}$ short tuples from $\mathfrak{C}$ corresponding to $\bar{x}$ and $\bar{y}$, respectively. Then $A^{-1}_{\pi,\bar{a},\bar{b}} = A_{\pi',\bar{b},\bar{a}}$, where $\pi'(\bar{y}, \bar{x}) = \pi(\bar{x}, \bar{y})$.*
ii) Let $n \geq 2$ be a natural number. Let $\bar{x}$, $\bar{y}$ and $\bar{x}_1, \dots, \bar{x}_n$ be disjoint, short tuples of variables of the same length. Then there exists a partial type $\Phi_n(\bar{x}, \bar{y}, \bar{x}_1, \dots, \bar{x}_n)$ such that for every partial types $\pi_1(\bar{x}_1, \bar{y}), \dots, \pi_n(\bar{x}_n, \bar{y})$ and tuple $\bar{a}$ corresponding to $\bar{x}$ one has

$$A_{\pi_1,\bar{a}} \cdot \dots \cdot A_{\pi_n,\bar{a}} = A_{\pi,\bar{a}},$$

where

$$\pi(\bar{x}, \bar{y}) = (\exists \bar{x}_1, \dots, \bar{x}_n)(\pi_1(\bar{x}_1, \bar{y}) \wedge \dots \wedge \pi_n(\bar{x}_n, \bar{y}) \wedge \Phi_n(\bar{x}, \bar{y}, \bar{x}_1, \dots, \bar{x}_n)).$$

Proof. (i) follows immediately from the fact that for any $\sigma \in \mathrm{Aut}(\mathfrak{C})$

$$\mathfrak{C} \models \pi(\sigma(\bar{a}), \bar{b}) \iff \mathfrak{C} \models \pi(\bar{a}, \sigma^{-1}(\bar{b})) \iff \mathfrak{C} \models \pi'(\sigma^{-1}(\bar{b}), \bar{a}).$$

(ii) We will show that for $n = 2$ the type $\Phi_2(\bar{x}, \bar{y}, \bar{x}_1, \bar{x}_2) := (\bar{x}\bar{x}_1 \equiv \bar{x}_2\bar{y})$ and for $n \geq 3$ the type $\Phi_n(\bar{x}, \bar{y}, \bar{x}_1, \dots, \bar{x}_n)$ defined as

$$(\exists \bar{z}_1, \dots, \bar{z}_{n-2})(\bar{x}\bar{z}_{n-2} \equiv \bar{x}_n\bar{y} \wedge \bar{z}_{n-2}\bar{z}_{n-3} \equiv \bar{x}_{n-1}\bar{y} \wedge \dots \wedge \bar{z}_2\bar{z}_1 \equiv \bar{x}_3\bar{y} \wedge \bar{z}_1\bar{x}_1 \equiv \bar{x}_2\bar{y})$$

is as required.

First, let us see that $A_{\pi_1,\bar{a}} \cdot \dots \cdot A_{\pi_n,\bar{a}} \subseteq A_{\pi,\bar{a}}$. Take σ from the left hand side, i.e. $\sigma = \sigma_1 \cdot \dots \cdot \sigma_n$, where $\models \pi_i(\sigma_i(\bar{a}), \bar{a})$. Then $\models \pi(\sigma(\bar{a}), \bar{a})$ is witnessed by $\bar{x}_i := \sigma_i(\bar{a})$ for $i = 1, \dots, n$ and $\bar{z}_i := (\sigma_1 \dots \sigma_{i+1})(\bar{a})$ for $i = 1, \dots, n-2$. So $\sigma \in A_{\pi,\bar{a}}$.

Finally, we will justify that $A_{\pi_1,\bar{a}} \cdot \ldots \cdot A_{\pi_n,\bar{a}} \supseteq A_{\pi,\bar{a}}$. Consider the case $n \geq 3$. Take any σ such that $\models \pi(\sigma(\bar{a}), \bar{a})$. Let $\bar{a}_1, \ldots, \bar{a}_n$ be witnesses for $\bar{x}_1, \ldots, \bar{x}_n$, and $\bar{b}_1, \ldots, \bar{b}_{n-2}$ be witnesses for $\bar{z}_1, \ldots, \bar{z}_{n-2}$, i.e.:

(1) $\models \pi_i(\bar{a}_i, \bar{a})$ for $i = 1, \ldots, n$, and

(2) $\sigma(\bar{a})\bar{b}_{n-2} \equiv \bar{a}_n\bar{a} \wedge \bar{b}_{n-2}\bar{b}_{n-3} \equiv \bar{a}_{n-1}\bar{a} \wedge \cdots \wedge \bar{b}_2\bar{b}_1 \equiv \bar{a}_3\bar{a} \wedge \bar{b}_1\bar{a}_1 \equiv \bar{a}_2\bar{a}$.

By (2), there are $\tau_1, \ldots, \tau_{n-1} \in \operatorname{Aut}(\mathfrak{C})$ mapping the right hand sides of the equivalences in (2) to the left hand sides. Then $\tau_1(\bar{a}_n) = \sigma(\bar{a})$, so $\tau_1^{-1}\sigma(\bar{a}) = \bar{a}_n$, so $\tau_1^{-1}\sigma \in A_{\pi_n,\bar{a}}$ by (1). Next, $\tau_1(\bar{a}) = \bar{b}_{n-2} = \tau_2(\bar{a}_{n-1})$, so $\tau_2^{-1}\tau_1(\bar{a}) = \bar{a}_{n-1}$, so $\tau_2^{-1}\tau_1 \in A_{\pi_{n-1},\bar{a}}$ by (1). We continue in this way, obtaining in the last step: $\tau_{n-1}(\bar{a}) = \bar{a}_1$, so $\tau_{n-1} \in A_{\pi_1,\bar{a}}$ by (1). Therefore,

$$\sigma = \tau_{n-1}(\tau_{n-1}^{-1}\tau_{n-2}) \ldots (\tau_2^{-1}\tau_1)(\tau_1^{-1}\sigma) \in A_{\pi_1,\bar{a}} \cdot \ldots \cdot A_{\pi_n,\bar{a}}.$$

For $n = 2$, in (2), we just have $\sigma(\bar{a})\bar{a}_1 \equiv \bar{a}_2\bar{a}$, so taking $\tau_1 \in \operatorname{Aut}(\mathfrak{C})$ which maps $\bar{a}_2\bar{a}$ to $\sigma(\bar{a})\bar{a}_1$, we get $\tau_1^{-1}\sigma \in A_{\pi_2,\bar{a}}$ and $\tau_1 \in A_{\pi_1,\bar{a}}$, hence $\sigma \in A_{\pi_1,\bar{a}} \cdot A_{\pi_2,\bar{a}}$. □

Corollary 4.8. *Let $\pi_1(\bar{x}, \bar{y}), \ldots, \pi_n(\bar{x}, \bar{y})$ be partial types, $\bar{a}$ a tuple corresponding to $\bar{x}$ and $\bar{y}$, and $\epsilon_1, \ldots, \epsilon_n \in \{-1, 1\}$.*
(i) Then

$$A^{\epsilon_1}_{\pi_1,\bar{a}} \cdot \ldots \cdot A^{\epsilon_n}_{\pi_n,\bar{a}} = \bigcap \left\{ A^{\epsilon_1}_{\varphi_1,\bar{a}} \cdot \ldots \cdot A^{\epsilon_n}_{\varphi_n,\bar{a}} : \pi_1 \vdash \varphi_1, \ldots, \pi_n \vdash \varphi_n \right\}.$$

ii) If $A^{\epsilon_1}_{\pi_1,\bar{a}} \cdot \ldots \cdot A^{\epsilon_n}_{\pi_n,\bar{a}}$ is contained in a relatively definable subset A of $\operatorname{Aut}(\mathfrak{C})$, then there are $\varphi_i(\bar{x}, \bar{y})$ implied by $\pi_i(\bar{x}, \bar{y})$ for $i = 1, \ldots, n$, such that $A^{\epsilon_1}_{\varphi_1,\bar{a}} \cdot \ldots \cdot A^{\epsilon_n}_{\varphi_n,\bar{a}} \subseteq A$.

Proof. This follows easily from Lemma 4.7, using compactness and the fact that $\mathfrak{C}$ is a monster model. □

Lemma 4.9. *Let $p(\bar{x}) \in S(\emptyset)$ with $\bar{x}$ short, $q \in S_p(\mathfrak{C})$, $M \prec \mathfrak{C}$ small, and $\bar{\alpha} \models q|_M$. Then $A_{q|_{\bar{\alpha}},\bar{\alpha}} A_{q|_{\bar{\alpha}},\bar{\alpha}} A^{-1}_{q|_{\bar{\alpha}},\bar{\alpha}} A^{-1}_{q|_{\bar{\alpha}},\bar{\alpha}} \bar{\alpha} \subseteq \{\bar{\beta} \subset \mathfrak{C} : d_L(\bar{\alpha}, \bar{\beta}) \leq 4\} \subseteq [\bar{\alpha}]_{E_L}$.*

Proof. Let us start from the following

Claim 1: For any $\bar{\beta} \models q|_{\bar{\alpha}}$, $d_L(\bar{\beta}, \bar{\alpha}) \leq 1$.

Proof. Take $\bar{\gamma} \models q|_{M\bar{\alpha}}$. Then $d_L(\bar{\gamma}, \bar{\alpha}) \leq 1$, so the conclusion follows from the fact that $\bar{\beta} \equiv_{\bar{\alpha}} \bar{\gamma}$. □(claim)

Now, consider any $\sigma_1, \sigma_2, \sigma_3, \sigma_4 \in A_{q|_{\bar{\alpha}},\bar{\alpha}}$. Then $\sigma_i(\bar{\alpha}) \models q|_{\bar{\alpha}}$, so, by the claim, we get $d_L(\sigma_i(\bar{\alpha}), \bar{\alpha}) \leq 1$. Therefore, $d_L(\sigma_4^{-1}(\bar{\alpha}), \bar{\alpha}) \leq 1$, so $d_L(\sigma_3^{-1}\sigma_4^{-1}(\bar{\alpha}), \sigma_3^{-1}(\bar{\alpha})) \leq 1$, so $d_L(\sigma_3^{-1}\sigma_4^{-1}(\bar{\alpha}), \bar{\alpha}) \leq 2$, so $d_L(\sigma_2\sigma_3^{-1}\sigma_4^{-1}(\bar{\alpha}), \sigma_2(\bar{\alpha})) \leq 2$, so $d_L(\sigma_2\sigma_3^{-1}\sigma_4^{-1}(\bar{\alpha}), \bar{\alpha}) \leq 3$, so $d_L(\sigma_1\sigma_2\sigma_3^{-1}\sigma_4^{-1}(\bar{\alpha}), \sigma_1(\bar{\alpha})) \leq 3$, so $d_L(\sigma_1\sigma_2\sigma_3^{-1}\sigma_4^{-1}(\bar{\alpha}), \bar{\alpha}) \leq 4$. □

The proof of the next lemma uses a version of the stabilizer theorem obtained in [5, Corollary 2.12]. We will not recall here all the terminology involved in [5, Corollary 2.12]; the reader may consult Subsections 2.1, 2.2, and 2.3 of [5]. Recall only that $\mathcal{L}_k$, $k \in \omega$, is a recursively defined notion of largeness of $\bigvee$-definable subsets of a group, which is invariant under left translations. Then

$\mathrm{St}_{\mathcal{L}_k}(Y) := \{g : \mathcal{L}_k(gY \cap Y)\}$. So, if Y is invariant under left translations by the elements of some subgroup of the group in question, then $\mathrm{St}_{\mathcal{L}_k}(Y)$ is invariant under both left and right translations by the elements of the same subgroup.

Lemma 4.10. *Assume* $\mathrm{Aut}(\mathfrak{C})$ *is relatively definably amenable. By Corollary 2.15, take the induced* $\mathrm{Aut}(\mathfrak{C})$*-invariant, finitely additive, probability measure* μ *on the Boolean algebra* $\mathcal{A}$ *generated by relatively type-definable subsets of* $\mathrm{Aut}(\mathfrak{C})$*. Suppose* $A \subseteq \mathrm{Aut}(\mathfrak{C})$ *is relatively type-definable with* $\mu(A) > 0$ *and* $A^4 := AAA^{-1}A^{-1} \subseteq A'$ *for some relatively definable* $A' \subseteq \mathrm{Aut}(\mathfrak{C})$*. Then there exists a relatively type-definable, generic, symmetric* $Y \subseteq \mathrm{Aut}(\mathfrak{C})$ *such that* $Y^8 \subseteq A'$*.*

Proof. By Lemma 4.7, relatively type-definable sets are closed under taking products and inversions, and one can easily check that also under left translations.

Claim 1: There exists a generic and symmetric set $S \subseteq \mathrm{Aut}(\mathfrak{C})$ such that:

(1) $S^{16} \subseteq AAA^{-1}A^{-1}$,
(2) $S = \{\sigma \in \mathrm{Aut}(\mathfrak{C}) : \mathrm{tp}(\sigma(\bar{a})/\bar{a}) \in \mathcal{P}\}$ for some $\mathcal{P} \subseteq S_{\bar{a}}(\bar{a})$ for some short tuple $\bar{a}$ (which is a tuple of finitely many conjugates by elements of $\mathrm{Aut}(\mathfrak{C})$ of the tuple over which A is relatively type-definable).

Proof. Apply [5, Corollary 2.12] for $G := \mathrm{Aut}(\mathfrak{C})$, A from the statement of Lemma 4.10, $\mathcal{B} := \{A\}$, $N := 16$, $\mathcal{D} := \mathcal{A}$, and $m := \mu$. As a result, we obtain a set $B' = A \cap \sigma_1[A] \cap \cdots \cap \sigma_n[A]$ for some σ_i's in $\mathrm{Aut}(\mathfrak{C})$ such that for some $l \in \mathbb{N}_{>0}$, $S := \mathrm{St}_{\mathcal{L}_{l-1}}(B')$ is generic, symmetric, and satisfies $S^{16} \subseteq AAA^{-1}A^{-1}$. Since A is relatively type-definable over some short tuple $\bar{\alpha}$, so is B', but over $\bar{a} := \bar{\alpha}\sigma_1(\bar{\alpha}) \dots \sigma_n(\bar{\alpha})$. Hence, $\mathrm{Aut}(\mathfrak{C}/\bar{a}) \cdot B' = B'$. Therefore, by the property of St_{l-1} recalled before Lemma 4.10, we get that

$$\mathrm{Aut}(\mathfrak{C}/\bar{a}) \cdot S \cdot \mathrm{Aut}(\mathfrak{C}/\bar{a}) = S,$$

which means that $S = \{\sigma \in \mathrm{Aut}(\mathfrak{C}) : \mathrm{tp}(\sigma(\bar{a})/\bar{a}) \in \mathcal{P}\}$ for some $\mathcal{P} \subseteq S_{\bar{a}}(\bar{a})$. □(claim)

Take any $p \in \mathcal{P}$. We can write $p = p(\bar{x}, \bar{a})$ for the obvious complete type $p(\bar{x}, \bar{y})$ over $\emptyset$. Then $(A_{p,\bar{a}} \cdot A_{p,\bar{a}}^{-1})^8 \subseteq (SS^{-1})^8 = S^{16} \subseteq AAA^{-1}A^{-1} \subseteq A'$. Hence, by Corollary 4.8(ii), there is $\psi_p(\bar{x}, \bar{y}) \in p(\bar{x}, \bar{y})$ for which $(A_{\psi_p,\bar{a}} \cdot A_{\psi_p,\bar{a}}^{-1})^8 \subseteq A'$.

Now, the complement of $\bigcup_{p\in\mathcal{P}} A_{\psi_p,\bar{a}}$ equals $\bigcap_{p\in\mathcal{P}} A_{\neg\psi_p,\bar{a}}$ which is clearly relatively type-definable. Thus, $\bigcup_{p\in\mathcal{P}} A_{\psi_p,\bar{a}} \in \mathcal{A}$. On the other hand, $S \subseteq \bigcup_{p\in\mathcal{P}} A_{\psi_p,\bar{a}}$ and S being generic implies that $\bigcup_{p\in\mathcal{P}} A_{\psi_p,\bar{a}}$ is generic. Therefore, $\mu(\bigcup_{p\in\mathcal{P}} A_{\psi_p,\bar{a}}) > 0$.

Let $\tilde{\mu}$ be the $\mathrm{Aut}(\mathfrak{C})$-invariant, (regular) Borel probability measure on $S_{\bar{c}}(\mathfrak{C})$ from which μ is induced. Then $\tilde{\mu}(\bigcup_{p\in\mathcal{P}}[\psi_p]) > 0$, so, by regularity, there is a compact $K \subseteq \bigcup_{p\in\mathcal{P}}[\psi_p]$ of positive measure. But K is covered by finitely many clopen sets $[\psi_p]$ one of which must be of positive measure, i.e. $\tilde{\mu}([\psi_p]) > 0$ for some $p \in \mathcal{P}$. Then $\mu(A_{\psi_p,\bar{a}}) > 0$. This implies that $Y := A_{\psi_p,\bar{a}} \cdot A_{\psi_p,\bar{a}}^{-1}$ is generic, and it is clearly symmetric. By Lemma 4.7, it is also relatively type-definable. Moreover, by the choice of ψ_p, $Y^8 \subseteq A'$, so we are done. $\square$

Corollary 4.11. *Assume* $\mathrm{Aut}(\mathfrak{C})$ *is relatively definably amenable. By Corollary 2.15, take the induced* $\mathrm{Aut}(\mathfrak{C})$*-invariant, finitely additive, probability measure* μ *on the Boolean algebra* $\mathcal{A}$ *generated by relatively type-definable subsets of* $\mathrm{Aut}(\mathfrak{C})$*. Suppose* $A \subseteq \mathrm{Aut}(\mathfrak{C})$ *is relatively type-definable and* $\mu(A) > 0$*. Then* $\mathrm{Autf}_{KP}(\mathfrak{C}) \subseteq AAA^{-1}A^{-1}$*.*

Proof. Take any A' relatively definable, symmetric, and such that $AAA^{-1}A^{-1} \subseteq A'$. Put $C_0 := A'$.

By Lemma 4.10, we obtain a relatively type-definable, generic, symmetric Y such that $(Y^4)^2 \subseteq A'$. So, by Corollary 4.8, there is a relatively definable, symmetric Y' satisfying $Y^4 \subseteq Y'$ and $Y'^2 \subseteq A'$. Put $C_1 := Y'$.

Next, we apply Lemma 4.10 to Y in place of A and Y' in place of A', and we obtain a relatively type-definable, generic, symmetric Z such that $(Z^4)^2 \subseteq Y'$. So, by Corollary 4.8, there is a relatively definable, symmetric Z' satisfying $Z^4 \subseteq Z'$ and $Z'^2 \subseteq Y'$. Put $C_2 := Z'$.

Continuing in this way, we obtain a family $\{C_i : i \in \omega\}$ of relatively definable, generic, symmetric subsets of $\mathrm{Aut}(\mathfrak{C})$ such that $C_{i+1}^2 \subseteq C_i$ for every $i \in \omega$. By Corollary 4.6, $\mathrm{Autf}_{KP}(\mathfrak{C}) \subseteq \bigcap_{i\in\omega} C_i \subseteq A'$. Since A' was an arbitrary relatively definable, symmetric set containing A^4, we get $\mathrm{Autf}_{KP}(\mathfrak{C}) \subseteq A^4$. $\square$

We have now all the ingredients to prove our main theorem.

Proof of Theorem 4.1. By Corollary 2.15, a measure $\tilde{\mu}$ on $S_{\bar{c}}(\mathfrak{C})$ witnessing relative definable amenability of $\mathrm{Aut}(\mathfrak{C})$ induces an $\mathrm{Aut}(\mathfrak{C})$-invariant, finitely additive, probability measure μ on the Boolean algebra $\mathcal{A}$ generated by relatively type-definable subsets of $\mathrm{Aut}(\mathfrak{C})$.

Consider any $p(\bar{x}) = \mathrm{tp}(\bar{\alpha}/\emptyset) \in S(\emptyset)$ with a short subtuple $\bar{\alpha}$ of $\bar{c}$. Choose a μ-wide type $q \in S_p(\mathfrak{C})$, i.e. $\tilde{\mu}([\varphi(\bar{x}', \bar{b})]) > 0$ (equivalently, $\mu(A_{\varphi,\bar{\alpha},\bar{b}}) > 0$) for any $\varphi(\bar{x}, \bar{b}) \in q$ (where $\bar{x}' \supset \bar{x}$ is the tuple of variables corresponding to $\bar{c}$). Take a small model $M \prec \mathfrak{C}$. Applying an appropriate automorphism of $\mathfrak{C}$ to q and M, and using $\mathrm{Aut}(\mathfrak{C})$-invariance of μ, we can assume that $\bar{\alpha} \models q|_M$.

Consider any $\varphi(\bar{x}, \bar{\alpha}) \in q|_{\bar{\alpha}}$. Then $\mu(A_{\varphi,\bar{\alpha}}) > 0$, so, by Corollary 4.11, we conclude that $\mathrm{Autf}_{KP}(\mathfrak{C}) \subseteq A_{\varphi,\bar{\alpha}}A_{\varphi,\bar{\alpha}}A^{-1}_{\varphi,\bar{\alpha}}A^{-1}_{\varphi,\bar{\alpha}}$. Therefore, by Corollary 4.8(i), we get

$$\mathrm{Autf}_{KP}(\mathfrak{C}) \subseteq \bigcap_{\varphi(\bar{x},\bar{\alpha})\in q|_{\bar{\alpha}}} A_{\varphi,\bar{\alpha}}A_{\varphi,\bar{\alpha}}A^{-1}_{\varphi,\bar{\alpha}}A^{-1}_{\varphi,\bar{\alpha}} = A_{q|_{\bar{\alpha}},\bar{\alpha}}A_{q|_{\bar{\alpha}},\bar{\alpha}}A^{-1}_{q|_{\bar{\alpha}},\bar{\alpha}}A^{-1}_{q|_{\bar{\alpha}},\bar{\alpha}}.$$

On the other hand, Lemma 4.9 tells us that

$$A_{q|_{\bar{\alpha}},\bar{\alpha}}A_{q|_{\bar{\alpha}},\bar{\alpha}}A^{-1}_{q|_{\bar{\alpha}},\bar{\alpha}}A^{-1}_{q|_{\bar{\alpha}},\bar{\alpha}}\bar{\alpha} \subseteq \{\bar{\beta} : d_L(\bar{\alpha}, \bar{\beta}) \leq 4\} \subseteq [\bar{\alpha}]_{E_L}.$$

Therefore, $[\bar{\alpha}]_{E_{KP}} = [\bar{\alpha}]_{E_L}$ has diameter at most 4. $\square$

Theorem 4.1 is a global result. It is natural to ask whether we can extend it to a local version (as in Proposition 4.2).

Question 4.12. *Is it true that if $p(\bar{x}) \in S(\emptyset)$ is amenable, then the Lascar strong types on $p(\bar{x})$ coincide with Kim-Pillay strong types? Does amenability of $p(\bar{x})$ imply that the Lascar diameter of $p(\bar{x})$ is at most 4?*

One could think that the above arguments should yield the positive answer to these questions. The problem is that, assuming only amenability of $p(\bar{x})$, we have the induced measure μ but defined only on the Boolean algebra of relatively $\bar{\alpha}$-type-definable subsets of $\mathrm{Aut}(\mathfrak{C})$, for a fixed $\bar{\alpha} \models p$. So, for the recursive proof of Corollary 4.11 to go through, starting from a set $A \subseteq \mathrm{Aut}(\mathfrak{C})$ relatively $\bar{\alpha}$-type-definable [where for the purpose of answering Question 4.12 via an argument as in the proof of Theorem 4.1, we can additionally assume that A is defined over $\bar{\alpha}$] of positive measure, we need to produce the desired Y also relatively $\bar{\alpha}$-type-definable [over $\bar{\alpha}$] (in order be able to continue our recursion). But this requires a strengthening of Lemma 4.10 to the version where for A relatively $\bar{\alpha}$-type-definable of positive measure one wants to obtain the desired Y which is also relatively $\bar{\alpha}$-type-definable; the variant with A and Y defined over $\bar{\alpha}$ would also be sufficient. Trying to follow the lines of the proof of Lemma 4.10, even if A is defined over $\bar{\alpha}$, Claim 1 requires a longer tuple $\bar{a}$ which produces the desired set Y which is relatively $\bar{a}$-type-definable, and this is the only obstacle to answer positively Question 4.12 via the above arguments.

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