LOCALLY COMPACT MODELS FOR APPROXIMATE RINGS

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Abstract. By an approximate subring of a ring we mean an additively symmetric subset X such that $X \cdot X \cup (X + X)$ is covered by finitely many additive translates of X. We prove that each approximate subring X of a ring has a locally compact model, i.e. a ring homomorphism $f \colon \langle X \rangle \to S$ for some locally compact ring S such that $f[X]$ is relatively compact in S and there is a neighborhood U of 0 in S with $f^{-1}[U] \subseteq 4X + X \cdot 4X$ (where $4X := X + X + X + X$). This S is obtained as the quotient of the ring $\langle X \rangle$ interpreted in a sufficiently saturated model by its type-definable ring connected component. The main point is to prove that this component always exists. In order to do that, we extend the basic theory of model-theoretic connected components of definable rings (developed in [GJK22] and [KR22]) to the case of rings generated by definable approximate subgrings and we answer a question from [KR22] in the more general context of approximate subrings. Namely, let X be a definable (in a structure M) approximate subring of a ring and $R := \langle X \rangle$. Let $\bar{X}$ be the interpretation of X in a sufficiently saturated elementary extension and $\bar{R} := \langle \bar{X} \rangle$. It follows from [MW15] that there exists the smallest M-type-definable subgroup of $(\bar{R}, +)$ of bounded index, which is denoted by $(\bar{R}, +)^{00}_M$. We prove that $(\bar{R}, +)^{00}_M + \bar{R} \cdot (\bar{R}, +)^{00}_M$ is the smallest M-type-definable two-sided ideal of $\bar{R}$ of bounded index, which we denote by $\bar{R}^{00}_M$. Then S in the first sentence of the abstract is just $\bar{R}/\bar{R}^{00}_M$ and $f : R \to \bar{R}/\bar{R}^{00}_M$ is the quotient map. In fact, f is the universal "definable" (in a suitable sense) locally compact model.

1. Introduction

A subset X of a group is called an *approximate subgroup* if it is symmetric (i.e. $e \in X$ and $X^{-1} = X$) and $XX \subseteq FX$ for some finite $F \subseteq \langle X \rangle$. Approximate subgroups were introduced by Tao in [Tao08] and have become one of the central objects in additive combinatorics. A breakthrough in the study of the structure of approximate subgroups was obtained by Hrushovski in [Hru12], where a locally compact model for any pseudofinite approximate subgroup (more generally, *near-subgroup*) X was obtained by using model-theoretic tools, and in consequence also a Lie model was found for some approximate subgroup commensurable with X. This paved the way for Breuillard, Green, and Tao to give a full classification of all finite approximate subgroups in [BGT12].

Let X be an approximate subgroup and $G := \langle X \rangle$. By a *locally compact [resp. Lie] model* of X we mean a group homomorphism $f \colon \langle X \rangle \to H$ for some locally compact [resp. Lie] group H such that $f[X]$ is relatively compact in S and there is a neighborhood U of the neutral element in S with $f^{-1}[U] \subseteq X^m$ for some $m < \omega$. (In this paper, locally compact spaces are Hausdorff by definition.)

By a *definable* (in some structure M) *approximate subgroup* we mean an approximate subgroup X of some group such that $X, X^2, X^3, \ldots$ are all definable in M and $\cdot|_{X^n \times X^n} : X^n \times X^n \to X^{2n}$ is definable in M as well. Naming the appropriate parameters, we will be assuming that the definable approximate subgroups are 0-definable (i.e. without parameters). If the approximate subgroup X is definable in M, then in the definition of a locally compact model, one usually additionally requires *definability* of f in the sense that for any open $U \subseteq H$

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and compact $C \subseteq H$ such that $C \subseteq U$, there exists a definable (in M) subset Y of G such that $f^{-1}[C] \subseteq Y \subseteq f^{-1}[U]$. Note that in the abstract situation of an arbitrary approximate subgroup X, we can always equip the ambient group with the *full structure* (i.e. add all subsets of all finite Cartesian powers as predicates), and then X becomes definable and the additional requirement of definability of locally compact models is automatically satisfied. In other words, definable approximate subgroups generalize abstract approximate subgroups.

It is folklore (see Corollary 3.4) that for a definable (in a structure M) approximate subgroup X, the existence of a definable locally compact model is equivalent to the existence of some (equivalently, the smallest) M-type-definable subgroup of $\bar{G} := \langle \bar{X} \rangle$ of bounded index, where $\bar{X}$ is the interpretation of X in a monster model extending M (see Subsection 2.2). This smallest subgroup is denoted by $\bar{G}^{00}_M$. By compactness, any type-definable subgroup of $\bar{G}$ is contained in some power $\bar{X}^m$; in particular, if $\bar{G}^{00}_M$ exists, it is necessarily contained in some $\bar{X}^m$. The existence of $\bar{G}^{00}_M$ together with the requirement $\bar{G}^{00}_M \subseteq \bar{X}^m$ for a given m is precisely equivalent to saying that there exists a sequence $(D_n)_{n<\omega}$ of definable, symmetric subsets of X^m with the properties $D_{n+1}D_{n+1} \subseteq D_n$ and D_n is generic (i.e. finitely many left translates of D_n cover X) for all $n < \omega$. We have that if a definable locally compact model for X exists (equivalently, $\bar{G}^{00}_M$ exists), then the quotient map $G \to \bar{G}/\bar{G}^{00}_M$ is the universal definable locally compact model (see Proposition 3.3).

Having a pseudofinite approximate subgroup X of a group M, one can equip M with a sufficiently rich structure (e.g. the full structure where all subsets of all finite Cartesian powers are added as predicates). Let $G = \langle X \rangle$. Hrushovski proved that for $\bar{X}$ being the interpretation of X in the monster model extending M and $\bar{G} := \langle \bar{X} \rangle$, the component $\bar{G}^{00}_M$ exists and is contained in $\bar{X}^4$. Then the quotient map $G \to \bar{G}/\bar{G}^{00}_M$ is the universal locally compact model for X. Next, using Yamabe's theorem, he deduced that there exists an approximate subgroup Y commensurable with X (i.e. finitely many left translates of Y cover X and vice versa) and contained in X^4 such that Y has a Lie model. He proved a much more general result for the so-called near-subgroups (see [Hru12, Theorem 4.2]). This was obtained as a consequence of a suitable "stabilizer theorem" in a stable context proved in [Hru12]. Some variants of Hrushovski's stabilizer theorem were established later in several papers by various authors. For example, Massicot and Wagner proved the existence of definable locally compact models for definably amenable approximate subgroups. More precisely, from [MW15, Theorem 12] it follows that if X is a definable (in a structure M) definably amenable approximate subgroup, then in the monster model the group $\bar{G} := \langle \bar{X} \rangle$ has the component $\bar{G}^{00}_M$ contained in $\bar{X}^4$ (see also Fact 4.2). *Definable amenability* of X means that there is an invariant under left translation, finitely additive measure μ on definable subsets of $\bar{G} := \langle \bar{X} \rangle$ such that $\mu(\bar{X}) = 1$. In particular, this applies in the case when $\bar{G}$ is abelian, as then $\bar{G}$ is an amenable group, and so X is definably amenable (even amenable) by [Hru20, Lemma 6.1].

Wagner conjectured (see [Mas18, Conjecture 0.15] and the paragraph after Theorem 1 in [MW15]) that a definable locally compact model always exists. This conjecture was refuted in [HKP22, Section 4] (even in the abstract situation, where definability can be erased). In [Hru20], Hrushovski proved the existence of locally compact and Lie models in a generalized sense involving quasi-homomorphisms, and used them to give complete classifications of approximate lattices in $\mathrm{SL}_n(\mathbb{Z})$ and $\mathrm{SL}_n(\mathbb{Q}_p)$. This work is very advanced; among various tools, it uses a new locally compact group attached to a theory invented by Hrushovski as a counterpart of the Ellis group (or rather its canonical Hausdorff quotient) of a first order theory which was defined and studied in [KPR18], [KNS19], and [KR22].

The goal of the present paper is to see whether definable locally compact models for definable approximate rings always exist. In contrast to approximate groups, our main result yields a positive answer in full generality. We obtain it as an easy corollary of our main theorem which concerns some fundamental issues on model-theoretic connected components of approximate rings, answering in particular the main question from [KR22], but in a more general context of approximate rings. Let us give some details.

In this paper, rings need not be commutative or unital. There are various possible definitions of approximate subrings. One can define an approximate subgring of a ring as an additively symmetric subset X such that $XX \cup (X+X) \subseteq (F \cup \{1\})X \cap (F+X)$ for some finite subset F of the subring $\langle X \rangle$ generated by X. We will work with a more general definition saying that $XX \cup (X+X) \subseteq F+X$ for some finite $F \subseteq \langle X \rangle$; in particular, X is additively an approximate subgroup. Define recursively X_n, $n < \omega$, by: $X_0 := X$ and $X_{n+1} := X_n X_n + (X_n + X_n)$. As we will see in Fact 2.1, if X is an approximate subring, then each X_n is covered by finitely many additive translates of X. Important structural results on finite approximate subrings were obtained by Tao in [Tao09].

By a *definable* (in some structure M) *approximate subring* we mean an approximate subring X such that $X_0, X_1, \dots$ are all definable in M and $+$ and $\cdot$ restricted to any X_n are also definable in M.

A [definable] locally compact model of a [definable] approximate subring X is defined as a counterpart of a [definable] locally compact model of a [definable] approximate subgroup (see the paragraph preceding Proposition 3.3). As in the case of definable approximate subgroups, the existence of a definable locally compact model is equivalent to the existence of a suitable model-theoretic ring component of the ring $\bar{R} := \langle \bar{X} \rangle$ generated by the interpretation $\bar{X}$ of X in the monster model. Namely, we observe in Corollary 3.4 that a definable locally compact model for X exists if and only if there exists some (equivalently, the smallest) M-type-definable two-sided ideal in $\bar{R}$. This smallest ideal is denoted by $\bar{R}^{00}_M$. By compactness, any type-definable subgroup of $(\bar{R},+)$ is contained in some $\bar{X}_m$; in particular, if $\bar{R}^{00}_M$ exists, it is contained in some $\bar{X}_m$. Various model-theoretic connected components of definable rings were defined and studied in [GJK22] and [KR22]. In particular, it was shown in [GJK22] that in the definition of $\bar{R}^{00}_M$ "two-sided ideal" can be replaced by "left ideal" or "right ideal" or "subring" and in each case we get the same notion. The proofs also work for $\bar{R} = \langle \bar{X} \rangle$. By compactness, one easily shows that the existence of $\bar{R}^{00}_M$ together with the requirement $\bar{R}^{00}_M \subseteq \bar{X}_m$ for a given m is precisely equivalent to saying that there exists a sequence $(D_n)_{n<\omega}$ of definable, additively symmetric subsets of X_m with the properties $D_{n+1}D_{n+1} + (D_{n+1} + D_{n+1}) \subseteq D_n$ and D_n is generic (i.e. finitely many left additive translates of D_n cover X) for all $n < \omega$.

In [KR22, Theorem 1.2], it was shown that for a unital definable ring $\bar{R}$ we have $(\bar{R},+)^{00}_M + \bar{R} \cdot (\bar{R},+)^{00}_M + \bar{R} \cdot (\bar{R},+)^{00}_M = \bar{R}^{00}_M$ (so we say that $(\bar{R},+)^{00}_R$ generates an ideal in $2\frac{1}{2}$ steps), and for a definable ring of finite characteristic we have $(\bar{R},+)^{00}_M + \bar{R} \cdot (\bar{R},+)^{00}_M = \bar{R}^{00}_M$ (i.e. $(\bar{R},+)^{00}_M$ generates an ideal in $1\frac{1}{2}$ steps). It was left as a question (see [KR22, Question 1.3]) if finitely many steps are enough for arbitrary definable rings (besides unital or finite characteristic rings a positive answer was also obtained for finitely generated rings, but with higher numbers of steps). It was also shown in Examples 8.1 and 8.2 of [KR22] that $1\frac{1}{2}$ steps is an optimal (i.e. cannot be decreased) bound on the number of steps needed to generate an ideal.

In this paper, we prove that $1\frac{1}{2}$ steps is enough not only for arbitrary definable rings (answering [KR22, Question 1.3]), but also for rings generated by definable approximate subrings, i.e. for $\bar{R} = \langle \bar{X} \rangle$ where X is a definable approximate subring. Namely, in Theorem 4.1, we show that $(\bar{R},+)^{00}_M + \bar{R} \cdot (\bar{R},+)^{00}_M = \bar{R}^{00}_M$; in particular, $\bar{R}^{00}_M$ exists. From this, we deduce in Corollary 4.11 that a definable locally compact model exists for an arbitrary definable approximate subring, and the quotient map $R \to \bar{R}/\bar{R}^{00}_M$ is the universal such model.

In fact, we show that for an arbitrary small subset A of the monster model $(\bar{R},+)^{00}_A + \bar{R} \cdot (\bar{R},+)^{00}_A$ is an invariant over A two-sided ideal of bounded index (actually the smallest one, denoted by $\bar{R}^{000}_A$). For a definable $\bar{R}$ it gives us $(\bar{R},+)^{00}_A + \bar{R} \cdot (\bar{R},+)^{00}_A = \bar{R}^{00}_A$. In our general context of $\bar{R} = \langle \bar{X} \rangle$ (where X is a 0-definable approximate subring), we get the last equality assuming that $R \subseteq \mathrm{dcl}(A)$ (so e.g. for $A = R$ or $A = M$).

2. Preliminaries

2.1. **Approximate rings.** By an approximate subring of a ring we mean an additively symmetric subset X of this ring such that $X \cdot X \cup (X+X) \subseteq F + X$ for some finite subset F of the ring generated by X, which will be denoted by $\langle X \rangle$. Then X is clearly additively an approximate subgroup. The sequence $(X_n)_{n<\omega}$ is defined recursively: $X_0 := X$ and $X_{n+1} := X_n X_n + (X_n + X_n)$. For $m \in \mathbb{N}$ let $m(X^{<m})$ denote the set of sums of m elements which are products of at most m elements of X. Adapting the proof of [Bre11, Lemma 5.5], we get the following fact.

Fact 2.1. *If X is an approximate subring, then for every $m \in \mathbb{N}_{>0}$, $m(X^{<m})$ is covered by finitely many additive translates of X. In particular, every X_n is covered by finitely many additive translates of X.*

Proof. The second part follows directly from the first, as X_n is contained in $m(X^{<m})$ for a sufficiently large m.

Let $F \subseteq \langle X \rangle$ be such that $XX \cup (X+X) \subseteq F + X$.

Claim 1. *For every $x \in \langle X \rangle$, xX is covered by finitely many additive translates of X.*

Proof. First, by induction on m, we show that for any $x_0, \dots, x_{m-1} \in X$ one has that $x_{m-1} \dots x_0 X$ is covered by finitely many additive translates of X. For $m = 1$ we have $x_0 X \subseteq XX \subseteq F + X$. For the induction step, assume that $x_{m-1} \dots x_0 X \subseteq G + X$ for some finite G. Then $x_m \dots x_0 X \subseteq x_m(G + X) = x_m G + XX \subseteq x_m G + F + X$ and $x_m G + F$ is clearly finite.

Next, by induction on m, we show that whenever $x_0, \dots, x_{m-1} \in \langle X \rangle$ are such that for every $i < m$, $x_i X \subseteq F_i + X$ for some finite F_i, then $(x_{m-1} + \dots + x_0)X$ is also covered by finitely many additive translates of X. For the induction step, assume that $(x_{m-1} + \dots + x_0)X \subseteq G + X$ for some finite G. Then $(x_m + \dots + x_0)X \subseteq F_m + X + G + X \subseteq F_m + G + F + X$ and $F_m + G + F$ is clearly finite. □(claim)

Claim 2. *$X^m \subseteq F_m + X$ for some finite F_m.*

Proof. We prove it by induction on m. For $m = 1$ it is trivial. For the induction step, assume that $X^m \subseteq F_m + X$ for some finite F_m. By Claim 1, for every $x \in F_m$ there exists a finite F_x such that $xX \subseteq F_x + X$. Hence, $F_m X \subseteq X + \bigcup_{x \in F_m} F_x$ and $G_m := \bigcup_{x \in F_m} F_x$ is finite. So $X^{m+1} = X^m X \subseteq F_m X + XX \subseteq G_m + X + F + X \subseteq G_m + F + F + X$ and $G_m + F + F$ is finite. □(claim)

Since $X^{<m} = X^1 \cup \dots \cup X^{m-1}$, by Claim 2, we get $X^{<m} \subseteq F'_m + X$ for some finite F'_m. By an easy induction, we conclude that $m(X^{<m})$ is covered by finitely many additive translates of X. □

2.2. **Model theory.** Let T be a complete first order theory in a language $\mathcal{L}$. For a model M of T and $A \subseteq M$, a *type* over A is a consistent collection of formulas with parameters from A. A *monster model* of T (often denoted by $\mathfrak{C}$) is a κ-saturated and strongly κ-homogeneous model of T for a sufficiently large cardinal κ; usually it suffices to assume that κ is a strong limit cardinal greater that $|\mathcal{L}|$ (i.e. the cardinality of the set all formulas in $\mathcal{L}$). *κ-saturation* means that every type over a set of parameters from $\mathfrak{C}$ of cardinality less that κ has a realization in $\mathfrak{C}$; *strong κ-homogeneity* means that every elementary map between subsets of $\mathfrak{C}$ of cardinality smaller than κ extends to an automorphism of $\mathfrak{C}$. It is a common thing in model theory to work in a fixed monster model, which always exists (by using model-theoretic compactness and a suitable recursive construction). A subset of $\mathfrak{C}$ is said to be *small* if its cardinality is smaller than κ; a cardinal is *bounded* if it is smaller than κ. It is very convenient to work in a monster model, especially when one deals with definable approximate groups or rings and with the model-theoretic connected components of the groups or rings generated by them.

Working in a model M of T, for $A \subseteq M$, an A-*definable* set is the set of realizations in M of a formula with parameters from A; a *definable set* is an M-definable set; instead of "$\emptyset$-definable" we will write "0-definable". Working in $\mathfrak{C}$, for a small $A \subseteq \mathfrak{C}$, an A-*type-definable* set is a set of realizations in $\mathfrak{C}$ of a type over A; a *type-definable set* is an A-type-definable set for some small $A \subseteq \mathfrak{C}$. (The empty set is also considered as type-definable if needed.) Finally, a subset of $\mathfrak{C}$ (or of a Cartesian power of $\mathfrak{C}$) is said to be A-*invariant*, if it is invariant under $\mathrm{Aut}(\mathfrak{C}/A)$ (= the pointwise stabilizer of A); in contrast to definability and type-definability, *invariance* means 0-invariance (i.e. invariance under $\mathrm{Aut}(\mathfrak{C})$). Throughout the paper, $\mathfrak{C}$ is always chosen as a monster model with respect to M, that is $\mathfrak{C} \succ M$ and the degree of saturation (i.e. κ above) of $\mathfrak{C}$ is bigger than $|M|$.

A group [ring] G is said to be A-*definable* if both the universe G and the group operation [resp. $\cdot$ and $+$] are A-definable. *Type-definable* and *invariant* groups [rings] are defined analogously (working in $\mathfrak{C}$).

Definable (in M) approximate subgroups and subrings were defined in the introduction. Adding finitely many parameters from M to the language, we can and do assume that they are 0-definable. More general notions are those of $\bigvee$-definable (or ind-definable) groups and rings, but we will not go into that in this paper.

For a definable subset D of M, by $\bar{D}$ we will usually denote its interpretation in $\mathfrak{C}$, but with one exception. If X is a definable (in M) approximate subgroup [or subring] and $R := \langle X \rangle$, then $\bar{X}$ is the interpretation of X in $\mathfrak{C}$, but $\bar{R}$ will stand for $\langle \bar{X} \rangle$. It may happen that R is definable in M, and if it is not the case that $R = X^m$ [or $R = X_m$ in the case of rings] for some m, then $\bar{R}$ is not the interpretation of the definable R in $\mathfrak{C}$. This is because, by saturation of $\mathfrak{C}$, the fact that $\bar{R}$ is definable is equivalent to $\bar{R} = \bar{X}^m$ [or $\bar{R} = \bar{X}_m$ in the case of rings] for some m.

For any $a \in \mathfrak{C}$ and $A \subseteq \mathfrak{C}$, by $\mathrm{tp}(a/A)$ we denote the *type of a over A*, that is the collection of all formulas over A realized by a. By $\mathrm{dcl}(A)$ we denote the *definable closure* of A, i.e. the collection of all elements which are fixed by $\mathrm{Aut}(\mathfrak{C}/A)$.

When D is a definable set in M and C is a compact space, then a function $f \colon D \to C$ is said to be *definable* if the preimages of any two disjoint closed subsets of C can be separated by a definable subset of D. (This is essentially saying that f is a continuous logic formula, but we will not use any continuous logic terminology in this paper.) By [GPP14, Lemma 3.2], this is equivalent to saying that f extends to an *M-definable map* $\bar{f} \colon \bar{D} \to C$ in the sense that the preimage of any closed subset of C is M-type-definable. Such an extension $\bar{f}$ is unique and given by $\bar{f}(a) = \bigcap_{\varphi(x) \in \mathrm{tp}(a/M)} \mathrm{cl}(f[\varphi(M) \cap D])$ (where cl denotes the closure in C). In Section 3, we extend these considerations to definable approximate subgroups and subrings.

2.3. **Model-theoretic connected components of definable groups and rings.**

We recall below some facts on model-theoretic connected components of definable groups and rings. While definable groups have played an important role in model theory for many years, the components of rings were introduced recently in [GJK22] where they were used to compute Bohr compactifications of some groups of matrices, e.g. both the discrete and continuous Heisenberg group. They were further studied in [KR22].

Let R be a 0-definable group [resp. ring], $\bar{R} = R(\mathfrak{C})$, and $A \subseteq \mathfrak{C}$ be a small set of parameters.

- $\bar{R}^0_A$ is the intersection of all A-definable, finite index subgroups [ideals] of $\bar{R}$.
- $\bar{R}^{00}_A$ is the smallest A-type-definable, bounded index subgroup [ideal] of $\bar{R}$.
- $\bar{R}^{000}_A$ is the smallest A-invariant, bounded index subgroup [ideal] of $\bar{R}$.

We did not specify whether the ideals above are left, right, or two-sided. This is because of Proposition 3.6, Corollary 3.7, and Proposition 3.10 from [GJK22] which tell us that

Fact 2.2. *The above components of the ring $\bar{R}$ do not depend on the choice of the version (left, right, or two-sided) of the ideals. Moreover, instead of "ideal" we can equivalently write "subring" in the above definitions.*

In the case of a definable group R, it is easy to see (cf. for example [Gis11, Lemma 2.2(3)]) that $\bar{R}^0_A, \bar{R}^{00}_A, \bar{R}^{000}_A$ are always normal subgroups of $\bar{R}$.

For a definable group [or ring] R and a small $A \subseteq \mathfrak{C}$, $\bar{R}^{000}_A \leqslant \bar{R}^{00}_A \leqslant \bar{R}^0_A$. It is easy to see (cf. [Gis11, Lemma 2.2(1)]) that all these components exist and their indices in $\bar{R}$ are in fact bounded by $2^{|\mathcal{L}|+|A|}$.

If S is a type-definable, normal subgroup [two-sided ideal] in $\bar{R}$ of bounded index, then $\bar{R}/S$ is equipped with the *logic topology*: closed sets are those whose preimages under the quotient map are type-definable. This makes the quotient $\bar{R}/S$ a compact (topological) group [ring] (for the case of groups see [Pil04, Section 2]; for rings it remains to check that multiplication is continuous which is an easy exercise).

A *compactification* of a (discrete) group [resp. ring] R is a homomorphism $f\colon R \to C$ with dense image, where C is a compact group [ring]. A *definable compactification* of R is a compactification which is a definable map as defined in Subsection 2.2. The *Bohr compactification* of R is the unique (up to isomorphism) universal compactification $h\colon R \to U$ of R (universality means that for any other compactification $f\colon R \to C$ there exists a unique continuous homomorphism $g : U \to C$ such that $f = g \circ h$); and similarly in the definable version.

By [GPP14, Proposition 3.4] and [GJK22, Proposition 3.28], we know that the quotient map $R \to \bar{R}/\bar{R}^{00}_M$ is the definable Bohr compactification of the group [ring] R. The idea of the proof is very simple. If $f\colon G \to C$ is a definable compactification, one extends it uniquely to an M-definable map $\bar{f}\colon \bar{R} \to C$ and checks that $\bar{f}$ is also a homomorphism which factors through the quotient map $\bar{R} \to \bar{R}/\bar{R}^{00}_M$. Similarly, the quotient map $R \to \bar{R}/\bar{R}^0_M$ is the universal definable profinite compactification of R. So for example the equality $\bar{R}^{00}_M = \bar{R}^0_M$ means precisely that both compactifications coincide. When R is equipped with the full structure, we can erase the adjective "definable" and we get classical notions of compactification (described in a model-theoretic way).

Using the classical fact that compact unital or finite characteristic rings are profinite, we get that whenever a 0-definable ring R is unital or of finite characteristic, then $\bar{R}^0_A = \bar{R}^{00}_A$ (see [KR22, Corollary 2.10]). The following is [KR22, Theorem 1.2]:

Fact 2.3. *Let R be a 0-definable ring and $A \subseteq \mathfrak{C}$ a small set of parameters.*

(1) If R is unital, then $(\bar{R},+)^{00}_A + \bar{R}\cdot(\bar{R},+)^{00}_A + \bar{R}\cdot(\bar{R},+)^{00}_A = \bar{R}^{000}_A = \bar{R}^{00}_A = \bar{R}^0_A$.

(2) If R is of positive characteristic (not necessarily unital), then $(\bar{R},+)^{00}_A + \bar{R}\cdot(\bar{R},+)^{00}_A = \bar{R}^{000}_A = \bar{R}^{00}_A = \bar{R}^0_A$.

It was asked in [KR22] whether a similar fact holds for arbitrary 0-definable R (except "$= \bar{R}^0_A$", which fails in general; e.g. in some rings with zero multiplication) and if yes, how many steps are needed. In Section 4, we will answer this question by proving that for every 0-definable ring R, $(\bar{R},+)^{00}_A + \bar{R}\cdot(\bar{R},+)^{00}_A = \bar{R}^{000}_A = \bar{R}^{00}_A$, so $1\frac{1}{2}$ steps always suffice. On the other hand, Examples 8.1 and 8.2 of [KR22] show that one cannot decrease the number of steps to 1 (i.e. $(\bar{R}\cup\{1\})\cdot(\bar{R},+)^{00}_A$ need not be an additive subgroup), even for commutative, unital rings of finite characteristic.

3. Model-theoretic connected components of definable approximate groups and rings

For a definable (in some M) approximate subgroup [subring] X, $R := \langle X \rangle$, $\bar{R} = \langle \bar{X} \rangle$, and a small set of parameters $A \subseteq \mathfrak{C}$, we define the following components.

- $\bar{R}^{00}_A$ is the smallest A-type-definable, bounded index subgroup [two-sided ideal] of $\bar{R}$.

- $\bar{R}^{000}_A$ is the smallest A-invariant, bounded index subgroup [two-sided ideal] of $\bar{R}$.

In contrast to definable groups, the existence of $\bar{R}^{00}_A$ for definable approximate subgroups [subrings] is a non-trivial issue.

Both above components for definable approximate subgroups were studied in Section 4 of [HKP22]. In particular, Proposition 4.3 of [HKP22] yields the existence and a description of $\bar{R}^{000}_A$ which implies that $[\bar{R} : \bar{R}^{000}_A] \leqslant 2^{|\mathcal{L}|+|A|}$. On the other hand, [HKP22, Subsection 4.3] yields an example where $\bar{R}^{00}_A$ does not exist. The existence of $\bar{R}^{00}_A$ is equivalent to the existence of some A-type-definable subgroup of $\bar{R}$ of bounded index (as then the intersection of all such subgroups is A-type-definable of index $\leqslant 2^{|\mathcal{L}|+|A|}$, so equals $\bar{R}^{00}_A$).

In the context of definable approximate subrings, we will prove in Section 4 that $(\bar{R},+)^{00}_A + \bar{R} \cdot (\bar{R},+)^{00}_A = \bar{R}^{000}_A$ which further equals $\bar{R}^{00}_A$ provided that $R \subseteq \operatorname{dcl}(A)$; in particular, in contrast to definable approximate subgroups, $\bar{R}^{00}_A$ always exists for definable approximate subrings (under the assumption that $R \subseteq \operatorname{dcl}(A)$). For completeness notice that the existence of $\bar{R}^{000}_A$ is clear: the intersection of all A-invariant, bounded index, two-sided ideals of $\bar{R}$ will be A-invariant and of bounded index $\leqslant [\bar{R} : (\bar{R},+)^{000}_A] \leqslant 2^{|\mathcal{L}|+|A|}$.

The proofs of statements 3.3, 3.5, and 3.6(i,ii) of [GJK22] go through with very minor adjustments to conclude with

Proposition 3.1. *The components $\bar{R}^{00}_A$ and $\bar{R}^{000}_A$ of the ring $\bar{R}$ do not depend on the choice of the version (left, right, or two-sided) of the ideals. Moreover, instead of "two-sided ideal" we can equivalently write "subring" in the above definitions.*

Let $\bar{R}$ be as in the first sentence of this section. If I is a type-definable, normal subgroup [two-sided ideal] in $\bar{R}$ of bounded index, then $\bar{R}/S$ is equipped with the *logic topology*: a subset $F \subseteq \bar{R}/S$ is closed if the sets $\pi^{-1}[F] \cap \bar{X}^m$ [resp. $\pi^{-1}[F] \cap \bar{X}_m$] are type-definable for every $m \in \omega$, where $\pi\colon \bar{R} \to \bar{R}/I$ is the quotient map. This makes the quotient $\bar{R}/I$ a locally compact topological group [resp. ring]. For groups it appeared first time in Section 7 of [HPP07], and then stood behind the model-theoretic approach to approximate subgroups. For rings one additionally has to check that multiplication is continuous, which is an easy exercise. Let us only remark that by compactness (or rather saturation of $\mathfrak{C}$), $I \subseteq \bar{X}^m$ [resp. $I \subseteq \bar{X}_m$] for some $m \in \omega$. A compact neighborhood of e [resp. of 0] is for example the set $\bar{X}^n/I$ [resp. $\bar{X}_n/I$] for any $n \geqslant m$, with an open neighborhood of e [resp. 0] contained in it being $\{a/I : a + I \subseteq \bar{X}^n\}$ [resp. $\{a/I : a + I \subseteq \bar{X}_n\}$]. The compact subsets of $\bar{R}/I$ are those with type-definable preimage under π (and so contained in some $\bar{X}^n$ [resp. $\bar{X}_n$]).

As was already explained in the introduction, one can extend the notion of a definable map from a definable set to a compact space to homomorphisms from groups [rings] generated by definable approximate subgroups [subrings] to locally compact groups. Namely, for a definable approximate subgroup [subring] X and a locally compact group [resp. ring] H, a homomorphism $f\colon R \to H$ such that $f[X]$ is relatively compact in H will be called *definable* if for any open $U \subseteq H$ and compact $C \subseteq H$ such that $C \subseteq U$, there exists a definable (in M) subset Y of R such that $f^{-1}[C] \subseteq Y \subseteq f^{-1}[U]$.

Lemma 3.2. *Let X be a definable approximate subgroup [subring] and H a locally compact group [ring]. Let $f\colon R \to H$ be a homomorphism such that $f[X]$ is relatively compact in H.*

(1) If f is definable, then it extends uniquely to a map $\bar{f}\colon \bar{R} \to H$ such that $\bar{f}^{-1}[C] \cap \bar{X}^m$ [resp. $\bar{f}^{-1}[C] \cap \bar{X}_m$] is M-type-definable for every m and for every closed $C \subseteq H$. This unique $\bar{f}$ is a homomorphism.

(2) Assume additionally that there is a neighborhood V of e [resp. of 0] in H such that $f^{-1}[V] \subseteq X^m$ [resp. $f^{-1}[V] \subseteq X_m$] for some m. Then, if f extends to some $\bar{f}\colon \bar{R} \to H$ as in (1), then f is definable.

Proof. Let us focus on the case of an approximate subgroup X; the case of an approximate subring is completely analogous (working with X_m in place of X^m).

(1) Let $H_m := \mathrm{cl}(f[X^m])$. Since $H_1 = \mathrm{cl}(f[X])$ is compact by assumption and $H_m = \mathrm{cl}(f[X]^m)$, we get that $H_m = H_1^m$ is also compact. Therefore, by the assumption of (1), $f|_{X^m} \colon X^m \to H_m$ is a definable map from a definable set to a compact space. So it extends uniquely to an M-definable function $\bar{f}_m \colon \bar{X}^m \to H_m$, as explained in the last paragraph of Subsection 2.2. By the explicit formulas for the $\bar{f}_m$'s, we see that $\bar{f}_1 \subseteq \bar{f}_2 \subseteq \dots$. So $\bar{f} := \bigcup_m \bar{f}_m$ is the desired extension of f. Its uniqueness follows from the uniqueness of the $\bar{f}_m$'s after noticing that for any other $\bar{f}' \colon \bar{R} \to H$ as in (1) we have $\bar{f}'[\bar{X}^m] \subseteq H_m$ (which holds as $\bar{f}'^{-1}[H_m] \cap \bar{X}^m$ is an M-type-definable set containing X^m, and so $\bar{f}'^{-1}[H_m] \cap \bar{X}^m = \bar{X}^m$).

To see that $\bar{f}$ is a homomorphism, one can apply the argument from [GPP14, Proposition 3.4]. Namely, since $\bar{f}_m \subseteq \bar{f}$ and any $a \in \bar{R} = \langle \bar{X} \rangle$ belongs to some $\bar{X}^m$, we have $\bar{f}(a) = \bigcap_{\varphi(x) \in \mathrm{tp}(a/M)} \mathrm{cl}(f[\varphi(M) \cap X^m])$. Consider any $a, b \in \bar{R}$. Choose m such that $a, b, ab \in \bar{X}^m$. Let $p = \mathrm{tp}(a/M)$, $q = \mathrm{tp}(b/M)$, and $r = \mathrm{tp}(ab/M)$. Then

$$\{\bar{f}(ab)\} = \bigcap_{\varphi(x) \in r} \mathrm{cl}(f[\varphi(M) \cap X^m]) \subseteq \bigcap_{\varphi(x) \in p, \psi(x) \in q} \mathrm{cl}(f[\varphi(M) \cdot \psi(M) \cap X^m]) =$$

$$\bigcap_{\varphi(x) \in p, \psi(x) \in q} \mathrm{cl}(f[\varphi(M) \cap X^m] \cdot f[\psi(M) \cap \bar{X}^m]) =$$

$$\bigcap_{\varphi(x) \in p, \psi(x) \in q} \mathrm{cl}(f[\varphi(M) \cap X^m]) \cdot \mathrm{cl}(f[\psi(M) \cap X^m]) =$$

$$\bigcap_{\varphi(x) \in p} \mathrm{cl}(f[\varphi(M) \cap X^m]) \cdot \bigcap_{\psi(x) \in q} \mathrm{cl}(f[\psi(M) \cap X^m]) = \{\bar{f}(a)\bar{f}(b)\},$$

where the third and fourth equality uses compactness of H_m.

(2) Consider a compact $C \subseteq H$ and an open $U \subseteq H$ such that $C \subseteq U$. Choose a neighborhood W of e such that $W^{-1}W \subseteq V$ (where V is from the assumption in (2)). Since C is compact, $C \subseteq \bigcup_{i<n} g_i W$ for some $n < \omega$ and $g_i \in H$. Hence, $f^{-1}[C] \subseteq \bigcup_{i<n} f^{-1}[g_i W]$. Pick $a_i \in f^{-1}[g_i W]$ (if there is any) for $i < n$. Then $f^{-1}[g_i W] \subseteq a_i f^{-1}[W^{-1}W] \subseteq a_i f^{-1}[V] \subseteq a_i X^m$. So $f^{-1}[C] \subseteq X^k$ for some k. On the other hand, by the property of $\bar{f}$, we have that $\bar{f}^{-1}[C] \cap \bar{X}^k$ and $\bar{f}^{-1}[H \backslash U] \cap \bar{X}^k$ are disjoint M-type-definable sets. So they can be separated by $\bar{Y}$ for some (M-)definable subset Y of X^k. Hence, $f^{-1}[C] \subseteq Y \subseteq f^{-1}[U]$. □

By a *definable locally compact model* of R we mean a definable homomorphism $f \colon R \to S$ for some locally compact group [resp. ring] S such that $f[X]$ is relatively compact in S and there is a neighborhood U of e [resp. 0] with $f^{-1}[U] \subseteq X^m$ [resp. $f^{-1}[U] \subseteq X_m$] for some $m < \omega$. It is well-known (at least for approximate groups) that for $A \subseteq M$, the quotient map $R \to \bar{R}/\bar{R}_A^{00}$ is a definable locally compact model of R. We give a proof below for the readers convenience, and we additionally prove universality of this model for $A := M$.

Proposition 3.3. *Let $A \subseteq M$ and assume that $\bar{R}_A^{00}$ exists. The quotient map $h \colon R \to \bar{R}/\bar{R}_A^{00}$ is a definable locally compact model, which for $A := M$ is universal in the sense that for any other definable locally compact model $f \colon R \to S$ there is a unique continuous homomorphism $g \colon \bar{R}/\bar{R}_M^{00} \to S$ such that $f = g \circ h$.*

Proof. We skip the proof that $\bar{R}/\bar{R}_A^{00}$ is a locally compact group [ring] (see the discussion after Proposition 3.1). Since $h[X] \subseteq \bar{X}/\bar{R}_A^{00}$ and the last set is compact, we get that $h[X]$ is relatively compact. As remarked above, if we choose m with $\bar{R}_A^{00} \subseteq \bar{X}^m$ [resp. $\bar{R}_A^{00} \subseteq \bar{X}_m$], then $U := \{a/\bar{R}_A^{00} : a + \bar{R}_A^{00} \subseteq \bar{X}^m\}$ [resp. $U := \{a/\bar{R}_A^{00} : a + \bar{R}_A^{00} \subseteq \bar{X}_m\}$] is an open neighborhood of e [resp. 0]; and clearly $h^{-1}[U] \subseteq X^m$ [resp. $h^{-1}[U] \subseteq X^m$]. To show definability of h, consider any compact $C \subseteq \bar{R}/\bar{R}_A^{00}$ and open $V \subseteq \bar{R}/\bar{R}_A^{00}$ such that $C \subseteq V$. Let $\bar{h} \colon \bar{R} \to \bar{R}/\bar{R}_A^{00}$

be the quotient map. Then $\bar{h}^{-1}[C]$ is a type-definable subset of some $\bar{X}^n$ which is disjoint from the type-definable set $\bar{h}^{-1}[(\bar{R}/\bar{R}_A^{00})\backslash V] \cap \bar{X}^n$. It remains to show that these disjoint sets are M-invariant, as then they are M-type-definable, so, being disjoint, they can be separated by $\bar{Y}$ for some definable subset Y of X^n; then clearly $h^{-1}[C] \subseteq Y \subseteq h^{-1}[V]$, as required. The fact that these type-definable sets are M-invariant follows from the fact that the relation of having the same type over M is the finest M-invariant, bounded equivalence relation on $\bar{R}$ and so it refines the relation of lying in the same coset of $\bar{R}_A^{00}$ (as $A \subseteq M$).

Observe that $R/\bar{R}_A^{00}$ is dense in $\bar{R}/\bar{R}_A^{00}$. Indeed, take a non-empty open $V \subseteq \bar{R}/\bar{R}_A^{00}$. Then the preimage $\bar{h}^{-1}[V]$ is a union of definable sets; and, as above we can find these sets to be definable over M. Since this union is non-empty, at least one of these M-definable sets is non-empty, and so it intersects R, which shows that $V \cap (R/\bar{R}_A^{00})$ is non-empty. By the density of $R/\bar{R}_A^{00}$, uniqueness in the universal property becomes clear.

For the existence, consider any definable locally compact model $f\colon R \to S$. By Lemma 3.2, f extends to a homomorphism $\bar{f}\colon \bar{R} \to S$ such that $\bar{f}^{-1}[C] \cap \bar{X}^k$ [resp. $\bar{f}^{-1}[C] \cap \bar{X}_k$] is M-type-definable for every k and for every closed $C \subseteq S$. Since by assumption $\ker(\bar{f}) \subseteq \bar{X}^n$ [resp. X_n] for some n, we get that $\ker(\bar{f})$ is an M-type-definable subgroup [two-sided ideal] of $\bar{R}$ of bounded index (it is bounded by the cardinality of the closure of $f[R]$ in S, so by $2^{2^{|R|}}$). Therefore, $\bar{R}_M^{00} \subseteq \ker(\bar{f})$, and so $\bar{f}$ factors through the quotient map $\bar{h}\colon \bar{R} \to \bar{R}/\bar{R}_M^{00}$, i.e. there is a homomorphism $g\colon \bar{R}/\bar{R}_M^{00} \to S$ such that $\bar{f} = g \circ \bar{h}$. Since $h \subseteq \bar{h}$ and $f \subseteq \bar{f}$, we get $f = g \circ h$. It remains to check that g is continuous. Take any closed $C \subseteq S$. Then $\bar{h}^{-1}[g^{-1}[C]] = \bar{f}^{-1}[C]$ has type-definable intersections with all the $\bar{X}^m$'s [resp. $\bar{X}_m$'s]. Therefore, $g^{-1}[C]$ is closed by the definition of the logic topology. □

Corollary 3.4. *The existence of a definable locally compact model of X is equivalent to the existence of $\bar{R}_M^{00}$.*

Proof. If $\bar{R}_M^{00}$ exists, then the quotient map $R \to \bar{R}/\bar{R}_M^{00}$ is a definable locally compact model. Conversely, if $f\colon R \to S$ is a definable locally compact model, then, by the last paragraph of the proof of Proposition 3.3, $\ker(\bar{f})$ is an M-type-definable subgroup [two-sided ideal] of $\bar{R}$ of bounded index. Hence, $\bar{R}_M^{00}$ exists. □

4. Generating in $1\frac{1}{2}$ steps and a locally compact model

Here, we prove the main results of this paper, answering the main question from [KR22] and providing locally compact models for arbitrary definable approximate subrings (so, in particular, abstract approximate subrings by taking the full structure). The goal is to prove:

Theorem 4.1. *Let X be a 0-definable (in M) approximate subring, $R := \langle X \rangle$, $\bar{R} = \langle \bar{X} \rangle$, and let $A \subseteq \mathfrak{C}$ be a small set of parameters. Then $(\bar{R},+)_A^{00} + \bar{R} \cdot (\bar{R},+)_A^{00} = \bar{R}_A^{000}$. Moreover, if $R \subseteq \operatorname{dcl}(A)$, then $\bar{R}_A^{00}$ exists and equals $\bar{R}_A^{000} = (\bar{R},+)_A^{00} + \bar{X}(\bar{R},+)_A^{00}$.*

First of all, we have

Fact 4.2. *If Z is a definably amenable 0-definable (in M) approximate subgroup, then $\langle \bar{Z} \rangle_A^{00}$ exists (where $\langle \bar{Z} \rangle$ is the group generated by $\bar{Z}$). Moreover, $\langle \bar{Z} \rangle_A^{00} \subseteq \bar{Z}^8$, and if $A = M$, then $\langle \bar{Z} \rangle_A^{00} \subseteq \bar{Z}^4$. In particular, $(\bar{R},+)_A^{00}$ exists and is contained in $8\bar{X}$, and if $A = M$, then it is contained in $4\bar{X}$.*

Proof. The part concerning Z follows from [MW15, Theorem 12 or Corollary 13] and [Mas18, Theorem 5.2]. If we work with $A = M$, then instead of [Mas18, Theorem 5.2], an easy compactness argument from the proof of Claim 1 of [KP19] in Case 2 (on page 1282) can be used to make sure that the parameters are taken from M, and it gives us $\langle \bar{Z} \rangle_M^{00} \subseteq \bar{Z}^4$. For the second part (concerning R), note that since the additive group generated by $\bar{X}$, say G, is abelian and so amenable, by [Hru20, Lemma 6.1], we get that $\bar{X}$ is an amenable (and so definably amenable) approximate subgroup, hence G_A^{00} exists and satisfies the desired inclusions by the first part. Then $G_A^{00} = (\bar{R}, +)_A^{00}$, because G is of bounded (even countable) index in $(\bar{R}, +)$ by Fact 2.1. □

We will need the notion of thick subset of $\bar{R}$, as given in [HKP22, Definition 4.1].

Definition 4.3. A definable, additively symmetric subset D of $\bar{R}$ is *thick* if for every sequence $(r_i)_{i<\lambda}$ of unbounded length which consists of elements of $\bar{R}$ there are $i < j < \lambda$ with $r_j - r_i \in D$.

Using compactness, one gets

Remark 4.4. A definable, symmetric subset D of $\bar{R}$ is thick if and only if for every $m \in \omega$ there exists a positive integer M such that for every $r_0, \dots, r_{M-1} \in \bar{X}_m$ there are $i < j < M$ with $r_j - r_i \in D$. For any M with this property, we will say that D is *M-thick* in $\bar{X}_m$.

Using this remark together with finite Ramsey theorem (exactly as in the proof of [Gis10, Lemma 1.2]), we get that the class of thick subsets of $\bar{R}$ is closed under finite intersections. Remark 4.4 also implies that in Definition 4.3 the adjective "unbounded" can be replaced by "uncountable".

The following basic observation will be crucial in the proof of the main lemma below. From now on, in this section, $H := (\bar{R}, +)_A^{00}$.

Remark 4.5. Every definable, additively symmetric subset of $\bar{R}$ which contains H is thick. Thus, H is the intersection of a downward directed family of A-definable thick subsets of $\bar{R}$.

Proof. Since $[\bar{R} : H] \leqslant 2^{|\mathcal{L}|+|A|}$, we have that for any $\lambda > 2^{|\mathcal{L}|+|A|}$, for every sequence $(r_i)_{i<\lambda}$ of elements of $\bar{R}$ there are $i < j < \lambda$ with $r_j - r_i \in H$. Hence, the same is true for any superset of H, and so all definable, additively symmetric supersets of H are thick. The second part follows from that, since H is clearly the intersection of the family of all A-definable, additively symmetric subsets of $\bar{R}$ containing H and this family is downward directed. □

We will also need the following definition and remark from [KR22].

Definition 4.6. We will say that two subgroups H_1 and H_2 of an abelian group G are *coset-independent* if any coset of H_1 intersects any coset of H_2. They are *coset-dependent* if they are not coset-independent.

Remark 4.7. Let G be an abelian group and $H_1, H_2 \leqslant G$. The following conditions are equivalent.

(i) H_1 and H_2 are coset-independent.
(ii) H_1 intersects any coset of H_2.
(iii) $H_1 + H_2 = G$.

Thus, H_1 and H_2 are coset-dependent if and only if $H_1 + H_2$ is a proper subgroup of G.

The next lemma is the technical core of the proof of Theorem 4.1. This lemma is a variant of Lemma 4.4 from [KR22], and its proof is a non-trivial elaboration on the proof of that lemma.

Lemma 4.8. *Let G be the intersection of all sets of the form $\bar{R}K/H$, where K ranges over all bounded index subgroups of $(\bar{R}, +)$ which are type-definable over some sets of parameters of cardinality at most $2^{2^{|\mathcal{L}|+|A|}}$. Then G is a subgroup of $(\bar{R}/H, +)$.*

Proof. It is clear that $0/H \in G$ and G is closed under additive inverses. Thus, we need to show that it is closed under $+$. So consider any $a, b \in G$, and we will show that $a + b \in G$.

The family of subgroups of $(\bar{R}, +)$ over which K ranges in the statement of the lemma will be denoted by $\mathcal{K}$.

By Remark 4.5, $H = \bigcap_{i\in I} D_i$ for some downward directed family $\{D_i\}_{i\in I}$ of A-definable thick subsets of $\bar{R}$. We can assume that $|I| \leqslant |\mathcal{L}|+|A|$. Using Remark 4.4 and the terminology introduced there, for every $i \in I$ and $m \in \omega$ we can choose a positive integer $M_{i,m}$ such that D_i is $M_{i,m}$-thick in $\bar{X}_m$.

For every $i \in I$, $s \in \bar{R}$, and $K \in \mathcal{K}$, define:

$$n_{i,s,K} := \max\{|Y| : Y \subseteq sK \text{ and for every distinct } x, y \in Y \text{ we have } x - y \notin D_i\}.$$

Note that since $sK \subseteq \bar{X}_k$ for some $k \in \omega$, we have $n_{i,s,K} < M_{i,k}$. Since for any fixed $m \in \omega$ and $K \in \mathcal{K}$ there is k such that $\bar{X}_m K \subseteq \bar{X}_k$, for every fixed $i \in I$ and $m \in \omega$ there exists a smallest $n_{i,m} \in \omega$ for which there exists $K_{i,m} \in \mathcal{K}$ such that

$$(\forall s \in \bar{X}_m)(b \in sK_{i,m} \Rightarrow n_{i,s,K_{i,m}} \leqslant n_{i,m}).$$

(In particular, if there is no $s \in \bar{X}_m$ for which $b \in sK_{i,m}$, then $n_{i,m} = 0$.) Put

$$K_{I,\omega} := \bigcap_{i\in I} \bigcap_{m\in\omega} K_{i,m}.$$

Since $|I| \leqslant |\mathcal{L}| + |A|$, we get $K_{I,\omega} \in \mathcal{K}$. By the above choices, we also have

$$(*) \qquad (\forall i \in I)(\forall m \in \omega)(\forall s \in \bar{X}_m)(b \in sK_{I,\omega} \Rightarrow n_{i,s,K_{I,\omega}} \leqslant n_{i,m}).$$

For $r \in \bar{R}$ let $g_r : \bar{R} \to \bar{R}/H$ be given by $g_r(x) := rx/H$. It is a group homomorphism. Note that $[\bar{R} : \ker(g_r)] \leqslant |\bar{R}/H| \leqslant 2^{|\mathcal{L}|+|A|}$.

Case 1. For every $K \in \mathcal{K}$ with $K \leqslant K_{I,\omega}$, there are $r, s \in \bar{R}$ with $a \in rK/H$ and $b \in sK/H$ such that $\ker(g_r) \cap K$ and $\ker(g_s) \cap K$ are coset-independent subgroups of K.

Then, since $g_r^{-1}(a) \cap K$ and $g_s^{-1}(b) \cap K$ are cosets of $\ker(g_r) \cap K$ and $\ker(g_s) \cap K$, respectively, they have a non-empty intersection, i.e. there is $k \in K$ with $rk/H = a$ and $sk/H = b$. Hence, $a + b = (r + s)k/H \in \bar{R}K/H$. Since this holds for every $K \in \mathcal{K}$ with $K \leqslant K_{I,\omega}$ (so also for every $K \in \mathcal{K}$), we conclude that $a + b \in G$.

Case 2. There exists $K \in \mathcal{K}$ with $K \leqslant K_{I,\omega}$ such that for all $r, s \in R$ with $a \in rK/H$ and $b \in sK/H$, $\ker(g_r) \cap K$ and $\ker(g_s) \cap K$ are coset-dependent subgroups of K.

By the definition of G, pick $r_0 \in R$ with $a \in r_0K/H$. By Remark 4.7, for any $s \in \bar{R}$ with $b \in sK/H$ (by the definition of G, at least one such s exists),

$$(**) \qquad \ker(g_{r_0}) \cap K \leqslant (\ker(g_{r_0}) \cap K) + (\ker(g_s) \cap K) \lneq K.$$

Put $L_s := (\ker(g_{r_0}) \cap K) + (\ker(g_s) \cap K)$. Since $[K : \ker(g_{r_0}) \cap K] \leqslant |\bar{R}/H| \leqslant 2^{|\mathcal{L}|+|A|}$, there are at most $2^{2^{|\mathcal{L}|+|A|}}$ possibilities for L_s when s varies as above. Let $K'_{I,\omega}$ be the intersection of all these L_s's. Since each L_s is type-definable over the parameters over which K is defined together with r_0, s, we see that $L_s \in \mathcal{K}$. Hence, $K'_{I,\omega} \in \mathcal{K}$.

Since there are at most $2^{2^{|\mathcal{L}|+|A|}}$ possibilities for L_s, by (**), there exists a set $E \subseteq K$ with $|E| \leqslant 2^{2^{|\mathcal{L}|+|A|}}$ such that

$$(***) \qquad (\forall s \in \bar{R})(b \in sK/H \Rightarrow (\exists k \in E)(k \in K \backslash L_s)).$$

Note that the condition $k \in K\backslash L_s$ implies that $sk \notin sL_s + H$.

Claim. *For every $m \in \omega$ there exists $i_m \in I$ such that for every $s \in \bar{X}_m$ with $b \in sK/H$ there is $k \in E$ such that $sk \notin sL_s + D_{i_m}$.*

Proof. Suppose this fails, which is witnessed by some $m \in \omega$. Note that the sets L_s are type-definable uniformly in s, that is there is a type $\pi(x, y)$ (with some fixed parameters) such that $L_s = \pi(\mathfrak{C}, s)$ for every s in question. Thus, since E is small and $\{D_i\}_{i\in I}$ is downward directed, by compactness (or rather saturation of $\mathfrak{C}$), there exists $s \in \bar{X}_m$ with $b \in sK/H$ such that $(\forall k \in E)(sk \in sL_s + H)$, a contradiction with $(***)$. □(claim)

Now, by the definition of G, we can find $s_0 \in \bar{R}$ for which $b \in s_0 K'_{I,\omega}/H$. Choose $m < \omega$ such that $s_0 \in \bar{X}_m$. By the claim, we get

$$(\forall s \in \bar{X}_m)(b \in sK/H \Rightarrow n_{i_m,s,L_s} < n_{i_m,s,K}).$$

Since $K'_{I,\omega} \leqslant L_s \leqslant K \leqslant K_{I,\omega}$, we conclude that

$$(\forall s \in \bar{X}_m)(b \in sK'_{I,\omega}/H \Rightarrow n_{i_m,s,K'_{I,\omega}} < n_{i_m,s,K_{I,\omega}}).$$

By $(*)$, this implies that

$$(\forall s \in \bar{X}_m)(b \in sK'_{I,\omega}/H \Rightarrow n_{i_m,s,K'_{I,\omega}} < n_{i_m,m}),$$

which contradicts the minimality of $n_{i_m,m}$ (as there is at least one $s \in \bar{X}_m$ with $b \in sK'_{I,\omega}/H$, namely s_0). □

In order to prove Theorem 4.1, we need one more non-trivial ingredient stated below. For a proof in the context of definable groups see the proof of Fact 2.2 of [KR22]. It works the same for definable approximate subgroups, as the facts on which it relies (i.e. [MW15, Theorem 12] and [Mas18, Theorem 5.2]) are stated for definable approximate subgroups. Also, instead of Lemmas 2.2(2) and 3.3 of [Gis11], one should use their versions for approximate subgroups stated in Propositions 4.3 and 4.5 of [HKP22].

Fact 4.9. *If Z is a definably amenable 0-definable approximate subgroup, then $\langle \bar{Z}\rangle^{00}_A = \langle \bar{Z}\rangle^{000}_A$. In particular, $(\bar{R},+)^{00}_A = (\bar{R},+)^{000}_A$.*

Now, we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. Let $\mathcal{K}$ be the family of all bounded index subgroups of $(\bar{R},+)$ which are type-definable over some sets of parameters of cardinality at most $2^{2^{|\mathcal{L}|+|A|}}$. Let $G := \bigcap_{K \in \mathcal{K}} \bar{R}K/H$, as in Lemma 4.8. By Lemma 4.8, G is a subgroup of $\bar{R}/H$ which is clearly A-invariant.

Since $|G| \leqslant |\bar{R}/H| \leqslant 2^{|\mathcal{L}|+|A|}$, G is an intersection of at most $2^{2^{|\mathcal{L}|+|A|}}$ sets $\bar{R}K/H$, i.e. $G = \bigcap_{K \in \mathcal{K}_0} \bar{R}K/H$ for some $\mathcal{K}_0 \subseteq \mathcal{K}$ of cardinality bounded by $2^{2^{|\mathcal{L}|+|A|}}$. Let $K_0 := H \cap \bigcap \mathcal{K}_0$. Then $K_0 \in \mathcal{K}$ and $G = \bar{R}K_0/H$.

Let $H_0 := \bigcap_{r \in \bar{R}} g_r^{-1}[G]$, where $g_r \colon H \to \bar{R}/H$ is given by $g_r(x) := rx/H$. Since G is a subgroup of $\bar{R}/H$ and G and H are both A-invariant, we get that H_0 is an A-invariant subgroup of H. It is clear that $K_0 \leqslant H_0$, so H_0 is of bounded index in $(\bar{R},+)$. Therefore, $(\bar{R},+)^{000}_A \leqslant H_0 \leqslant H = (\bar{R},+)^{00}_A$. Since by Fact 4.9 $(\bar{R},+)^{00}_A = (\bar{R},+)^{000}_A$, we conclude that $H_0 = H$.

On the other hand, $\bar{R}H_0/H = G$ (($\subseteq$) follows by the definition of H_0, while ($\supseteq$) follows from the fact that $K_0 \leqslant H_0$ and $\bar{R}K_0/H = G$).

Putting the last two paragraphs together, we get $G = \bar{R}H/H$. Hence, $\pi^{-1}[G] = H + \bar{R}H$ is an A-invariant, bounded index subgroup of $\bar{R}$, where $\pi \colon \bar{R} \to \bar{R}/H$ is the quotient map. It follows that $H + \bar{R}H$ is closed under left multiplication by the elements of $\bar{R}$, and so it is a left ideal. Therefore, by Proposition 3.1 and Fact 4.9, we get

$$\bar{R}^{000}_A \subseteq H + \bar{R}H = (\bar{R},+)^{000}_A + \bar{R}(\bar{R},+)^{000}_A \subseteq \bar{R}^{000}_A + \bar{R}\bar{R}^{000}_A = \bar{R}^{000}_A,$$

and hence $H + \bar{R}H = \bar{R}^{000}_A$ as required.

For the "moreover" part, it is enough to prove that

$$(\bar{R},+)^{00}_A + \bar{R}(\bar{R},+)^{00}_A = (\bar{R},+)^{00}_A + \bar{X}(\bar{R},+)^{00}_A.$$

Indeed, since the right hand side is A-type-definable and the left hand side equals $\bar{R}^{000}_A$ by the first part of the theorem, we conclude that both sides are equal to $\bar{R}^{00}_A$, which will complete the proof.

The inclusion ($\supseteq$) is obvious. So we prove ($\subseteq$). By Fact 2.1, we can choose a countable subset Y of R so that $Y + \bar{X} = \bar{R}$. It is enough to show that

$$(\forall y \in Y)(y(\bar{R},+)^{00}_A \subseteq (\bar{R},+)^{00}_A).$$

So pick $y \in Y$. Let $l_y \colon \bar{R} \to \bar{R}$ be given by $l_y(t) := yt$. This is a $\{y\}$-invariant group homomorphism. Choose $m \in \omega$ so that $(\bar{R},+)^{00}_A \subseteq \bar{X}_m$. Since $l_y|_{\bar{X}_m}$ is $\{y\}$-definable, and, by the assumption that $R \subseteq \operatorname{dcl}(A)$ we have $y \in \operatorname{dcl}(A)$, we get $l_y[(\bar{R},+)^{00}_A] = (l_y[\bar{R}],+)^{00}_A$. On the other hand, $(l_y[\bar{R}],+)^{00}_A \leqslant (\bar{R},+)^{00}_A$, for if not, then $(l_y[\bar{R}],+)^{00}_A \cap (\bar{R},+)^{00}_A$ would be a proper A-type-definable subgroup of $(l_y[\bar{R}],+)^{00}_A$ of bounded index. Therefore, $y(\bar{R},+)^{00}_A = l_y[(\bar{R},+)^{00}_A] = (l_y[\bar{R}],+)^{00}_A \subseteq (\bar{R},+)^{00}_A$. ☐

The next corollary answers positively Question 1.3 of [KR22].

Corollary 4.10. *If $\bar{R}$ is definable, then $(\bar{R},+)^{00}_A + \bar{R}\cdot(\bar{R},+)^{00}_A = \bar{R}^{000}_A = \bar{R}^{00}_A$ for an arbitrary small $A \subseteq \mathfrak{C}$.*

Proof. It follows from Theorem 4.1, because $(\bar{R},+)^{00}_A + \bar{R}\cdot(\bar{R},+)^{00}_A$ is A-type-definable. ☐

The next corollary yields the existence and a description of the universal definable locally compact model for an arbitrary definable approximate subgring.

Corollary 4.11. *Let X be a 0-definable (in M) approximate subring, $R := \langle X\rangle$, and $\bar{R} = \langle \bar{X}\rangle$. Then X has a definable locally compact model. More precisely, the quotient map $h \colon R \to \bar{R}/\bar{R}^{00}_M$ is the universal definable locally compact model of X, and $U := \{a/\bar{R}^{00}_M : a + \bar{R}^{00}_M \subseteq 4\bar{X} + \bar{X}\cdot 4\bar{X}\}$ is an open neighborhood of $0/\bar{R}^{00}_M$ such that $h^{-1}[U] \subseteq 4X + X\cdot 4X$.*

Proof. By the "moreover" part of Theorem 4.1 and Proposition 3.3, we get that $\bar{R}^{00}_M$ exists and the quotient map $h \colon R \to \bar{R}/\bar{R}^{00}_M$ is the universal definable locally compact model of X.

By Fact 4.2, we know that $(\bar{R},+)^{00}_M \subseteq 4\bar{X}$. So, by the "moreover" part of Theorem 4.1, we get $\bar{R}^{00}_M = (\bar{R},+)^{00}_A + \bar{X}(\bar{R},+)^{00}_A \subseteq 4\bar{X} + \bar{X}\cdot 4\bar{X}$. Thus, $0/\bar{R}^{00}_M \in U$. The fact that U is open follows easily from the definition of the logic topology on $\bar{R}/\bar{R}^{00}_M$. The fact that $h^{-1}[U] \subseteq 4X + X\cdot 4X$ is obvious by the definition of U. ☐

5. The 0-components

In the case of a 0-definable (in the monster model $\mathfrak{C}$) ring $\bar{R}$, we have Fact 2.2 for $\bar{R}^0_A$, and we know by Corollary 2.10 of [KR22] that $\bar{R}^{00}_A = \bar{R}^0_A$ whenever $\bar{R}$ is unital or of positive characteristic. In particular, in those two cases, $\bar{R}/\bar{R}^{00}_A = \bar{R}/\bar{R}^0_A$ is a profinite ring. In this subsection, we explain that all of this drastically fails for approximate subrings.

A natural counterpart of $\bar{R}^0_A$ for approximate subrings is as follows. From now on, let X be a 0-definable (in M) approximate subring, $R := \langle X\rangle$, $\bar{R} = \langle\bar{X}\rangle$, and let $A \subseteq \mathfrak{C}$ be a small set of parameters.

Definition 5.1. $\bar{R}^0_{A,ideal}$ is the intersection of all A-definable two-sided ideals of $\bar{R}$ of countable (equivalently, bounded) index. $\bar{R}^0_{A,ring}$ is the intersection of all A-definable subrings of $\bar{R}$ of countable index.

The existence of $\bar{R}^0_{A,ideal}$ [resp. $\bar{R}^0_{A,ring}$] is clearly equivalent to the existence of some A-definable two-sided ideal [resp. subring] of countable index. We will see in the examples below that it may happen that $\bar{R}^0_{A,ring}$ does not exist as well as that it exists but $\bar{R}^0_{A,ideal}$ does not. Even $(\bar{R},+)^0_A$ need not exist. Moreover, even for unital $\bar{R}$, $\bar{R}/\bar{R}^{00}_A$ need not be totally disconnected. If $\bar{R}$ is of positive characteristic, then $\bar{R}/\bar{R}^{00}_A$ is totally disconnected but need not have a basis of neighborhoods of 0 consisting of open ideals. Let us go to some details.

Note that "$\bar{R}^0_{A,ideal}$ exists" implies "$\bar{R}^0_{A,ring}$ exists" implies "$(\bar{R},+)^0_A$ exists".

Example 5.2. Let $M := (\mathbb{R},+,\cdot,0,1)$ and $X := [-1,1]$ which is clearly a 0-definable approximate subring (and here $R = \mathbb{R}$). Then $(\bar{R},+)^0_M$ does not exist. Also, $\bar{R}^{00}_M = (\bar{R},+)^{00}_M = \bigcap_{n\in\omega} \bar{I}_n =: \mu$, where $I_n := [-\frac{1}{n}, \frac{1}{n}]$ and $\bar{I}_n$ is the interpretation of I_n in $\mathfrak{C}$ (i.e. μ is the subgroup of the infinitesimals of $\bar{R}$), and $\bar{R}/\bar{R}^{00}_M$ is isomorphic to $\mathbb{R}$ as a topological ring, so it is not totally disconnected.

Proof. By compactness, the fact that $(\bar{R},+)^0_M$ does not exist is equivalent to the fact that there is no definable subgroup of $(R,+)$ contained in some $nX = [-n,n]$ and whose finitely many additive translates cover nX. And the right hand side clearly holds, as the only subgroup of $(R,+)$ contained in some $[-n,n]$ is $\{0\}$.

For the second part, the analysis of Example 3.2 of [GJK22] applies with minor adjustments (note that still we have a well-defined standard part map st: $\bar{R} \to \mathbb{R}$ and we show that $\ker(\mathrm{st}) = \bar{R}^{00}_M$). □

If one prefers to work in the abstract context, one can equip the reals with the full structure (where all subsets of all finite Cartesian powers are added as predicates on M). Then, by the same reason as above, $(\bar{R},+)^0_M$ does not exist. Regarding the second part, we get a continuous epimorphism from $\bar{R}/\bar{R}^{00}_M$ to $\mathbb{R}$ which implies that $\bar{R}/\bar{R}^{00}_M$ is not totally disconnected.

Example 5.3. Let $M := \mathbb{F}_p((t))$ be the field of formal Laurent series (over the finite field $\mathbb{F}_p$) equipped with the full structure. Let X be the subset (in fact, additive subgroup) consisting of the series of the form $\sum_{i=-1}^{\infty} a_i t$. This is clearly a 0-definable approximate subring, and $R := \langle X \rangle = \mathbb{F}_p((t))$. Then $\bar{R}^0_{M,ideal}$ does not exist, while $\bar{R}^0_{M,ring}$ does exist. The ring $\bar{R}/\bar{R}^{00}_M$ is totally disconnected but does not have a basis of neighborhoods of 0 consisting of open ideals.

Proof. The existence of $\bar{R}^0_{M,ideal}$ is equivalent to the existence of a definable ideal of R contained in some X_m and whose finitely many additive translates cover X_m. But R is a field, so it does not have such an ideal. Thus, $\bar{R}^0_{M,ideal}$ does not exist. On the other hand, since the set $\mathbb{F}_p[[t]]$ of all formal power series is a definable subring of R whose p translates cover X, we get that $\bar{R}^0_{M,ring}$ exists. Since the additive group of $\bar{R}/\bar{R}^{00}_M$ is a torsion, locally compact abelian group, we get that it is totally disconnected (e.g. see [Arm81, Theorem 3.5]). Let st: $\bar{R} \to R$ be the standard part map (where R is equipped with the usual valuation topology which makes it a locally compact field). Then $\ker(\mathrm{st}) = \bigcap_{n\in\omega} \bar{I}_n$, where I_n is the set of formal power series of the form $\sum_{i=n}^{\infty} a_i t^i$. We have that $\bar{R}/\ker(\mathrm{st})$ is topologically isomorphic to R, and the obvious map $\bar{R}/\bar{R}^{00}_M \to \bar{R}/\ker(\mathrm{st})$ is a continuous epimorphism. Thus, if $\bar{R}/\bar{R}^{00}_M$ had a basis of neighborhoods of 0 consisting of open ideals, then the images of these ideals would form a basis at 0 in R consisting of open ideals, which is a non-sense as R is a non-discrete field. □

Theorem 1.1 of [KR22] tells us that if $\bar{R}$ is definable and H is an A-definable subgroup of $(\bar{R},+)$ of finite index, then $H + \bar{R}H$ contains an A-definable two-sided ideal of finite index. From that it is deduced that $(\bar{R},+)^0_A + \bar{R}(\bar{R},+)^0_A = \bar{R}^0_A$ (see Theorem 1.5 and Proposition 3.4(2) of [KR22]). In our general context of $\bar{R}$ generated by $\bar{X}$, both observations fail: Example

5.3 is a counter-example to both statements (where in the first statement we replace "finite index" by "countable index", and in the second one we assume that $(\bar{R},+)^0_A$ exists) . However, the following remains unclear.

Question 5.4. *Suppose $(\bar{R},+)^0_A$ exists. Is it true that $(\bar{R},+)^0_A + \bar{R}(\bar{R},+)^0_A$ is a subgroup of $(\bar{R},+)$?*

Recall that Corollary 8.11 of [KR22] yields an example of a definable, commutative, unital ring $\bar{R}$ and a 0-type-definable subgroup H of $(\bar{R},+)$ which is an intersection of a countable descending sequence of definable subgroups of finite index (so $(\bar{R},+)^0_\varnothing \leqslant H$), but $\bar{R}H$ does not additively generate a subgroup in finitely many steps; in particular, $H+\bar{R}H$ is not a subgroup.

Proposition 5.5. *Assume that $\bar{R}$ is of positive characteristic. Then $(\bar{R},+)^0_A$ exists and coincides with $(\bar{R},+)^{00}_A$. Thus, if also $R \subseteq \operatorname{dcl}(A)$, then $(\bar{R},+)^0_A + \bar{R}(\bar{R},+)^0_A = \bar{R}^{00}_A$ (is a subgroup).*

Proof. $\bar{R}/(\bar{R},+)^{00}_A$ is a torsion, locally compact abelian group, and as such it has a basis $\{H_i\}_{i\in I}$ of neighborhoods of 0 consisting of open (so clopen) subgroups (see [Arm81, Theorem 3.5]). Take m such that $(\bar{R},+)^{00}_A \subseteq \bar{X}_m$ (e.g. $m=3$ works, but it does not matter here). Then $U := \{a/(\bar{R},+)^{00}_A : a+(\bar{R},+)^{00}_A \subseteq \bar{X}_m\}$ is an open neighborhood of 0 in $\bar{R}/(\bar{R},+)^{00}_A$, and without loss of generality we can assume that each H_i is contained in U. Let $\pi\colon \bar{R} \to \bar{R}/(\bar{R},+)^{00}_A$ be the quotient map. Then both $\pi^{-1}[H_i] \subseteq \bar{X}_m$ and its complement in $\bar{X}_m$ are type-definable and so definable sets. Hence, $\pi^{-1}[H_i] \subseteq \bar{X}_m$ are definable subgroup of $(\bar{R},+)$ of bounded index (for $i \in I$). And clearly $(\bar{R},+)^{00}_A$ is the intersection of all of them, so $(\bar{R},+)^{00}_A = (\bar{R},+)^0_A$. Thus, the second part of the proposition follows by Theorem 4.1. □

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