GENERALIZED LOCALLY COMPACT MODELS FOR APPROXIMATE GROUPS

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Abstract. We give a proof of the existence of *generalized definable locally compact models* for arbitrary approximate subgroups via an application of topological dynamics in model theory. Our construction is simpler and shorter than the original one by Hrushovski [Hru20] and it uses only basic model theory (mostly spaces of types and realizations of types). The main tools are Ellis groups from topological dynamics considered for suitable spaces of types. However, we need to redevelop some basic theory of topological dynamics for suitable "locally compact flows" in place of (compact) flows. We also prove that the generalized definable locally compact model which we constructed is universal in an appropriate category. We note that the main result yields structural information on definable generic subsets of definable groups, with a more precise structural result for generics in the universal cover of $\mathrm{SL}_2(\mathbb{R})$.

1. Introduction

A subset X of a group is called an *approximate subgroup* if it is symmetric (i.e. $e \in X$ and $X^{-1} = X$) and $XX \subseteq FX$ for some finite $F \subseteq \langle X \rangle$; if F can be chosen of size K, we say that X is a *K-approximate subgroup*. Approximate subgroups were introduced by Tao in [Tao08]. They naturally originate in the classical (in additive number theory and combinatorics) study of finite subsets of groups which have small doubling (or tripling, or n-pling) property, where recall that a finite subset X of a group has *doubling at most* K, if $|XX| \leqslant K|X|$. It is well known (e.g. see [Bre11, Proposition 2.3]) that in the abelian context if X has doubling at most K, then $X - X$ is a K^5-approximate subgroup; a similar result holds for subsets of small tripling in arbitrary groups [Bre11, Proposition 2.2]. The study of the structure of subsets with small doubling goes back to Freiman's theorem saying that the subsets of $\mathbb{Z}$ with small doubling are "close" to generalized arithmetic progressions [Fre64]. An important motivation for introducing approximate subgroups also came from connections and applications to expanders, random walks, and spectral gaps [BG08], and later to geometric group theory [BGT12]. For a historical background on approximate subgroups the reader may consult [Bre14; Toi19]. An advantage of Tao's definition is that it is more algebraic than the notion of small doubling, and applies also to infinite subsets. This is needed for example in the theory of approximate latices (which are certain special approximate subgroups living in locally compact groups) and aperiodic order. The study of approximate latices goes back to the seminal monograph of Meyer [Mey72] from 1972. A short historical background on approximate lattices is outlined in the introduction to the very recent paper [Mac23].

There is a long list of authors and important papers in the subject, including many applications within and outside mathematics, e.g. to quasicrystals. In this introduction, we will only mention a few milestone contributions after [Tao08] which are relevant for this paper.

A symmetric compact neighborhood of the neutral element in a locally compact group is always an approximate subgroup. Let X be an approximate subgroup and $G := \langle X \rangle$. By a *locally compact [resp. Lie] model* of X we mean a group homomorphism $f \colon \langle X \rangle \to H$ for some

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locally compact [resp. Lie] group H such that $f[X]$ is relatively compact in H and there is a neighborhood U of the neutral element in H with $f^{-1}[U] \subseteq X^m$ for some $m < \omega$. It is easy to show that if $f\colon \langle X \rangle \to H$ is a locally compact model of X, then X can be recovered up to commensurability as the preimage of any compact neighborhood of the identity in H.

A breakthrough in the study of the structure of approximate subgroups was obtained by Hrushovski in [Hru12], where a locally compact model for any pseudofinite approximate subgroup (more generally, *near-subgroup*) X was obtained by using model-theoretic tools, and in consequence also a Lie model was found for some approximate subgroup commensurable with X and contained in X^4. This paved the way for Breuillard, Green, and Tao to give a full classification of all finite approximate subgroups in [BGT12].

From the model-theoretic point of view, it is natural to consider *definable approximate subgroups* and their *definable locally compact models*. This is also useful in applications (e.g. in the proof of the main theorem of [BGT12]). These notions are recalled in Definition 2.4. We would like to emphasize here that in the abstract situation of an arbitrary approximate subgroup X, we can always equip the ambient group with the *full structure* (i.e. add all subsets of all finite Cartesian powers as predicates), and then X becomes definable and the additional requirement of definability of locally compact models is automatically satisfied. In other words, definable approximate subgroups generalize abstract approximate subgroups.

Massicot and Wagner [MW15] proved the existence of definable locally compact models for all definably amenable definable approximate subgroups, and Wagner conjectured that a locally compact model exists for an arbitrary approximate subgroup. Literally, this conjecture is false; a counter-example can be found for example in [HKP22, Section 4]. However, in another breakthrough paper [Hru20], Hrushovski weakened the notion of locally compact [and Lie] model of an approximate subgroup X by replacing a homomorphism by a quasi-homomorphism $f\colon \langle X \rangle \to H$ with a compact, normal, symmetric error set S (meaning that $f(y)^{-1}f(x)^{-1}f(xy) \in S$ for all $x, y \in \langle X \rangle$) whose preimage under f is contained in an absolute (i.e. independent of X) power of X, and he proved the existence of such *generalized definable locally compact models* for arbitrary approximate subgroups (where the notion of definability is also weakened appropriately). This is a structural result on arbitrary approximate subgroups, as each approximate subgroup can be recovered up to commensurability as the preimage of any compact neighborhood of the (compact, symmetric) error set via a generalized locally compact model. This allowed Hrushovski to deduce the existence of suitable generalized Lie models and obtain full classifications of approximate lattices in some contexts, e.g., in $\mathrm{SL}_n(\mathbb{R})$ and $\mathrm{SL}_n(\mathbb{Q}_p)$. Very recently, Machado wrote an impressive paper [Mac23] with a complete structure theorem for approximate lattices in linear algebraic groups over local fields and a uniqueness result for generalized locally compact models with certain extra properties.

The proof in [Hru20] of the existence of generalized definable locally compact (and Lie) models is based on a new theory developed by Hrushovski including *definability patterns structures* and *local logics*, which is difficult and may be inaccessible to non model theorists.

We prove the existence of generalized definable locally compact models via topological dynamics methods in a model-theoretic context. The main idea is to extend the fundamental theory of Ellis groups to the context of suitable locally compact flows, and then the desired generalized definable locally compact model is a certain (explicitly defined) quasi-homomorphism to the canonical Hausdorff quotient of the Ellis group. Our proof is much shorter and uses only standard model theory (e.g. externally definable sets, [external] types, realizations of types). In fact, if one is not interested in obtaining any definability property of generalized locally compact models, then one can just equip the group $G := \langle X \rangle$ (generated by the given abstract approximate subgroup X) with the full structure and then our proof uses a suitable locally compact subflow of the Stone-Čech compactification βG of G, and so there is no need to use externally definable sets and external types. Our construction of the generalized definable locally compact model is supposed to be fully self-contained. In particular, we will provide almost all the proofs while developing the theory of Ellis groups for suitable locally compact flows in Section 3; only a few easy proofs that are identical to the proofs in the classical context of compact flows are omitted with precise references to where they can be found.

We also prove universality of our generalized definable locally compact model in a suitable category. As a consequence, we obtain a characterization for a quasi-homomorphism to be a generalized definable locally compact model. While the usual notion of definability of a map from a definable set to a compact space has a characterization coming from continuous logic (namely, a factorization through a suitable space of types), the modified notion of definablity used in generalized definable locally compact models is not so transparent and our characterization explains its nature.

It is also interesting to consider the special case when the approximate subgroup X in question generates a group G in finitely many steps. Then the target space of our generalized locally compact model is compact, and it is in fact the generalized Bohr compactification of G defined by Glasner (see [Gla76]); more generally, for definable X and G the target space of our generalized definable locally compact model is a certain canonical compact group associated to G (see [KP17]). This special case can be seen as a structural result on arbitrary definable generic subsets of definable groups. We will discuss it in Section 5.

In Section 2, we give the necessary preliminaries, including all basic definitions in model theory. Section 3 is devoted to our construction of a generalized definable locally compact model of an arbitrary definable approximate subgroup. In Section 4, we prove universality of our model and discuss related things. In Section 5, we focus on the situation when the approximate subgroup in question generates a group in finitely many steps, so in fact the situation of a definable, symmetric, generic subset of a definable group. We explain why the main result can be thought of as a structural result on such generic subsets and we use it to obtain more precise structural information on generics in the universal cover of $\mathrm{SL}_2(\mathbb{R})$. Moreover, our analysis of $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ leads to an answer to some natural question stated at the end of Section 4, and also shows that the weakening of Newelski's conjecture proposed in Section 5 holds for $\widetilde{\mathrm{SL}_2(\mathbb{R})}$.

We finish this introduction with a brief history connecting Hrushovski's approach from [Hru20] and our approach via topological dynamics. Topological dynamics methods were introduced to model theory by Newelski in [New09]. Since then many papers have appeared in this subject, in particular some connections and applications to model-theoretic components of groups and to strong types were obtained in [KP17; KPR18; KR20; KNS19]. Motivated by this work, Hrushovski developed in [Hru19] a parallel theory of definability patterns structures. Then, in [Hru20], he redeveloped it in the context of local logics introduced by himself in [Hru20], and used it to prove the existence of generalized definable locally compact models. In this paper, we return to the topological dynamics approach, but for locally compact flows instead of usual compact flows, and we provide a shorter and simpler proof of Hrushovski's theorem with further information on universality.

2. Preliminaries

In this section, we recall some basic notions from model theory and topological dynamics to make the main construction self-contained.

2.1. **Model theory.**

Let us fix a *language* (or *signature*) L, i.e. a collection of relation, function, and constant symbols. Using those symbols together with quantifiers, variables, and logical symbols, one constructs recursively the set of all *L-formulas*; *L-sentences* are L-formulas without free variables. An *L-structure* is a set M together with interpretations of all the symbols of L. For example, if L consists of just one binary function symbol, then any group is an L-structure. Let us fix an arbitrary L-structure M.

For any L-sentence φ, $M \models \varphi$ means that φ is true in M. For any subset A of M we can expand the language L to L_A be adding constant symbols for the members of A, which are then interpreted in M as the corresponding elements of A. For an L_A-formula $\varphi(x)$, $\varphi(M)$ denotes the *set of realizations* of $\varphi(x)$ in M, i.e. $\varphi(M) := \{a \in M^{|x|} : M \models \varphi(a)\}$. By an *A-definable subset* of M [more generally, of a Cartesian power M^n] we mean the set of realizations in M of an L_A-formula $\varphi(x)$ with one [resp. n] free variables x. By a *definable subset* we mean an M-definable subset. For example, the centralizer of an element of a group is a definable subset of this group.

An L-structure N is an *elementary superstructure* of M (symbolically, $M \prec N$) if $M \subseteq N$ and for every L-formula $\varphi(x_1, \dots, x_n)$ and tuple $(a_1, \dots, a_n) \in M^n$ we have $M \models \varphi(a_1, \dots, a_n) \iff N \models \varphi(a_1, \dots, a_n)$.

By a *type* over $A \subseteq M$ in variables x we mean a consistent collection $\pi(x)$ of L_A-formulas, where $\pi(x)$ being *consistent* means that for any finitely many formulas $\varphi_1(x), \dots, \varphi_n(x) \in \pi(x)$ we have $M \models (\exists x)(\varphi_1(x) \wedge \dots \wedge \varphi_n(x))$. The compactness theorem tells us that this is equivalent to the property that $\pi(x)$ has a realization a in some $N \succ M$, i.e. $N \models \varphi(a)$ for all $\varphi(x) \in \pi(x)$, which will be denoted by $a \models \pi$. A *complete type* over A in variables x is a type $p(x)$ over A such that for every L_A-formula $\varphi(x)$ we have $\varphi(x) \in p$ or $\neg\varphi(x) \in p$. This is equivalent to saying that $p = \mathrm{tp}(a/A) := \{\varphi(x) \text{ an } L_A\text{-formula} : N \models \varphi(a)\}$ for some tuple a in some $N \succ M$. The set of all complete types over A in variables x is denoted by $S_x(A)$. This is a compact, zero-dimensional topological spaces with a basis of open sets given by the L_A-formulas, i.e. any L_A-formula $\varphi(x)$ yields a basic open set $[\varphi(x)] := \{p \in S(A) : \varphi(x) \in p\}$. Identifying formulas (modulo equivalence) with the definable sets that they define, complete types over A can be treated as ultrafilters in the Boolean algebra of A-definable subsets of $M^{|x|}$, and then the topology on $S_x(A)$ is just the Stone space topology. We will often omit x in $S_x(A)$.

For a given cardinal κ, we say that $N \succ M$ is *κ-saturated* if for every $B \subseteq N$ of cardinality $< \kappa$, every $p \in S(B)$ has a realization in N. Using the compactness theorem, for every κ there exists $N \succ M$ which is κ-saturated. In this paper, we will work with $N \succ M$ which is $|M|^+$-saturated, and since it is very convenient to work with realizations of types from $S(N)$, we will be taking them in an $|N|^+$-saturated $\mathfrak{C} \succ N$.

An *externally definable* subset D of M is the intersection of M with a definable subset of N (where $N \succ M$ is $|M|^+$-saturated), that is $D = M \cap \varphi(N)$ for some formula $\varphi(x)$ with parameters from N. This definition does not depend on the choice of N. By a *complete external type* over M we mean an ultrafilter on the Boolean algebra of externally definable subsets of M; all these types form a Stone space $S_{\mathrm{ext}}(M)$. It is very convenient to identify $S_{\mathrm{ext}}(M)$ with a space of complete types in the usual sense. In order to do that, take an $|M|^+$-saturated $N \succ M$. Then $S_{\mathrm{ext}}(M)$ is homeomorphic with the space $S_M(N)$ of all complete types $p \in S(N)$ which are *finitely satisfiable* in M, i.e. for any $\varphi(x) \in p$, $\varphi(x)$ is realized by some element or tuple of elements of M; more precisely, $S_M(N) \ni p \mapsto \{\varphi(M) : \varphi(x) \in p\} \in S_{\mathrm{ext}}(M)$ is a homeomorphism.

One can also restrict the context to a given formula $\varphi(x)$ with parameters from M or to the set of realizations $X := \varphi(M)$. By $S_{\varphi(x)}(N)$ or $S_X(N)$ we denote the space of complete types $p \in S(N)$ which contain the formula $\varphi(x)$[1]; $S_{X,M}(N)$ will stand for the space of complete types over N which contain $\varphi(x)$ and are finitely satisfiable in M. Then $S_{X,M}(N)$ is

[1]There is a clash of notation here, as $S_M(N)$ and $S_X(N)$ have two different meanings when $X = M$. This should not cause any confusion, as the symbol $S_M(N)$ will always denote the space of complete types over N

homeomorphic with the space $S_{X,\mathrm{ext}}(M)$ of ultrafilters on the Boolean algebra of externally definable subsets of X. All of it applies also to any superset C of N (contained in $\mathfrak{C}$) in place of N. In particular, we have the spaces $S_M(C)$ and $S_{X,M}(C)$ homeomorphic with $S_{\mathrm{ext}}(M)$ and $S_{X,\mathrm{ext}}(M)$, respectively.

In this paper, we will need to extend this context to so-called $\bigvee$-definable sets, i.e. unions of possibly infinitely many definable sets. More precisely, let $\{X_i\}_{i\in I}$ be an upward directed family of A-definable sets for some $A \subseteq M$ (i.e. I is a directed set with respect to some pre-order $\leqslant$, and $X_i \subseteq X_j$ whenever $i \leqslant j$), and let $G := \bigcup_{i\in I} X_i$. Then by $S_{G,M}(N)$ we mean $\bigcup_{i\in I} S_{X_i,M}(N)$ with the topology inherited from $S_M(N)$. Since each $S_{X_i,M}(N)$ is clearly an open subset of $S_M(N)$, we get that $U \subseteq S_{G,M}(N)$ is open if and only if $U \cap S_{X_i,M}(N)$ is open in $S_{X_i,M}(N)$ for all $i \in I$; so $F \subseteq S_{G,M}(N)$ is closed if and only if $F \cap S_{X_i,M}(N)$ is closed in $S_{X_i,M}(N)$ for all $i \in I$. As each $S_{X_i,M}(N)$ is clearly a clopen subset of $S_{G,M}(N)$ which is a compact (Hausdorff) space, we get

Fact 2.1. *$S_{G,M}(N)$ is a locally compact (Hausdorff) space.*

In this paper, compact and locally compact spaces are Hausdorff by definition.

Note that the space $S_{G,M}(N)$ is naturally homeomorphic with the space $S_{G,\mathrm{ext}}(M)$ of those ultrafilters on the Boolean algebra of externally definable subsets of G (i.e. subsets which are intersections of G with the sets definable with parameters from N) which are concentrated on some X_i. It is also heomeomorphic with the space of those ultrafilters on the Boolean algebra of subsets of G generated by all externally definable subsets of the X_i's, $i \in I$, which are concentrated on some X_i.

As before, the above discussion applies also to any superset C of N in place of N. In particular, we have the locally compact space $S_{G,M}(C)$ homeomorphic with $S_{G,\mathrm{ext}}(M)$.

By $S_G(M)$ we mean $\bigcup_{i\in I} S_{X_i}(M)$ with the topology inherited from $S(M)$, where $S_{X_i}(M)$ is the space of complete types over M containing a formula defining X_i. As above, this is a locally compact space which is witnessed by the clopen compact sets $S_{X_i}(M)$.

If it is not specified, all the parameters and elements are taken from $\mathfrak{C}$. Sets of parameters are usually denoted by capital letters, while elements or tuples of elements by lower case letters. For any a, b, A, we will write $a \equiv_A b$ to express that $\mathrm{tp}(a/A) = \mathrm{tp}(b/A)$.

For any $A \subseteq B$ and a we say that $\mathrm{tp}(a/B)$ is a *coheir* over A if it is finitely satisfiable in A (i.e. any formula in $\mathrm{tp}(a/B)$ has a realization in A). The following remark will be used many times.

Remark 2.2. If $a \equiv_A b$ and $\mathrm{tp}(c/A, a, b)$ is a coheir over A, then $a \equiv_{A,c} b$.

Proof. If not, then there is an L_A-formula $\varphi(x, y)$ such that $\mathfrak{C} \models \varphi(a, c) \wedge \neg\varphi(b, c)$. Since $\mathrm{tp}(c/A, a, b)$ is a coheir over A, there is $c' \in A$ such that $\mathfrak{C} \models \varphi(a, c') \wedge \neg\varphi(b, c')$, so $a \not\equiv_A b$, a contradiction. □

Note that if a type $\pi(x)$ (over any set of parameters) is finitely satisfiable in A (meaning that any finite collection of formulas in $\pi(x)$ has a common realization in A), then it extends to a global type $p \in S(\mathfrak{C})$ finitely satisfiable in A. For that it is enough to take any ultrafilter $\mathcal{U}$ on the Boolean algebra of all subsets of A such that $\{\varphi(\mathfrak{C}) \cap A : \varphi(x) \in \pi\} \subseteq \mathcal{U}$ and to define p as $\{\varphi(x) \in L_{\mathfrak{C}} : \varphi(\mathfrak{C}) \cap A \in \mathcal{U}\}$.

Fact 2.3. *For any type $p \in S_M(N)$ and superset B of N there is a unique extension $\tilde{p} \in S(B)$ of p which is finitely satisfiable in M.*

Proof. Existence follows from the above paragraph. Uniqueness follows from the fact that $\{\varphi(M) : \varphi(x) \in p\}$ is an ultrafilter on the Boolean algebra of externally definable subsets of M (which in turn follows from $|M|^+$-saturation of N). □

finitely satisfiable in M, whereas the symbol $S_X(N)$ (for $X := \varphi(M)$) will always denote the space of complete types over N which contain the formula $\varphi(x)$ defining X.

The main object of interest in this paper will be a definable approximate subgroup.

Definition 2.4. (1) A *definable* (in some structure M) *approximate subgroup* is an approximate subgroup X of some group such that $X, X^2, X^3, \dots$ are all definable in M and $\cdot|_{X^n \times X^n} : X^n \times X^n \to X^{2n}$ is definable in M for every positive $n \in \mathbb{N}$.

(2) If the approximate subgroup X is definable in M, then by a *definable locally compact model* of X we mean a locally compact model $f : \langle X \rangle \to H$ of X such that for any open $U \subseteq H$ and compact $C \subseteq H$ with $C \subseteq U$, there exists a definable (in M) subset Y of G such that $f^{-1}[C] \subseteq Y \subseteq f^{-1}[U]$.

The model theory context in this paper will be the following: X will be an approximate subgroup definable in a structure M, $N \succ M$ an $|M|^+$-saturated elementary extension of M, $\mathfrak{C} \succ N$ a big (at least $|N|^+$-saturated) elementary extension of N (the so-called *monster model*), $G := \langle X \rangle$ — the group generated by X, $\bar{X} = X(\mathfrak{C})$ — the interpretation of X in $\mathfrak{C}$, $\bar{G} := \langle \bar{X} \rangle$ — the group generated by $\bar{X}$. Thus, we can use the above notation $S_{G,M}(N)$ for the family $\{X_i\}_{i \in I} := \{X^n : n \in \omega\}$.

Regarding the monster model $\mathfrak{C}$, besides saturation one usually also assumes strong homogeneity with respect to a sufficiently big cardinal. Using the compactness theorem, it is easy to construct $\mathfrak{C} \succ N$ which is $|N|^+$-saturated and *strongly* $|N|^+$*-homogeneous* (see [Hod93, Theorem 10.2.1]) which means that for any subset $A \subseteq \mathfrak{C}$ of cardinality at most $|N|$, any elementary map $f : A \to \mathfrak{C}$ (that is $\mathfrak{C} \models \varphi(a) \iff \mathfrak{C} \models \varphi(f(a))$ for every formula $\varphi(x) \in L$ and finite tuple a from A) extends to an automorphism of $\mathfrak{C}$. Although the arguments in this paper do not require strong $|N|^+$-homogeneity of $\mathfrak{C}$, it is convenient to assume it and use (without even mentioning) the fact that then for every A of cardinality at most $|N|$ and finite tuples a, b we have that $a \equiv_A b$ if and only if $b = f(a)$ for some $f \in \mathrm{Aut}(\mathfrak{C}/A)$ (the pointwise stabilizer of A).

2.2. **Topological dynamics.** Topological dynamics studies *flows*, that is pairs (G, Y) where Y is a compact space and G is a topological group acting continuously on Y. We focus on the case when G is discrete; then continuity of the action just means that the action is by homeomorphisms.

In this paper, we will have to extend the context to the case when Y is a certain special locally compact space on which G acts by homeomorphisms, namely $Y := S_{G,M}(N)$ from the end of the last subsection. We will develop all the necessary theory in this context providing all the details (including proofs) in Section 3. So here we only briefly recall some notions and facts in the classical context of (compact) flows. They will not be used in the main construction (except Fact 2.6), and we give them only to show what is well-known in topological dynamics. This classical context of (compact) flows is however sufficient when the approximate subgroup X generates G in finitely many steps, and this is the context of Section 5.

In the rest of this subsection, (G, X) will be an arbitrary flow (so X and G have nothing to do with the approximate subgroup X considered above). Classical references for Ellis semigroups and groups are [Aus88; Gla76]. A very good concise exposition with proofs can be found in Appendix A of [Rze18].

Definition 2.5. The *Ellis semigroup* of the flow (G, X), denoted by $E(X)$, is the closure of the collection of functions $\{\pi_g : g \in G\}$ (where $\pi_g : X \to X$ is given by $\pi_g(x) := gx$) in the space X^X equipped with the product topology, with composition as the semigroup operation.

$E(X)$ is a compact left topological semigroup (i.e. the semigroup operation is continuous in the left coordinate, meaning that for any fixed p the function $x \mapsto xp$ is continuous). We will usually notionally identify π_g with g for $g \in G$, although the map $g \mapsto \pi_g$ need not be injective.

The following fundamental fact was proved by Ellis (e.g. see Corollary 2.10 and Propositions 3.5 and 3.6 of [Ell69], or Fact A.8 of [Rze18]).

Fact 2.6. *Let S be a semigroup equipped with a quasi-compact T_1 topology such that for any $s_0 \in S$ the map $s \mapsto ss_0$ is a continuous and closed mapping (the latter follows immediately from continuity and compactness if S is Hausdorff). Then there is a minimal left ideal $\mathcal{M}$ in S (i.e. a minimal nonempty set such that $S\mathcal{M} = \mathcal{M}$), and every such $\mathcal{M}$ satisfies the following.*

i) For any $p \in \mathcal{M}$, $Sp = \mathcal{M}p = \mathcal{M}$ is closed.

ii) $\mathcal{M}$ is the disjoint union of the sets $u\mathcal{M}$ with u ranging over $J(\mathcal{M}) := \{u \in \mathcal{M} : u^2 = u\}$.

iii) For each $u \in J(\mathcal{M})$, $u\mathcal{M}$ is a group with identity element u, where the group operation is the restriction of the semigroup operation on S.

iv) All the groups $u\mathcal{M}$ (for $u \in J(\mathcal{M})$) are isomorphic, even when we vary the minimal left ideal $\mathcal{M}$.

Applying this to $S := E(X)$, the isomorphism type of the groups $u\mathcal{M}$ (or just any of these groups) from the above fact is called the *Ellis group* of the flow X. [2]

Definition 2.7. For $B \subseteq E(X)$ and $a \in E(X)$, $a \circ B$ is defined as the set of all points $c \in E(X)$ for which there exist nets $(b_i)_{i\in I}$ in B and $(g_i)_{i\in I}$ in G such that $\lim g_i = a$ and $\lim g_i b_i = c$.

Basic properties of $\circ$ are contained in Facts A.25-A.29 of [Rze18]. In particular, $a \circ B$ is closed.

Now, choose any minimal left ideal $\mathcal{M}$ of $E(X)$ and an idempotent $u \in \mathcal{M}$.

Definition 2.8. For $A \subseteq u\mathcal{M}$, define $\mathrm{cl}_\tau(A) := (u \circ A) \cap u\mathcal{M}$.

For the proofs of the facts listed below see Facts A.30-A.40 in [Rze18].

Fact 2.9. cl_τ *is a closure operator on $u\mathcal{M}$. The topology given by* cl_τ *is called the* τ-topology.

Fact 2.10. *$u\mathcal{M}$ with the τ-topology is a compact T_1 semitopological group (i.e. multiplication is separately continuous) which does not depend (up to topological isomorphism) on the choice of $\mathcal{M}$ and $u \in J(\mathcal{M})$.*

Fact 2.11. *$H(u\mathcal{M}) := \bigcap_V \mathrm{cl}_\tau(V)$, where V ranges over the τ-neighborhoods of u in $u\mathcal{M}$, is a τ-closed normal subgroup of $u\mathcal{M}$, and $u\mathcal{M}/H(u\mathcal{M})$ is a compact Hausdorff topological group. Moreover, if F is a subgroup of $u\mathcal{M}$ such that $u\mathcal{M}/F$ is a Hausdorff space, then $H(u\mathcal{M}) \leqslant F$.*

An ambit is a flow (G, X, x_0) with a distinguished point $x_0 \in X$ with dense orbit. An important classical G-flow is the universal G-ambit βG (see [Gla76, Proposition I.2.6]), i.e. the space of ultrafilters on the Boolean algebra of all subsets of G with the action of G by left translation and the distinguished ultrafilter being the principal ultrafilter of the neutral element. Then the Ellis semigroup $E(\beta G)$ is naturally isomorphic to $(\beta G, *)$, where $*$ is given by $U \in p * q \iff \{g \in G : g^{-1}U \in q\} \in p$ (the isomorphism is recalled below in a more general context). Model theory provides a transparent and very useful formula for $*$. Namely, treat G as a group definable in $M := G$ equipped with the full structure (i.e. with predicates for all subsets of all finite Cartesian powers of G). Then $\beta G = S_{G,\mathrm{ext}}(M)$ is naturally identified with the space of types $S_G(M)$ and it turns out that $p * q = \mathrm{tp}(ab/M)$, where $b \models q$, $a \models p$, and $\mathrm{tp}(a/M, b)$ is the unique extension of p which is a coheir over M. More generally, if we have a group G definable in a structure M, then $S_{G,\mathrm{ext}}(M)$ is a G-ambit with the action of G by left translation and the distinguished element being the ultrafilter of the neutral element. Identifying $S_{G,\mathrm{ext}}(M)$ with $S_{G,M}(N)$ (where $N \succ M$ is $|M|^+$-saturated), it turns out that the Ellis semigroup $E(S_{G,M}(N))$ is isomorphic to $(S_{G,M}(N), *)$ with $*$ given by $p * q := \mathrm{tp}(ab/N)$, where $b \models q$, $a \models p$, and $\mathrm{tp}(a/N, b)$ is the unique extension of p which is a coheir over M (an isomorphism $(S_{G,M}(N), *) \to E(S_{G,M}(N))$ is given by $p \mapsto l_p$, where $l_p(q) := p * q$). For the details see [New09, Section 4].

[2] This terminology is used by model theorists. Let E be the Ellis group (in our sense) of the universal G-ambit βG with neutral element v. In topological dynamics, the Ellis group of a pointed minimal G-flow (X, x_0) with $vx_0 = x_0$ is the subgroup of the elements $\eta \in \mathrm{E}$ for which $\eta x_0 = x_0$, but we will not use this definition.

3. Generalized definable locally compact model

This section is devoted to a new self-contained construction of a generalized definable locally compact model of an arbitrary definable approximate subgroup. Let us start from the context and precise definition of generalized definable locally compact models.

For a map $f\colon G \to H$ from a group (or even semigroup) G to a group H, $\mathrm{error}_r(f) := \{f(y)^{-1}f(x)^{-1}f(xy) : x, y \in G\}$ and $\mathrm{error}_l(f) := \{f(xy)f(y)^{-1}f(x)^{-1} : x, y \in G\}$. For $C \subseteq H$, we write $f\colon G \to H : C$ if $\mathrm{error}_r(f) \cup \mathrm{error}_l(f) \subseteq C$ and we say that f is a *quasi-homomorphism with an error set C*. Note that if C is normal in H, i.e. closed under conjugation by the elements of H (which will be the case in our context), then $\mathrm{error}_r(f) \subseteq C$ if and only if $\mathrm{error}_l(f) \subseteq C$. Also, if $f\colon G \to H : C$, then $f(e_G) \in C^{-1}$ and $f(x^{-1}) \in f(x)^{-1}C^{-2}$ for all $x \in G$. Sometimes one assumes that $f(e_G) = e_H$, and this will be satisfied in our construction.

From now on, take the situation and notation described at the end of Subsection 2.1.

Definition 3.1. A *generalized definable locally compact model of X* is a quasi-homomorphism $f\colon G \to H : C$ for some symmetric, normal, compact subset C of a locally compact group H such that:

(1) for every compact $V \subseteq H$ there is $i \in \mathbb{N}$ with $f^{-1}[V] \subseteq X^i$;
(2) for every $i \in \mathbb{N}$, $f[X^i]$ is relatively compact (i.e., with compact closure) in H;
(3) there is $l \in \mathbb{N}$ such that for any compact $Z, Y \subseteq H$ with $C^lY \cap C^lZ = \emptyset$ the preimages $f^{-1}[Y]$ and $f^{-1}[Z]$ can be separated by a definable set, meaning that there is a definable set D such that $f^{-1}[Y] \subseteq D$ and $f^{-1}[Z] \cap D = \emptyset$.

If we drop item (3), we get the notion of *generalized locally compact model*.

Remark 3.2. Item (2) of the above definition is equivalent to the property that $f[X]$ is relatively compact.

Proof. We have $f[X^2] \subseteq f[X]^2C$, so, by compactness of $\mathrm{cl}(f[X])$ and C, we get $\mathrm{cl}(f[X^2]) \subseteq \mathrm{cl}(f[X])^2C$. More generally, by induction, $\mathrm{cl}(f[X^i]) \subseteq \mathrm{cl}(f[X])^iC^{i-1}$ for all $i \geqslant 1$, and since the last set is compact, so is $\mathrm{cl}(f[X^i])$. □

Remark 3.3. If $f\colon G \to H : C$ is a generalized definable locally compact model of X, then there is $l \in \mathbb{N}$ such that for any compact $Z, Y \subseteq H$ with $C^lY \cap C^lZ = \emptyset$ there are disjoint definable subsets D_1 and D_2 of some X^n with $f^{-1}[Y] \subseteq D_1$ and $f^{-1}[Z] \subseteq D_2$.

Proof. It follows from items (1) and (3) of the definition. □

Fact 3.4. *Let $f\colon G \to H : C$ be a generalized locally compact model of X.*

(1) For every neighborhood U of e_H, $f^{-1}[UC]$ is generic in the sense that finitely many left translates of $f^{-1}[UC]$ cover X.
(2) For every relatively compact neighborhood U of e_H, $Y := f^{-1}[UC]$ is commensurable with X and YY^{-1} is an approximate subgroup commensurable with X.

Proof. (1) Take an open neighborhood W of e_H such at $W^{-1}W \subseteq U$. By compactness of $\mathrm{cl}(f[X])$, we have that $\mathrm{cl}(f[X])$ is covered by finitely many translates $a_1W, \dots, a_nW$.

For every $i \leqslant n$ with $f^{-1}[a_iW] \neq \emptyset$ choose $g_i \in f^{-1}[a_iW]$. We will show that X is covered by the finitely many translates $g_if^{-1}[UC]$ for $i \leqslant n$ such that $f^{-1}[a_iW] \neq \emptyset$.

Consider any $g \in X$; then $g \in f^{-1}[a_iW]$ for some $i \leqslant n$, i.e. $f(g) \in a_iW$. Write g as g_ih. Then $f(g) = f(g_i)f(h)f(h)^{-1}f(g_i)^{-1}f(g_ih) \in a_iWf(h)C$. Hence, the last set has a nonempty intersection with a_iW. So, as C is symmetric, $f(h) \in W^{-1}WC \subseteq UC$. Therefore, $h \in f^{-1}[UC]$, and so $g \in g_if^{-1}[UC]$.

(2) The fact that finitely many left translates of $f^{-1}[UC]$ cover X follows from (1). The fact that finitely many left translates of X cover $f^{-1}[UC]$ follows from item (1) of Definition 3.1 and the assumption that X is an approximate subgroup (which clearly implies that X^i is

covered by finitely many left translates of X for every every $i \in \mathbb{N}$). The very final part about YY^{-1} easily follows, as $Y \subseteq YY^{-1} \subseteq X^i$ for some i (the inclusion $Y \subseteq YY^{-1}$ holds, because $e_G \in Y$, which in turn is true as $f(e_G) \in C^{-1} = C \subseteq UC$). □

Thus, as mentioned in the introduction, a generalized locally compact model $f\colon G \to H : C$ of X allows us to recover X up to commensurability as the preimage of any compact neighborhood of C. Indeed, if V is a neighborhood of C, then, by compactness of C (and local compactness of H), there is a compact neighborhood U of e_H such that $UC \subseteq V$. On the other hand, $V \subseteq VC$. Assuming that V is compact, by Fact 3.4(2), both $f^{-1}[UC]$ and $f^{-1}[VC]$ are commensurable with X, and hence so is $f^{-1}[V]$.

3.1. **Topological dynamics of $S_{G,M}(N)$.** Recall that for $N \subseteq C \subseteq \mathfrak{C}$ by $S_M(C)$ we denote the space of complete types over C which are finitely satisfiable in M, and by $S_{G,M}(C)$ the subspace of $S_M(C)$ consisting of all types concentrated on some X^n. For a formula $\varphi(x)$ in L_C (with $N \subseteq C \subseteq \mathfrak{C}$) such that $\varphi(\mathfrak{C}) \subseteq \bar{X}^n$ for some $n \in \mathbb{N}$, we have that $[\varphi(x)] := \{p \in S_M(C) : \varphi(x) \in p\} \subseteq S_{X^n,M}(C)$ is a basic open set in $S_{G,M}(C)$. For any $g \in \bar{G}$, by $\varphi(g^{-1}x)$ [resp. $\varphi(xg^{-1})$] we mean an $L_{C,g}$-formula defining the set $g\varphi(\mathfrak{C})$ [resp. $\varphi(\mathfrak{C})g$]. Note that by the definability of the approximate subgroup X, it is clear that the sets $g\varphi(\mathfrak{C})$ and $\varphi(\mathfrak{C})g$ are indeed definable over C, g. More precisely, explicitly $\varphi(g^{-1}x)$ can be chosen to be $(\exists y)(\varphi(y) \wedge \psi(g, y, x))$, where $\psi(t, y, x)$ defines the graph of the group operation on G restricted to X^n for some n such that $\varphi(\mathfrak{C}) \subseteq \bar{X}^n$ and $g \in \bar{X}^n$.

The goal of this subsection is to extend the classical theory briefly mentioned in Subsection 2.2 to the action of G on the locally compact space $S_{G,M}(N)$ by left translation, that is $g\,\mathrm{tp}(a/N) := \mathrm{tp}(ga/N)$ (note that this is a well-defined action). First of all, this action is by homeomorphisms, because a basis of open sets in $S_{G,M}(N)$ consists of the sets of the form $[\varphi(x)]$ for formulas $\varphi(x)$ in L_N with $\varphi(\mathfrak{C}) \subseteq \bar{X}^n$ for some n, and $g[\varphi(x)] = [\varphi(g^{-1}x)]$ is still a basic open set for any $g \in G$.

Define a binary operation $*$ on $S_{G,M}(N)$ by

$$p * q := \mathrm{tp}(ab/N), \text{ where } b \models q,\ a \models p, \text{ and } \mathrm{tp}(a/N, b) \text{ is a coheir over } M.$$

Lemma 3.5. *$(S_{G,M}(N), *)$ is a left topological semigroup, that is, $*$ is well-defined, associative, and left continuous.*

Proof. The existence of a pair (a, b) as in the definition of $*$ follows from Fact 2.3. The fact that $\mathrm{tp}(ab/N)$ is a coheir over M follows from the fact that the types $\mathrm{tp}(a/N, b)$ and $\mathrm{tp}(b/N)$ are coheirs over M. Now, take pairs (a, b) and (a', b') both as in the definition of $*$. Thus, $b' \equiv_N b$, so, by $|N|^+$-saturation of $\mathfrak{C}$, we can find a'' such that $(a'', b) \equiv_N (a', b')$. Then $\mathrm{tp}(a''/N, b)$ is an extension of p which is a coheir over M. Therefore, $\mathrm{tp}(a''/N, b) = \mathrm{tp}(a/N, b)$ by Fact 2.3. Hence, $\mathrm{tp}(ab/N) = \mathrm{tp}(a''b/N) = \mathrm{tp}(a'b'/N)$. We have proved that $*$ is well-defined.

To check that $*$ is associative, consider any $p, q, r \in S_{G,M}(N)$ and pick $a \models p$, $b \models q$, and $c \models r$ such that both $\mathrm{tp}(b/N, c)$ and $\mathrm{tp}(a/N, b, c)$ are coheirs over M. Then $\mathrm{tp}(a/N, bc)$ is a coheir over M, so $abc \models p * (q * r)$. On the other hand, $\mathrm{tp}(a/N, b)$ and $\mathrm{tp}(ab/N, c)$ are both coheirs over M, so $abc \models (p * q) * r$. Thus, $p * (q * r) = (p * q) * r$.

It remains to show left continuity of $*$. Fix $q \in S_{G,M}(N)$ and pick $b \models q$. Then $b \in \bar{X}^m$ for some m. Consider any basic open set $U = [\varphi(x)] \subseteq S_{X^n,M}(N)$ for some n. The goal is to show that $V := \{p \in S_{G,M}(N) : p * q \in U\}$ is open. It is clear that $V \subseteq S_{X^{n+m},M}(N)$ (see Remark 3.8). By Fact 2.3, the restriction map $r\colon S_{X^{n+m},M}(N, b) \to S_{X^{n+m},M}(N)$ is a homeomorphism. So it is enough to show that $r^{-1}[V]$ is open.

For any a such that $\mathrm{tp}(a/N, b)$ is a coheir over M we have

$$\mathrm{tp}(a/N, b) \in r^{-1}[V] \iff \mathrm{tp}(ab/N) \in U \iff \mathfrak{C} \models \varphi(ab).$$

Therefore, $r^{-1}[V] = [\varphi(xb)]$ is a basic open set in $S_{X^{n+m},M}(N, b)$. □

Note that G naturally embeds as a group into $S_{G,M}(N)$ via $g \mapsto \mathrm{tp}(g/N)$, which we will be using without mentioning. Also, $gp = \mathrm{tp}(g/N) * p$ for all $g \in G$ and $p \in S_{G,M}(N)$.

Remark 3.6. For every n the set X^n is dense in $S_{X^n,M}(N)$, and G is dense in $S_{G,M}(N)$.

Proof. The second part follows from the first. The first part is clear, as for any nonempty basic open set $[\varphi(x)]$ in $S_{X^n,M}(N)$ there is $a \in \varphi(M) \subseteq X^n$. □

For $p \in S_{G,M}(N)$ let $l_p \colon S_{G,M}(N) \to S_{G,M}(N)$ be defined by $l_p(q) := p * q$. Although the next proposition will not be used in the rest of the construction, we include it for completeness of the whole picture.

Proposition 3.7. *The assignment l given by $p \mapsto l_p$ yields an isomorphism between $S_{G,M}(N)$ and the Ellis semigroup $E(S_{G,M}(N))$ defined in the same way as for (compact) flows in Subsection 2.2.*

Proof. The fact that for every $p \in S_{G,M}(N)$ the function l_p belongs to $E(S_{G,M}(N))$ follows from left continuity of $*$ and the observation that $p = \lim g_i$ for some net (g_i) of elements of G which holds by Remark 3.6. The fact that l is a homomorphism follows from associativity of $*$: $l_{p*q}(r) = (p * q) * r = p * (q * r) = l_p(q * r) = (l_p \circ l_q)(r)$.

To show that l is onto, consider any $\eta \in E(S_{G,M}(N))$. Then $\eta = \lim \pi_{g_i}$ for some net (g_i) of elements of G (where $\pi_{g_i}(p) := g_i p$ for $p \in S_{G,M}(N)$). Set $p := \eta(\mathrm{tp}(e/N)) = \lim \pi_{g_i}(\mathrm{tp}(e/N)) = \lim \mathrm{tp}(g_i/N)$. By left continuity of $*$, for every $q \in S_{G,M}(N)$ we have $p * q = \lim \mathrm{tp}(g_i/N) * q = \lim \pi_{g_i}(q) = \eta(q)$. So $\eta = l_p$.

Injectivity of l is clear, because $l_p(\mathrm{tp}(e/N)) = p * \mathrm{tp}(e/N) = p$. Continuity of l follows trivially from left continuity of $*$.

It remains to show that l is an open map. For that it is enough to show that if U is an open subset of some $S_{X^n,M}(N)$, then $l[U]$ is open in $E(S_{G,M}(N))$. Since l is a bijection and $l_p(\mathrm{tp}(e/N)) = p$, we see that for any $\eta \in E(S_{G,M}(N))$ the condition $\eta \in l[U]$ is equivalent to $\eta(\mathrm{tp}(e/N)) \in U$. Thus, $l[U]$ is open. □

The following property of the semigroup operation $*$, which follows immediately from the definition of $*$ and the assumption that X is symmetric, will play an essential role in the rest of the construction.

Remark 3.8. Whenever $q \in S_{X^n,M}(N)$, $r \in S_{X^m,M}(N)$, and $p * q = r$, then $p \in S_{X^{n+m},M}(N)$.

Lemma 3.9. *There exists a left ideal $\mathcal{M}$ of $S_{G,M}(N)$ for which the set $\mathcal{M} \cap S_{X,M}(N)$ is minimal (nonempty).*

Proof. By compactness of $S_{X,M}(N)$ and Zorn's lemma, it is enough to show that for every $s \in S_{X,M}(N)$ the set $(S_{G,M}(N) * s) \cap S_{X,M}(N)$ is closed. By Remark 3.8, $(S_{G,M}(N) * s) \cap S_{X,M}(N) = (S_{X^2,M}(N) * s) \cap S_{X,M}(N)$.

Since $r_s \colon S_{X^2,M}(N) \to S_{X^3,M}(N)$ given by $p \mapsto p * s$ is a continuous map between compact Hausdorff spaces, we get that $S_{X^2,M}(N) * s = r_s[S_{X^2,M}(N)]$ is closed, and so is $(S_{X^2,M}(N) * s) \cap S_{X,M}(N)$. □

Proposition 3.10. *There exists a minimal left ideal in $S_{G,M}(N)$.*

Proof. We can clearly find a left ideal $\mathcal{M}$ as in the conclusion of Lemma 3.9 which is of the form $S_{G,M}(N) * s_0$ for some $s_0 \in S_{X,M}(N)$. We will show that it is minimal. For that take any $s \in \mathcal{M}$. It is enough to show that $(S_{G,M}(N) * s) \cap S_{X,M}(N) \neq \varnothing$ (as then $s_0 \in (S_{G,M}(N) * s) \cap S_{X,M}(N)$ by the choice of $\mathcal{M}$).

We have that $s \in S_{X^n,M}(N)$ for some n; then $s = \mathrm{tp}(b/N)$ for some $b \in \bar{X}^n$.

Claim 1. *$\bar{X} \cdot b^{-1} \cap G \neq \varnothing$.*

Proof. Since X is an approximate subgroup, $X^n \subseteq Xg_1 \cup \cdots \cup Xg_n$ for some $g_1, \ldots, g_k \in G$. Hence, $\bar{X}^n \subseteq \bar{X}g_1 \cup \cdots \cup \bar{X}g_n$, i.e. $\bar{X}^n \subseteq \bar{X}G$. Since X is symmetric, we get that $(\bar{X}^n)^{-1} \subseteq \bar{X}^{-1}G$, so $b^{-1} \in \bar{X}^{-1}G$, that is $\bar{X}b^{-1} \cap G \neq \varnothing$. □(claim)

By this claim, $\bar{X} \cdot b^{-1} \cap G$ extends to an ultrafilter on the Boolean algebra of externally definable subsets of G which is concentrated on X^{n+1}. This ultrafilter corresponds to a unique $\mathrm{tp}(a/N, b)$ finitely satisfiable in M. Then $\mathrm{tp}(a/N) * \mathrm{tp}(b/N) = \mathrm{tp}(ab/N) \in S_{X,M}(N)$, so $(S_{G,M}(N) * s) \cap S_{X,M}(N) \neq \varnothing$. □

Lemma 3.11. *Any minimal left ideal of $S_{G,M}(N)$ is closed and intersects $S_{X,M}(N)$.*

Proof. Let $\mathcal{M}$ be a minimal left ideal of $S_{G,M}(N)$. The proof of Proposition 3.10 shows that any left ideal (in particular $\mathcal{M}$) of $S_{G,M}(N)$ intersects $S_{X,M}(N)$. To show closedness of $\mathcal{M}$, first note that $\mathcal{M} = S_{G,M}(N) * s$ for some $s \in S_{G,M}(N)$. Of course, $s \in S_{X^n,M}(N)$ for some n. By Remark 3.8, for every $m \in \mathbb{N}$, $(S_{G,M}(N) * s) \cap S_{X^m,M}(N) = (S_{X^{n+m},M}(N) * s) \cap S_{X^m,M}(N)$, and the last set is closed by compactness of $S_{X^{n+m},M}(N)$ and left continuity of $*$. □

From now on, we will often skip writing $*$.

Lemma 3.12. *Let $\mathcal{M}$ be an arbitrary minimal left ideal of $S_{G,M}(N)$. Then $J(\mathcal{M}) := \{u \in \mathcal{M} : u^2 = u\}$ is nonempty and $\mathcal{M}$ is the union of all $u\mathcal{M}$ with u ranging over $J(\mathcal{M})$.*

Proof. Consider any $p \in \mathcal{M}$. Then $p \in S_{X^n,M}(N)$ for some n. By minimality of $\mathcal{M}$, the set $P := \{q \in \mathcal{M} : qp = p\}$ is nonempty. Thus, by left continuity of $*$ and Remark 3.8, P is a nonempty closed subsemigroup of $\mathcal{M}$ contained in $S_{X^{2n},M}(M)$, so it is compact. By Zorn's lemma, there exists a minimal closed subsemigroup K of P.

Consider any $u \in K$. We will show that $u^2 = u$. Then, since $u \in P$, we get $p = up = u(up) \in u\mathcal{M}$, so we will be done.

Let $Q := \{q \in K : qu = u\}$. By compactness of K and left continuity of $*$, Ku is a (nonempty) closed subsemigroup of K, so $Ku = K$ as K is minimal. Hence, $Q \neq \varnothing$. Since Q is a closed subsemigroup of K, we get that $Q = K$, in particular $u \in Q$. □

The proofs of the next three lemmas are essentially identical to the proofs in the classical context. We only recall the proof of the first one, as the other two are not needed in our construction. For the proofs in the classical context see [Rze18, Fact A.8].

Lemma 3.13. *For any minimal left ideal $\mathcal{M}$ of $S_{G,M}(N)$ and $u \in J(\mathcal{M})$, the set $u\mathcal{M}$ is a group (with $*$ as group operation).*

Proof. $u\mathcal{M}$ is clearly closed under $*$, $u \in u\mathcal{M}$ is a neutral element in $u\mathcal{M}$, and $*$ is associative. Now, consider any $p \in u\mathcal{M}$. By minimality of $\mathcal{M}$, there is $q \in \mathcal{M}$ with $qp = u$. Then $(uq)p = u^2 = u$. Thus, $u\mathcal{M}$ is a semigroup with left identity and left inverses, and so it is a group. □

Lemma 3.14. *For every minimal left ideal $\mathcal{M}$ of $S_{G,M}(N)$ and any distinct $u, v \in J(\mathcal{M})$, $u\mathcal{M} \cap v\mathcal{M} = \varnothing$.*

Lemma 3.15. *For any minimal left ideals $\mathcal{M}, \mathcal{N}$ of $S_{G,M}(N)$ and $u \in J(\mathcal{M}), v \in J(\mathcal{N})$ the groups $u\mathcal{M}$ and $v\mathcal{N}$ are isomorphic.*

Therefore, the isomorphism type of all these groups $u\mathcal{M}$ (or just any of these groups separately) can be called the *Ellis group* of $S_{G,M}(N)$.

Now, the goal is to equip the Ellis group with a topology, which will be called the τ-topology. We will do it in the same way as in the classical context. Below, for $P \subseteq S_{G,M}(N)$ the closure of Q will be denoted by $\overline{Q}$, while for a subset Q of the Ellis group the closure with respect to the τ-topology will be denoted by $\mathrm{cl}_\tau(Q)$.

Definition 3.16. For any $p \in S_{G,M}(N)$ and $Q \subseteq S_{G,M}(N)$ we define $p \circ Q$ as the set of all $r \in S_{G,M}(N)$ for which there are nets $(g_i)_{i \in I}$ in G and $(q_i)_{i \in I}$ in Q such that $\lim_i g_i = u$ and $\lim_i g_i q_i = r$.

All the easy observations A.25 – A.35 from [Rze18] work with exactly the same proofs for $S_{G,M}(N)$ in place of the Ellis semigroup of a compact flow. The only place where a slight elaboration is needed is the proof of [Rze18, A.25(1)], because it uses compactness to pass to a convergent subnet. So let us prove it in our context.

Remark 3.17. For any $Q \subseteq S_{G,M}(N)$ and $p, r \in S_{G,M}(N)$ we have $(p \circ Q)r = p \circ (Qr)$.

Proof. The inclusion $(\subseteq)$ follows from left continuity of $*$. Namely, consider any $s \in p \circ Q$. Then there are nets $(g_i)_i$ in G and $(q_i)_i$ in Q such that $\lim_i g_i = p$ and $\lim_i g_i q_i = s$. By left continuity of $*$, we get $sr = (\lim_i g_i q_i)r = \lim_i g_i(q_i r) \in p \circ (Qr)$.

For the opposite inclusion consider any $s \in p \circ (Qr)$. Then there are nets $(g_i)_i$ in G and $(q_i r)_i$ in Qr (where $q_i \in Q$) such that $\lim_i g_i = p$ and $\lim g_i q_i r = s$. There are $m, n \in \mathbb{N}$ such that $r \in S_{X^n,M}(N)$ and $s \in S_{X^m,M}(N)$. By Remark 3.8, passing to suitable final segments of the nets in question, we can assume that $g_i q_i \in S_{X^{n+m},M}(N)$ for all i. Since $S_{X^{n+m},M}(N)$ is compact, passing to a subnet, we can assume that $\lim_i g_i q_i$ exists. Then $\lim_i g_i q_i \in p \circ Q$ and, by left continuity of $*$, we get $s = \lim_i g_i q_i r = (\lim_i g_i q_i)r \in (p \circ Q)r$. □

In particular, the counterparts of Facts A.30 and A.32 from [Rze18] yield

Lemma 3.18. *Given a minimal left ideal $\mathcal{M} \trianglelefteq S_{G,M}(N)$ and idempotent $u \in \mathcal{M}$, the operator cl_τ on subsets of $u\mathcal{M}$ given by $\mathrm{cl}_\tau(Q) := (u\mathcal{M}) \cap (u \circ Q) = u(u \circ Q)$ is a closure operator on $u\mathcal{M}$.*

Now, fix a minimal left ideal $\mathcal{M}$ of $S_{G,M}(N)$ and $u \in J(\mathcal{M})$.

Definition 3.19. By the *τ-topology* we mean the topology on the Ellis group $u\mathcal{M}$ given by the closure operator cl_τ from Lemma 3.18.

Fact A.33 of [Rze18] tells us that the τ-topology on $u\mathcal{M}$ is coarser than the subspace topology inherited from $S_{G,M}(N)$. The next lemma (see Fact A.35 of [Rze18]) yields an important connection between limits in both these topologies.

Lemma 3.20. *If $(a_i)_i$ is a net in $u\mathcal{M}$ converging to $a \in \overline{u\mathcal{M}}$, then $(a_i)_i$ converges to ua in the τ-topology.*

The next four technical lemmas will be very useful throughout the paper.

Lemma 3.21. *Let $Q \subseteq u\mathcal{M}$ be contained in a closed subset of $S_{G,M}(N)$ of the form $[\pi(x)] := \{p \in S_{G,M}(N) : \pi(x) \subseteq p\}$, where $\pi(x)$ is a (partial) type with parameters from N. Then, each element $q \in \mathrm{cl}_\tau(Q)$ can be written as $\mathrm{tp}(ab/N)$ for some $a, b \in \bar{G}$ with $a \models u$ and $b \models \pi(x)$.*

Proof. Consider any $q \in \mathrm{cl}_\tau(V)$. Then $q = \lim_i g_i q_i$ for some nets $(g_i)_i$ in G and $(q_i)_i$ in Q with $\lim_i g_i = u$. Note that $u \in S_{X^m,M}(N)$ and $q \in S_{X^n,M}(N)$ for some $m, n \in \mathbb{N}$. Consider any formulas: $\varphi(x) \in u$ with $\varphi(\mathfrak{C}) \subseteq \bar{X}^m$, $\psi(x)$ implied by $\pi(x)$, and $\theta(x) \in q$ with $\theta(\mathfrak{C}) \subseteq \bar{X}^n$, all with parameters from N. Then there is an index i such that $g_i \models \varphi(x)$ and $\theta(x) \in g_i q_i$. Take any $b_i \models q_i$. Then $\mathfrak{C} \models \varphi(g_i) \wedge \psi(b_i) \wedge \theta(g_i b_i)$, which implies that $g_i \in \bar{X}^m$ and $b_i \in \bar{X}^{n+m}$. Hence, by compactness (or rather $|N|^+$-saturation of $\mathfrak{C}$), there are $a \models u$ (so $a \in \bar{X}^m$) and $b \models \pi(x)$ with $b \in \bar{X}^{n+m}$ such that $ab \models q$. □

Lemma 3.22. *Let $V \subseteq u\mathcal{M}$ be τ-closed and contained in $S_{X^n,M}(N)$ for some n. Then V is quasi-compact in the τ-topology.*

Proof. We need to show that any net $(p_i)_{i \in I}$ in V has a τ-convergent subnet. By compactness of $S_{X^n,M}(N)$, the net $(p_i)_{i \in I}$ has a subnet $(q_j)_{j \in J}$ convergent to some $r \in S_{X^n,M}(N)$ in the usual topology on $S_{X^n,M}(N)$. By Lemma 3.20, $\tau\text{-}\lim_j q_j = ur$, and, by τ-closedness of V, $ur \in V$. □

Lemma 3.23. *Let $V \subseteq u\mathcal{M}$ be contained in $S_{X^n,M}(N)$ for some n. Take m such that $u \in S_{X^m,M}(N)$*[3]*. Then $\mathrm{cl}_\tau(V) \subseteq S_{X^{n+m},M}(N)$.*

Proof. Consider any $p \in \mathrm{cl}_\tau(V)$. By Lemma 3.21, $p = \mathrm{tp}(ab/N)$ for some $a \in \bar{X}^m$ and $b \in \bar{X}^n$. Hence, $p \in S_{X^{n+m},M}(N)$. □

Lemma 3.24. *Let $q \in u\mathcal{M}$. Take n, m such that $q \in S_{X^n,M}(N)$ and $u \in S_{X^m,M}(N)$*[3]*. Then q has a τ-open neighborhood V contained in $S_{X^{n+m},M}(N)$.*

Proof. Let

$$P := S_{X^{n+m},M}(N)^c \cap u\mathcal{M},$$

where $S_{X^{n+m},M}(N)^c$ denotes the complement of $S_{X^{n+m},M}(N)$ in $S_{G,M}(N)$.

Claim 1. *$S_{X^n,M}(N) \cap \mathrm{cl}_\tau(P) = \emptyset$.*

Proof. Take any $p \in \mathrm{cl}_\tau(P)$. By Lemma 3.21, $p = \mathrm{tp}(ab/N)$ for some $a \in \bar{X}^m$ and $b \notin \bar{X}^{n+m}$. Therefore, $p \notin S_{X^n,M}(N)$, as required. □(claim)

Let

$$V := u\mathcal{M} \setminus \mathrm{cl}_\tau(P).$$

By Claim 1 and the above choices, $q \in S_{X^n,M}(N) \cap u\mathcal{M} \subseteq V \subseteq S_{X^{n+m},M}(N)$. In particular, V is a τ-open neighborhood of q. □

Definition 3.25. Let us say that a topological space P is *quasi locally compact* if every point $p \in P$ has a neighborhood U whose closure is quasi-compact.

Proposition 3.26. *The Ellis group $u\mathcal{M}$ is a quasi locally compact T_1 space in the τ-topology.*

Proof. The fact that it is T_1 is easy: $\mathrm{cl}_\tau(\{p\}) = u(u \circ \{p\}) = \{u(up)\} = \{p\}$. Quasi local compactness follows from Lemmas 3.22, 3.23, 3.24. Namely, consider any $q \in u\mathcal{M}$. Then $q \in S_{X^n,M}(N)$ for some n. Also, $u \in S_{X^m,M}(N)$ for some m. By Lemma 3.24, q has a τ-open neighborhood V contained in $S_{X^{n+m},M}(N)$. By Lemma 3.23, $\mathrm{cl}_\tau(V) \subseteq S_{X^{n+2m},M}(N)$. Hence, $\mathrm{cl}_\tau(V)$ is quasi-compact by Lemma 3.22. □

Proposition 3.27. *$u\mathcal{M}$ equipped with the τ-topology is a semitopological group, i.e. group operation is separately continuous.*

Proof. The argument from Fact A.36 of [Rze18] works without any changes. □

The proof of Fact A.37 of [Rze18] applies to our context, so we get that all Ellis groups of $S_{G,M}(N)$ (for varying minimal left ideals $\mathcal{M}$ and idempotents $u \in J(\mathcal{M})$) are in fact topologically isomorphic. So the Ellis group of $S_{G,M}(N)$ is a well-defined semitopological group associated with $S_{G,M}(N)$.

Definition 3.28. Define $H(u\mathcal{M})$ as $\bigcap \mathrm{cl}_\tau(V)$ with V ranging over all τ-neighborhoods of u.

Proposition 3.29. *$H(u\mathcal{M})$ is a τ-closed normal subgroup of $u\mathcal{M}$, and $u\mathcal{M}/H(u\mathcal{M})$ is a locally compact (so Hausdorff) topological group.*

Proof. This is an elaboration on the proof of Fact A.40 (so, in fact, Fact A.12) of [Rze18].

Exactly as in the proof of [Rze18, Fact A.12], we get that $H(u\mathcal{M})$ is a τ-closed normal subsemigroup containing u. Hence, for every $h \in H(u\mathcal{M})$ both $hH(u\mathcal{M})$ and $H(u\mathcal{M})h$ are subsemigroups of $H(u\mathcal{M})$.

[3]In Lemma 3.32, we will see that $u \in S_{X^2,M}(N)$.

Claim 1. *For every $h \in H(u\mathcal{M})$, both $hH(u\mathcal{M})$ and $H(u\mathcal{M})h$ contain an idempotent.*

Proof. Fix $h \in H(u\mathcal{M})$ and consider $hH(u\mathcal{M})$ (the case of $H(u\mathcal{M})h$ is analogous). By Lemmas 3.23 and 3.24, there is a τ-neighborhood V of u in $u\mathcal{M}$ such that $\mathrm{cl}_\tau(V) \subseteq S_{X^k,M}(N)$ for some k. Then, by Lemma 3.22, $\mathrm{cl}_\tau(V)$ is quasi-compact. Hence, $H(u\mathcal{M})$ is τ-closed, quasi-compact, and T_1. Therefore, by Proposition 3.27 (which implies that multiplication on the left or on the right by a fixed element is a homeomorphism), $hH(u\mathcal{M})$ is τ-closed, quasi-compact, T_1, and the map $hH(u\mathcal{M}) \to hH(u\mathcal{M})$ given by $s \mapsto ss_0$ is continuous and closed for every $s_0 \in hH(u\mathcal{M})$. Hence, $hH(u\mathcal{M})$ contains an idempotent by Fact 2.6. □(claim)

Since the only idempotent in the group $u\mathcal{M}$ is u, we conclude from the above claim that $H(u\mathcal{M})$ is a subgroup of $u\mathcal{M}$. By Proposition 3.26, $u\mathcal{M}$ is quasi locally compact, and so it is *weakly quasi locally compact* in the sense that every $p \in u\mathcal{M}$ has a quasi-compact neighborhood. This property is easily seen to be preserved under taking group quotients of semitopological groups, so $u\mathcal{M}/H(u\mathcal{M})$ is weakly quasi locally compact.

The last paragraph of the proof of [Rze18, Fact A.12] applies to our context, so $u\mathcal{M}/H(u\mathcal{M})$ is Hausdorff.

By the last two paragraphs, $u\mathcal{M}/H(u\mathcal{M})$ is locally compact. On the other hand, since $u\mathcal{M}$ is a semitopological group, so is $u\mathcal{M}/H(u\mathcal{M})$. Therefore, by Ellis joint continuity theorem [Ell57, Theorem 2], we get that $u\mathcal{M}/H(u\mathcal{M})$ is jointly continuous and inversion is continuous. Thus, $u\mathcal{M}/H(u\mathcal{M})$ is a locally compact topological group. □

3.2. **The main theorem.** Recall that we are in the situation and notation described at the end of Subsection 2.1. Let $\mathcal{M}$ be a minimal left ideal of $S_{G,M}(N)$ and u an idempotent in $\mathcal{M}$. Let $F\colon G \to u\mathcal{M}$ be given by $F(g) := ugu$ and $\hat{F}\colon S_{G,M}(N) \to u\mathcal{M}$ be the extension of F given by $\hat{F}(p) := upu$. Let $f\colon G \to u\mathcal{M}/H(u\mathcal{M})$ be given by $f(g) := ugu/H(u\mathcal{M})$ and $\hat{f}\colon S_{G,M}(N) \to u\mathcal{M}/H(u\mathcal{M})$ be the extension of f given by $\hat{f}(p) := upu/H(u\mathcal{M})$. In particular, $f = \pi F$ where $\pi\colon u\mathcal{M} \to u\mathcal{M}/H(u\mathcal{M})$ is the quotient map.

The group $u\mathcal{M}/H(u\mathcal{M})$ is always equipped with the quotient topology induced by the τ-topology on $u\mathcal{M}$, and cl denotes the closure operator in this quotient topology.

The following sets will play a key role.

$$F_n := \{x_1y_1^{-1}\dots x_ny_n^{-1} : x_i, y_i \in \bar{G} \text{ and } x_i \equiv_M y_i \text{ for all } i \leqslant n\}$$
$$\tilde{F}_n := \{\mathrm{tp}(a/N) \in S_{G,M}(N) : a \in F_n \text{ such that } \mathrm{tp}(a/N) \text{ is a coheir over } M\}$$
$$\tilde{F} := ((\tilde{F}_7 \cap u\mathcal{M})/H(u\mathcal{M}))^{u\mathcal{M}/H(u\mathcal{M})}$$
$$C := \mathrm{cl}(\tilde{F}) \cup \mathrm{cl}(\tilde{F})^{-1}$$

Here is the main result, i.e. our version of Hrushovski's [Hru20, Theorem 4.2].

Theorem 3.30. *The above function f is a generalized definable locally compact model of X with the compact, normal, symmetric error set C defined above, which is witnessed by $l = 2$ (see Definition 3.1). Moreover, $f^{-1}[C] \subseteq X^{30}$ and there is a compact neighborhood U of the neutral element in $u\mathcal{M}/H(u\mathcal{M})$ such that $f^{-1}[U] \subseteq X^{14}$ and $f^{-1}[UC] \subseteq X^{34}$.*

The proof of Theorem 3.30 starts after the proof of Lemma 3.38 below.

Lemma 3.31. *$F_1 = \{xy^{-1} : x, y \in \bar{X}$ with $x \equiv_M y\}$. In particular, $F_n \subseteq \bar{X}^{2n}$ is M-type-definable (that is, the set of realizations of a type over M), and so $\tilde{F}_n \subseteq S_{X^{2n},M}(N)$ is closed.*

Proof. Only ($\subseteq$) requires a proof. Take any $a, b \in \bar{G}$ with $a \equiv_M b$. Then $a \in \bar{X}^n$ for some n. Since $X^n \subseteq XS$ for some finite $S \subseteq G$, we have that $\bar{X}^n \subseteq \bar{X}S$. So $ac \in \bar{X}$ for some $c \in S^{-1}$. As $a \equiv_M b$ and $c \in M$, also $bc \in \bar{X}$ and $ac \equiv_M bc$. So $ab^{-1} = (ac)(bc)^{-1} \in \{xy^{-1} : x, y \in \bar{X}$ with $x \equiv_M y\}$. The rest easily follows. □

Lemma 3.32. *(1)* $u \in \tilde{F}_1 \subseteq S_{X^2,M}(N)$.
(2) If $p \in \tilde{F}_n \cap u\mathcal{M}$, *then* $p^{-1} \in \tilde{F}_{n+1} \cap u\mathcal{M}$.
(3) $\hat{F}^{-1}[S_{X^n,M}(N) \cap u\mathcal{M}] \subseteq S_{X^{n+4},M}(N)$.
(4) $F^{-1}[S_{X^n,M}(N) \cap u\mathcal{M}] \subseteq X^{n+4}$.

Proof. (1) $u^2 = u$ implies that there are a and b realizing u such that $ab \models u$. So $ab \equiv_M b$, hence $a = (ab)b^{-1} \in F_1$. Therefore, $u \in \tilde{F}_1$ which is contained in $S_{X^2,M}(N)$ by Lemma 3.31.

(2) Since $pp^{-1} = u$, there are $a \models p$ and $b \models p^{-1}$ such that $ab \models u$. By assumption, $a \in F_n$, and, by (1), $ab \in F_1$. Therefore, $b \in F_{n+1}$.

(3) Take any $p \in \hat{F}^{-1}[S_{X^n,M}(N)]$, i.e. $upu \in S_{X^n,M}(N)$. Then $abc = d \in \bar{X}^n$ for some $a \models u$, $b \models p$, and $c \models u$. So $b = a^{-1}dc^{-1} \in \bar{X}^2\bar{X}^n\bar{X}^2 = \bar{X}^{n+4}$ by (1). Hence, $p \in S_{X^{n+4},M}(N)$.

(4) Take any $g \in F^{-1}[S_{X^n,M}(N)]$. By (3), $\mathrm{tp}(g/N) \in S_{X^{n+4},M}(N)$, so $g \in \bar{X}^{n+4}$. As $g \in G$, we get $g \in X^{n+4}$. □

Lemma 3.33. *There exists a* τ*-open neighborhood* V *of* u *in* $u\mathcal{M}$ *such that* $V \subseteq S_{X^4,M}(N)$.

Proof. This is a direct consequence of Lemmas 3.32(1) and 3.24. □

Lemma 3.34. *(1)* $(\tilde{F}_7 \cap u\mathcal{M})^{u\mathcal{M}} \subseteq \tilde{F}_8 \cap u\mathcal{M} \subseteq S_{X^{16},M}(N) \cap u\mathcal{M}$.
(2) $\mathrm{cl}_\tau((\tilde{F}_7 \cap u\mathcal{M})^{u\mathcal{M}}) \subseteq \tilde{F}_9 \cap u\mathcal{M} \subseteq S_{X^{18},M}(N) \cap u\mathcal{M}$ *is quasi-compact.*
(3) $\mathrm{cl}(\tilde{F}) = \pi[\mathrm{cl}_\tau((F_7 \cap u\mathcal{M})^{u\mathcal{M}})] \subseteq (\tilde{F}_9 \cap u\mathcal{M})/H(u\mathcal{M})$ *is compact.*
(4) C *is compact, normal, symmetric, and contained in* $(\tilde{F}_{10} \cap u\mathcal{M})/H(u\mathcal{M})$.

Proof. (1) Take $p \in \tilde{F}_7 \cap u\mathcal{M}$ and $q \in u\mathcal{M}$. The goal is to show that $qpq^{-1} \in \tilde{F}_8$ (the last inclusion follows from Lemma 3.31).

By the definition of $*$, we can find $\alpha \models q$, $\beta \models q^{-1}$, and $a_1, b_1, \dots, a_7, b_7$ with $a_i \equiv_M b_i$ for all $i \leqslant 7$ and $\mathrm{tp}(\alpha/N, a_{\leqslant 7}, b_{\leqslant 7}, \beta)$ a coheir over M such that $\alpha(\prod_{i\leqslant 7} a_i b_i^{-1})\beta \models qpq^{-1}$. We have $\alpha(\prod_{i\leqslant 7} a_i b_i^{-1})\beta = (\prod_{i\leqslant 7} a_i^\alpha (b_i^\alpha)^{-1})\alpha\beta$. Now, by Lemma 3.32(1), $\alpha\beta \models qq^{-1} = u \in \tilde{F}_1$, so $\alpha\beta \in F_1$. On the other hand, since $\mathrm{tp}(\alpha/M, a_i, b_i)$ is a coheir over M and $a_i \equiv_M b_i$, by Remark 2.2, we get that $a_i^\alpha \equiv_M b_i^\alpha$. Therefore, $(\prod_{i\leqslant 7} a_i^\alpha (b_i^\alpha)^{-1})\alpha\beta \in F_8$, so $qpq^{-1} \in \tilde{F}_8$.

(2) By Lemma 3.31, the sets F_1 and F_8 are M-type-definable. By Lemma 3.32(1), $u \in \tilde{F}_1$, and, by (1), $(\tilde{F}_7 \cap u\mathcal{M})^{u\mathcal{M}} \subseteq \tilde{F}_8$. Thus, using Lemma 3.21, we get that $\mathrm{cl}_\tau((\tilde{F}_7 \cap u\mathcal{M})^{u\mathcal{M}}) \subseteq \tilde{F}_9$ which is contained in $S_{X^{18},M}(N)$ by Lemma 3.31. Then quasi-compactness follows from Lemma 3.22.

(3) follows from (2) and Hausdorffness of $u\mathcal{M}/H(u\mathcal{M})$.

(4) Compactness follows from (3) and the fact that $u\mathcal{M}/H(u\mathcal{M})$ is a topological group. Normality is immediate, also using that $u\mathcal{M}/H(u\mathcal{M})$ is a topological group. The inclusion $C \subseteq (\tilde{F}_{10} \cap u\mathcal{M})/H(u\mathcal{M})$ follows from (3) and Lemma 3.32(2). □

Lemma 3.35. $H(u\mathcal{M}) \subseteq \tilde{F}_3 \cap u\mathcal{M} \subseteq S_{X^6,M}(N) \cap u\mathcal{M}$.

Proof. The second inclusion is by Lemma 3.31. The first one is essentially contained in the proof of Theorem 0.1(2) of [KP17], but we repeat it here for the reader's convenience.

By Lemma 3.32(1), $u \in \tilde{F}_1$. By Lemma 3.31, F_2 is M-type-definable. So let ρ be the partial type over M defining F_2 and closed under conjunction. Consider any $\varphi(x) \in \rho$. Let

$$V := [\neg\varphi(x)] \cap u\mathcal{M},$$

where $[\neg\varphi(x)]$ is the clopen subset of $S_{G,M}(N)$ consisting of all types containing $\neg\varphi(x)$.

Claim 1. *(1)* $u \notin \mathrm{cl}_\tau(V)$.
(2) $\mathrm{cl}_\tau(u\mathcal{M}\setminus \mathrm{cl}_\tau(V)) \subseteq \mathrm{cl}_\tau(u\mathcal{M}\setminus V) \subseteq \widetilde{F}_3^\varphi$, *where* $\widetilde{F}_3^\varphi := \{\mathrm{tp}(ab/N) \in S_{G,M}(N) : a \in F_1$ *and* $b \models \varphi(x)\}$.

Proof. (1) Suppose for a contradiction that $u \in \mathrm{cl}_\tau(V)$. By Lemma 3.21, there are $a \models u$ and $b \models \neg\varphi(x)$ with $ab \models u$. Then $a \in F_1$ and $ab \in F_1$, so $b \in F_2$, and hence $b \models \varphi(x)$, a contradiction.

(2) We need to check that $\mathrm{cl}_\tau(u\mathcal{M}\backslash V) \subseteq \widetilde{F}_3^\varphi$. Consider any $p \in \mathrm{cl}_\tau(u\mathcal{M}\backslash V)$. As before, there are $a \models u$ and $b \models \varphi(x)$ such that $ab \models p$. Then $a \in F_1$, so $\mathrm{tp}(ab/N) \in \widetilde{F}_3^\varphi$. □(claim)

Notice that $\bigcap_{\varphi(x)\in\rho} \widetilde{F}_3^\varphi = \widetilde{F}_3$. So, by the claim,

$$H(u\mathcal{M}) = \bigcap \{\mathrm{cl}_\tau(U) : U\ \tau\text{-neighborhood of } u\} \subseteq \bigcap_{\varphi(x)\in\rho} \widetilde{F}_3^\varphi \cap u\mathcal{M} = \widetilde{F}_3 \cap u\mathcal{M},$$

which completes the proof. □

Lemma 3.36. *(1) Every compact $K \subseteq u\mathcal{M}/H(u\mathcal{M})$ is contained in $(S_{X^n,M}(N) \cap u\mathcal{M})/H(u\mathcal{M})$ for some $n \in \mathbb{N}$ and $\pi^{-1}[K]$ is quasi-compact.*

(2) Every quasi-compact $K \subseteq u\mathcal{M}$ is contained in $S_{X^n,M}(N)$ for some $n \in \mathbb{N}$.

Proof. (1) Take V from Lemma 3.33. Then $U := \pi[V] \subseteq (S_{X^4,M}(N) \cap u\mathcal{M})/H(u\mathcal{M})$ is open in $u\mathcal{M}/H(u\mathcal{M})$, so K is covered by finitely many translates of U. Since all the translating elements are in some $(S_{X^m,M}(N)\cap u\mathcal{M})/H(u\mathcal{M})$, we get that $K \subseteq (S_{X^{m+4},M}(N)\cap u\mathcal{M})/H(u\mathcal{M})$. So the τ-closed subset $\pi^{-1}[K]$ of $u\mathcal{M}$ is contained in $S_{X^{m+4},M}(N)H(u\mathcal{M})$ which in turn is contained in $S_{X^{m+10},M}(N)$ by Lemma 3.35. So the final paragraph of the proof of Proposition 3.26 shows that $\pi^{-1}[K]$ is quasi-compact.

(2) follows from (1) and Lemma 3.35, as $\pi[K]$ is compact. □

The next two lemmas will be needed only in the proof of definability of f.

Lemma 3.37. *If $p = \mathrm{tp}(a/N)$ and $q = \mathrm{tp}(b/N)$ belong to $u\mathcal{M}$ and $a \in F_n b$, then $p \in (\tilde{F}_{n+2} \cap u\mathcal{M})q$.*

Proof. As $qq^{-1} = u$, we can find $b' \models q^{-1}$ such that $bb' \models u$. So $b' = b^{-1}\alpha$ for some $\alpha \models u$. Then $pq^{-1} = \mathrm{tp}(a''b^{-1}\alpha/N)$ for some $a'' \equiv_N a$. As $a \in F_n b$, we have $a'' \in F_n b''$ for some $b'' \equiv_N b$, i.e. $a'' = cb''$ for some $c \in F_n$. Thus, $pq^{-1} = \mathrm{tp}(cb''b^{-1}\alpha/N)$. Since by Lemma 3.32(1) we know that $\alpha \in F_1$, we conclude that $cb''b^{-1}\alpha \in F_{n+2}$, so $p \in (\tilde{F}_{n+2} \cap u\mathcal{M})q$. □

Lemma 3.38. *$\hat{F}[\overline{\hat{F}^{-1}[K]}] \subseteq (\tilde{F}_7 \cap u\mathcal{M})K$ for every τ-closed, quasi-compact subset K of $u\mathcal{M}$, where $\overline{\hat{F}^{-1}[K]}$ denotes the closure of $\hat{F}^{-1}[K]$ in $S_{G,M}(N)$.*

Proof. Consider any $p \in \overline{\hat{F}^{-1}[K]}$ and $q = \hat{F}(p) = upu$. Then $q = \mathrm{tp}(\alpha a\beta/N)$ for some $\alpha \models u$, $a \models p$, $\beta \models u$. Also, $p = \lim p'_j$ for some net $(p'_j)_j$ from $\hat{F}^{-1}[K]$. Take a net $(g'_k)_k$ in G such that $\lim g'_k = u$.

By Lemma 3.36(2), $K \subseteq S_{X^n,M}(N)$ for some n, so Lemma 3.32(3) implies that $\hat{F}^{-1}[K] \subseteq S_{X^{n+4},M}(N)$. Moreover, since $u \in S_{X^2,M}(N)$, taking an end segment of $(g'_k)_k$, we can assume that $g'_k \in X^2$ for all k. Then all $g'_k u p'_j$ belong to $S_{X^{n+8},M}(N)$. By compactness of $S_{X^{n+8},M}(N)$, we can find subnets $(g_i)_{i\in I}$ of $(g'_k)_k$ and $(p_i)_{i\in I}$ of $(p'_j)_j$ such that $\lim_i g_i u p_i$ exists.

Clearly, $\lim g_i = u$, $\lim p_i = p$, $up_i u \in u\hat{F}^{-1}[K]u = \hat{F}[\hat{F}^{-1}[K]] = K$, and $r := \lim g_i u p_i u = (\lim g_i u p_i)u$ exists. Hence, by Definition 3.16, we get $r \in u \circ K$, so $ur \in u(u \circ K) = \mathrm{cl}_\tau(K) = K$, as K is τ-closed.

Since $ur = u(\lim g_i u p_i)u$, by compactness (or rather $|N|^+$-saturation of $\mathfrak{C}$), we get $ur = \mathrm{tp}(\gamma\delta\epsilon b\beta/N)$ for some γ, δ, ϵ realizing u and $b \models p$ (note that β can be chosen the same as at the beginning of the proof.)

Put $x := \alpha a\beta$ and $y := \gamma\delta\epsilon b\beta$. Then $x = \alpha ab^{-1}\epsilon^{-1}\delta^{-1}\gamma^{-1}y \in F_5 y$, because $a \equiv_M b$ and $\alpha, \epsilon, \delta, \gamma \in F_1$ by Lemma 3.32(1). Therefore, using Lemma 3.37, we get

$$q = \mathrm{tp}(x/N) \in (\tilde{F}_7 \cap u\mathcal{M})\,\mathrm{tp}(y/N) = (\tilde{F}_7 \cap u\mathcal{M})ur.$$

As we observed above that $ur \in K$, we get $q \in (\tilde{F}_7 \cap u\mathcal{M})K$. □

Proof of Theorem 3.30. By Lemma 3.34(4), we already know that C is compact, normal, and symmetric. Let us divide the proof into numbered parts.

(1) *C is an error set of f.*

By normality of C, it is enough to show that $\mathrm{error}_r(f) \subseteq C$. We will show more, namely that $\mathrm{error}_r(f) \subseteq (\tilde{F}_3 \cap u\mathcal{M})/H(u\mathcal{M})$. For that take any $g, h \in G$ and we need to show that $F(h)^{-1}F(g)^{-1}F(gh) \in \tilde{F}_3 \cap u\mathcal{M}$. The left hand side equals $(uhu)^{-1}(ugu)^{-1}ughu = (uhu)^{-1}(ugu)^{-1}ghu$.

Claim 1. $(ugu)^{-1} = \mathrm{tp}(xy^{-1}g^{-1}/N)$ *for some* $x \equiv_N y$*, and similarly for* h *in place of* g*.*

Proof. Let $\alpha \models u$. Then $g\alpha \models gu$. Let $a \models (ugu)^{-1}$ be such that $\mathrm{tp}(a/N, \alpha)$ is a coheir over M. Then $u = (ugu)^{-1}ugu = (ugu)^{-1}gu = \mathrm{tp}(ag\alpha/N)$. Put $x := ag\alpha$ and $y := \alpha$. Then $x \equiv_N y$ (as each of these elements realizes u) and $a = xy^{-1}g^{-1}$. □(claim)

By this claim, we conclude that $F(h)^{-1}F(g)^{-1}F(gh) = \mathrm{tp}(zt^{-1}h^{-1}xy^{-1}g^{-1}gh\alpha/N)$ for some $z \equiv_N t$, $x \equiv_N y$, and $\alpha \models u$. But $zt^{-1}h^{-1}xy^{-1}g^{-1}gh\alpha = zt^{-1}x^{h^{-1}}(y^{h^{-1}})^{-1}\alpha \in F_3$, because $z \equiv_M t$, $x^{h^{-1}} \equiv_M y^{h^{-1}}$ (as $x \equiv_M y$ and $h \in M$), and $\alpha \in F_1$ (by Lemma 3.32(1)). Therefore, $F(h)^{-1}F(g)^{-1}F(gh) \in \tilde{F}_3 \cap u\mathcal{M}$.

(2) *There is a τ-open neighborhood V of u in $u\mathcal{M}$ such that $V \subseteq S_{X^4,M}(N)$. For any such V, $U := \pi[V]$ is an open neighborhood of the neutral element in $u\mathcal{M}/H(u\mathcal{M})$ and $f^{-1}[U] \subseteq X^{14}$. Thus, $f^{-1}[U] \subseteq X^{14}$ also holds for U replaced by any (in particular by a compact) neighborhood of the neutral element in $u\mathcal{M}/H(u\mathcal{M})$ contained in U.*

The existence of V is by Lemma 3.33. Then $U := \pi[V]$ is an open neighborhood of $u/H(u\mathcal{M})$ in $u\mathcal{M}/H(u\mathcal{M})$. By Lemma 3.35, $H(u\mathcal{M}) \subseteq S_{X^6,M}(N)$. So $\pi^{-1}[U] = VH(u\mathcal{M}) \subseteq S_{X^4,M}(N)S_{X^6,M}(N) \subseteq S_{X^{10},M}(N)$. Hence, $f^{-1}[U] = F^{-1}[\pi^{-1}[U]] \subseteq F^{-1}[S_{X^{10},M}(N)]$, and the last preimage is contained in X^{14} by Lemma 3.32(4).

(3) *For every compact $K \subseteq u\mathcal{M}/H(u\mathcal{M})$ there is $k \in \mathbb{N}$ with $f^{-1}[K] \subseteq X^k$.*

Consider any compact $K \subseteq u\mathcal{M}/H(u\mathcal{M})$. By Lemma 3.36(1), $K \subseteq (S_{X^n,M}(N) \cap u\mathcal{M})/H(u\mathcal{M})$ for some n. So

$$f^{-1}[K] = F^{-1}[\pi^{-1}[K]] \subseteq F^{-1}[S_{X^n,M}(N)H(u\mathcal{M})] \subseteq F^{-1}[S_{X^{n+6},M}(N)] \subseteq X^{n+10},$$

where the second inclusion follows from Lemma 3.35 and the last one by Lemma 3.32(4).

(4) *$f[X^i]$ is relatively compact for every $i \in \mathbb{N}$.*

By Remark 3.2, it is enough to show it for $i = 1$. We have

$$f[X] = \pi[F[X]] \subseteq \pi[uS_{X,M}(N)u] \subseteq \pi[S_{X^5,M}(N) \cap u\mathcal{M}] \subseteq \pi[\mathrm{cl}_\tau(S_{X^5,M}(N) \cap u\mathcal{M})],$$

where the second inclusion follows from Lemma 3.32(1). By Lemmas 3.32(1) and 3.23, we have $\mathrm{cl}_\tau(S_{X^5,M}(N) \cap u\mathcal{M}) \subseteq S_{X^7,M}(N)$. Hence, $\mathrm{cl}_\tau(S_{X^5,M}(N) \cap u\mathcal{M})$ is quasi-compact by Lemma 3.22. Thus, $\pi[\mathrm{cl}_\tau(S_{X^5,M}(N) \cap u\mathcal{M})]$ is compact, and so is the closure of $f[X]$.

(5) $f^{-1}[C] \subseteq X^{30}$.

By Lemma 3.34(4), $C \subseteq (\tilde{F}_{10} \cap u\mathcal{M})/H(u\mathcal{M})$. Hence,

$$f^{-1}[C] = F^{-1}[\pi^{-1}[C]] \subseteq F^{-1}[(\tilde{F}_{10} \cap u\mathcal{M})H(u\mathcal{M})] \subseteq F^{-1}[\tilde{F}_{13} \cap u\mathcal{M}] \subseteq F^{-1}[S_{X^{26},M}(N) \cap u\mathcal{M}] \subseteq X^{30},$$

where the second inclusion follows from Lemma 3.35, the third one from Lemma 3.31, and the last one from Lemma 3.32(4).

(6) *For U from (2) we have $f^{-1}[UC] \subseteq X^{34}$. Thus, the same holds for U replaced by any (in particular by a compact) neighborhood of the neutral element in $u\mathcal{M}/H(u\mathcal{M})$ contained in U.*

We have

$$f^{-1}[UC] = F^{-1}[\pi^{-1}[UC]] \subseteq F^{-1}[S_{X^4,M}(N)S_{X^{20},M}(N)S_{X^6,M}(N)] = F^{-1}[S_{X^{30},M}(N)] \subseteq X^{34},$$

where the first inclusion follows from the choice of U and Lemmas 3.34(4) and 3.35, and the last one from Lemma 3.32(4).

(7) *For any compact* $Z, Y \subseteq u\mathcal{M}/H(u\mathcal{M})$ *with* $C^2Y \cap C^2Z = \emptyset$ *there is a formula* $\varphi(x)$ *over* M *such that* $\hat{f}^{-1}[Y] \subseteq [\varphi(x)]$ *and* $\hat{f}^{-1}[Z] \subseteq [\neg\varphi(x)]$*, and so the preimages* $f^{-1}[Y]$ *and* $f^{-1}[Z]$ *are separated by the definable set* $\varphi(M)$*.*

By Lemma 3.36(1), $\pi^{-1}[Y]$ and $\pi^{-1}[Z]$ are quasi-compact and clearly τ-closed. On the other hand, since $(\tilde{F}_7 \cap u\mathcal{M})/H(u\mathcal{M}) \subseteq C$, we have that

$$(\tilde{F}_7 \cap u\mathcal{M})(\tilde{F}_7 \cap u\mathcal{M})\pi^{-1}[Y] \cap (\tilde{F}_7 \cap u\mathcal{M})(\tilde{F}_7 \cap u\mathcal{M})\pi^{-1}[Z] = \emptyset.$$

So the following claim will complete the proof of (7) and of the whole theorem.

Claim 2. *For any* τ*-closed, quasi-compact* $Z, Y \subseteq u\mathcal{M}$ *such that the sets* $(\tilde{F}_7 \cap u\mathcal{M})Y$ *and* $(\tilde{F}_5 \cap u\mathcal{M})(\tilde{F}_7 \cap u\mathcal{M})Z$ *are disjoint the preimages* $\hat{F}^{-1}[Y]$ *and* $\hat{F}^{-1}[Z]$ *can be separated by a clopen* $[\varphi(x)] \subseteq S_{G,M}(N)$ *where* $\varphi(x)$ *is a formula with parameters from* M*.*

Proof. Let $\rho\colon S_{G,M}(N) \to S_G(M)$ be the restriction map. By the definition of the topologies on type spaces, ρ is a continuous map. We claim that it is enough to show

$$(*) \qquad \rho[\overline{\hat{F}^{-1}[Y]}] \cap \rho[\overline{\hat{F}^{-1}[Z]}] = \emptyset,$$

To see that $(*)$ is enough, note that by Lemma 3.36(2) and 3.32(3) both $\overline{\hat{F}^{-1}[Y]}$ and $\overline{\hat{F}^{-1}[Z]}$ are contained in some $S_{X^n,M}(N)$. Hence, by $(*)$ and compactness of $S_{X^n,M}(N)$, $\rho[\overline{\hat{F}^{-1}[Y]}]$ and $\rho[\overline{\hat{F}^{-1}[Z]}]$ are disjoint closed subsets of $S_{X^n}(M)$, and so they can be separated by a basic open set $[\varphi(x)] \subseteq S_{X^n}(M)$ for some formula in L_M. Taking the preimages under ρ, we see that the clopen $[\varphi(x)] \subseteq S_{G,M}(N)$ separates $\hat{F}^{-1}[Y]$ and $\hat{F}^{-1}[Z]$.

Let us prove $(*)$. Suppose it fails, i.e. there are $p \in \overline{\hat{F}^{-1}[Y]}$ and $q \in \overline{\hat{F}^{-1}[Z]}$ such that $\rho(p) = \rho(q)$. So, taking $\alpha \models p$ and $\beta \models q$, we have $\alpha \equiv_M \beta$. Pick them so that $\operatorname{tp}(\alpha/N, \beta)$ is a coheir over M; then $\operatorname{tp}(\alpha, \beta/N)$ is a coheir over M. Next, $\hat{F}(p) = upu = \operatorname{tp}(\gamma_1\alpha\gamma_2/N)$ and $\hat{F}(q) = uqu = \operatorname{tp}(\gamma_1\beta\gamma_2/N)$ for some $\gamma_1, \gamma_2 \models u$ (note that we can choose the same γ_1, γ_2 in both formulas: first we chose $\gamma_2 \models u$ such that $\operatorname{tp}(\alpha, \beta/N, \gamma_2)$ is a coheir over M, and then $\gamma_1 \models u$ such that $\operatorname{tp}(\gamma_1/N, \alpha, \beta, \gamma_2)$ is a coheir over M). Put $x := \gamma_1\alpha\gamma_2$ and $y := \gamma_1\beta\gamma_2$. Using Lemma 3.32(1), we conclude that $xy^{-1} = \gamma_1\alpha\beta^{-1}\gamma_1^{-1} \in F_3$, so $x \in F_3y$. By Lemma 3.37, this implies that $\hat{F}(p) = \operatorname{tp}(x/N) \in (\tilde{F}_5 \cap u\mathcal{M})\operatorname{tp}(y/N) = (\tilde{F}_5 \cap u\mathcal{M})\hat{F}(q)$. On the other hand, by Lemma 3.38, we have $\hat{F}(p) \in (\tilde{F}_7 \cap u\mathcal{M})Y$ and $\hat{F}(q) \in (\tilde{F}_7 \cap u\mathcal{M})Z$. Thus, we conclude that $\hat{F}(p)$ is in the intersection of $(\tilde{F}_7 \cap u\mathcal{M})Y$ and $(\tilde{F}_5 \cap u\mathcal{M})(\tilde{F}_7 \cap u\mathcal{M})Z$, which contradicts the assumption of the claim. □(claim)

□

3.3. **Around the main theorem.** In this subsection, we discuss some improvements or variants of Theorem 3.30.

Concrete numbers in the statement of Theorem 3.30.

In [Hru20, Theorem 4.2], Hrushovski produced a generalized definable locally compact model f of X with an error set C such that $f^{-1}[C] \subseteq X^{12}$, while in our theorem $f^{-1}[C] \subseteq X^{30}$.

The proof of part (1) inside the proof of Theorem 3.30 shows that $\operatorname{error}_r(f) \subseteq (\tilde{F}_3 \cap u\mathcal{M})/H(u\mathcal{M})$. Analogously, one can show that $\operatorname{error}_l(f) \subseteq (\tilde{F}_3 \cap u\mathcal{M})/H(u\mathcal{M})$. Therefore, if we dropped the definability requirement from the definition of generalized definable locally compact model (i.e. item (3) of Definition 3.1), then we could decrease our error set C by taking $\tilde{F} := ((\tilde{F}_3 \cap u\mathcal{M})/H(u\mathcal{M}))^{u\mathcal{M}/H(u\mathcal{M})}$ in place of $\tilde{F} := ((\tilde{F}_7 \cap u\mathcal{M})/H(u\mathcal{M}))^{u\mathcal{M}/H(u\mathcal{M})}$, and setting $C := \operatorname{cl}_\tau(\tilde{F}) \cup \operatorname{cl}_\tau(\tilde{F})^{-1}$ as before. After this modification, our proofs yield

$f^{-1}[C] \subseteq X^{22}$ and $f^{-1}[UC] \subseteq X^{26}$. A question is whether after this modification item (3) of Definition 3.1 still holds for some l (maybe greater than 2). By the proof of part (7) in the proof of Theorem 3.30, it would hold with $l = 4$ if the answer to the second question below was positive.

Question 3.39. *(1) Does $\widetilde{F}_n * \widetilde{F}_m = \widetilde{F}_{n+m}$?*
(2) Does $(\widetilde{F}_n \cap u\mathcal{M}) * (\widetilde{F}_m \cap u\mathcal{M}) = \widetilde{F}_{n+m} \cap u\mathcal{M}$?

And a final question is whether we could use yet smaller C obtained by replacing $\tilde{F} := ((\tilde{F}_7 \cap u\mathcal{M})/H(u\mathcal{M}))^{u\mathcal{M}/H(u\mathcal{M})}$ by $\tilde{F} := ((\tilde{F}_1 \cap u\mathcal{M})/H(u\mathcal{M}))^{u\mathcal{M}/H(u\mathcal{M})}$. For this C our proof would give us $f^{-1}[C] \subseteq X^{18}$.

Definability over X

In [Hru20, Theorem 4.2], Hrushovski obtains separation by two sets definable over X, while we got separation by a set definable over M. However, assuming that our approximate subgroup X is $\varnothing$-definable (i.e. all the X^n's and all the restrictions $\cdot|_{X^n \times X^n}$ are $\varnothing$-definable), it is not difficult to modify our error set C to get separation by a set definable over X, and then we get separation by subsets of some X^n which are definable over X (see Remark 3.3). We explain the necessary modification of C below.

Notice that, by $\varnothing$-definability of the approximate subgroup X, definability over X is equivalent to definability over $G := \langle X \rangle$. Let us modify the definition of C by replacing $F_n := \{x_1 y_1^{-1} \dots x_n y_n^{-1} : x_i, y_i \in \bar{G}$ and $x_i \equiv_M y_i$ for all $i \leqslant n\}$ by $F'_n := \{x_1 y_1^{-1} \dots x_n y_n^{-1} : x_i, y_i \in \bar{G}$ and $x_i \equiv_G y_i$ for all $i \leqslant n\}$. Then $\tilde{F}'_n$, $\tilde{F}'$, and C' are defined using F'_n in the same way as the corresponding objects without primes are defined at the beginning of Subsection 3.2.

We claim that Theorem 3.30 holds with C replaced by C' with the stronger conclusion that for any compact $Z, Y \subseteq u\mathcal{M}/H(u\mathcal{M})$ with $C^2 Y \cap C^2 Z = \varnothing$ the preimages $f^{-1}[Y]$ and $f^{-1}[Z]$ can be separated by an X-definable set.

Since the sets with primes are supersets of the corresponding sets without primes, most of the statements from Subsection 3.2 automatically imply the corresponding statements for sets with primes. However, the proofs of some of the statements require minor adjustments. Let us discuss three such situations.

(1) $F'_1 = \{xy^{-1} : x, y \in \bar{X}$ with $x \equiv_G y\} \subseteq \bar{X}^2$;
(2) $(\tilde{F}'_7 \cap u\mathcal{M})^{u\mathcal{M}} \subseteq \tilde{F}'_8 \cap u\mathcal{M}$;
(3) $(*)$ from the proof of part (7) in the proof of Theorem 3.30 but for $\rho\colon S_{G,M}(N) \to S_G(M)$ replaced by $\rho'\colon S_{G,M}(N) \to S_G(G)$ being the restriction map.

Proof. (1) The proof of Lemma 3.31 adapts, because the set S from that proof is contained in G, and so $c \in G$.

(2) Since all types in $S_{G,M}(N)$ are finitely satisfiable in G, the proof of Lemma 3.34 adapts choosing $\alpha \models q$ so that $\mathrm{tp}(\alpha/N, a_{\leqslant 7}, b_{\leqslant 7}, \beta)$ is a coheir over G.

(3) Replacing ρ by ρ', the proof of $(*)$ works as before (with $\alpha \equiv_G \beta$ in place of $\alpha \equiv_M \beta$). ☐

An error set for $\hat{f}$

Elaborating on the proof of part (1) in the proof of Theorem 3.30, we obtain the following, where $S_{G,M}(N)$ is equipped with its semigroup structure.

Proposition 3.40. *The function $\hat{f}\colon S_{G,M}(N) \to u\mathcal{M}/H(u\mathcal{M})$ (given by $\hat{f}(p) := upu/H(u\mathcal{M})$) is a quasi-homomorphism with $\mathrm{error}_r(\hat{f}) \cup \mathrm{error}_l(\hat{f}) \subseteq (\widetilde{F}_5 \cap u\mathcal{M})/H(u\mathcal{M})$, and so*

$$\hat{C} := \mathrm{cl}\left(((\widetilde{F}_5 \cap u\mathcal{M})/H(u\mathcal{M}))^{u\mathcal{M}/H(u\mathcal{M})}\right) \cup \mathrm{cl}\left(((\widetilde{F}_5 \cap u\mathcal{M})/H(u\mathcal{M}))^{u\mathcal{M}/H(u\mathcal{M})}\right)^{-1}$$

is a compact, normal, symmetric error set of $\hat{f}$*.*

Proof. We will only explain how to prove that $\mathrm{error}_r(\hat{f}) \subseteq (\widetilde{F}_5 \cap u\mathcal{M})/H(u\mathcal{M})$. The proof that $\mathrm{error}_l(\hat{f}) \subseteq (\widetilde{F}_5 \cap u\mathcal{M})/H(u\mathcal{M})$ is similar. The rest follows as at the beginning of the proof of Theorem 3.30, using an obvious variant of Lemma 3.34(4).

We need to show that $(uqu)^{-1}(upu)^{-1}pqu \subseteq \tilde{F}_5$ for all $p, q \in S_{G,M}(N)$. We have $pqu = \mathrm{tp}(g'h'\alpha/N)$ for some $g' \models p$, $h' \models q$, and $\alpha \models u$. An obvious extension of Claim 1 in the proof of Theorem 3.30 yields

$$(upu)^{-1} = \mathrm{tp}(xy^{-1}g^{-1}/N) \text{ for some } x \equiv_N y \text{ and } g \models p,$$
$$(uqu)^{-1} = \mathrm{tp}(zt^{-1}h^{-1}/N) \text{ for some } z \equiv_N t \text{ and } h \models q.$$

Looking at the proof of the aforementioned Claim 1, we can choose all the above data so that $\mathrm{tp}(t/N, x, y, g, g', h', \alpha)$, $\mathrm{tp}(h/N, t, x, y, g, g', h', \alpha)$, $\mathrm{tp}(zt^{-1}h^{-1}/N, x, y, g, g', h', \alpha)$, and $\mathrm{tp}(xy^{-1}g^{-1}/N, g', h', \alpha)$ are all coheirs over M. Then,

$$(uqu)^{-1}(upu)^{-1}pqu = \mathrm{tp}(zt^{-1}h^{-1}xy^{-1}g^{-1}g'h'\alpha/N) =$$
$$\mathrm{tp}(zt^{-1}x^{h^{-1}}(y^{h^{-1}})^{-1}(g^{h^{-1}})^{-1}g'^{h^{-1}}h^{-1}h'\alpha/N) \in \tilde{F}_5,$$

because $z \equiv_M t$, $x^{h^{-1}} \equiv_M y^{h^{-1}}$ (as $x \equiv_M y$ and $\mathrm{tp}(h/M, x, y)$ is a coheir over M), $g^{h^{-1}} \equiv_M g'^{h^{-1}}$ (as $g \equiv_M g'$ and $\mathrm{tp}(h/M, g, g')$ is a coheir over M), $h \equiv_M h'$, and $\alpha \in F_1$. □

In fact, all the steps of the proof of Theorem 3.30 can be stated for $\hat{f}$ in place of f and $S_{X,M}(N)$ in place of X (step (7) was already stated in this way to be able to apply it directly in Section 4). The proofs are essentially the same, using Lemma 3.32(3) instead of Lemma 3.32(4). Of course, in item (7) we have to work with the same C as in Theorem 3.30 (which clearly contains $\hat{C}$ defined above). So this shows that $\hat{f}$ is something that we could call a "generalized definable locally compact model of $S_{X,M}(N)$".

Expressing the proof in terms of Boolean algebras

The point of this short subsection is to indicate that the above construction could be also performed in two other (related) contexts – a more set-theoretic and a more combinatorial one – which are briefly outlined in the next two paragraphs.

Suppose X is an abstract (rather than definable) approximate subgroup X and one is interested in finding a generalized locally compact model of X. Taking $M := G = \langle X \rangle$ equipped with the full structure, $S_{G,M}(N)$ becomes the subspace of βG (the space of ultrafilters on the Boolean algebra of all subsets of G) which consists of the ultrafilters concentrated on some X^n (for varying n). So no model theory is involved in those objects. In this situation, one should be able to completely eliminate model theory from our construction of the generalized locally compact model by using the classical formula for the convolution product on βG (recalled in the last paragraph of Subsection 2.2) in place of the one in terms of realizations of types, but we find it more technical and less intuitive. And this context does not give any meaningful definability property of the resulting generalized locally compact model.

After the first author's conference talk on Theorem 3.30, Sergei Starchenko suggested that it could be interesting to modify our construction of the generalized definable locally compact model of a definable approximate subgroup X by replacing the Boolean algebra generated by externally definable subsets of G by a smaller (or even smallest possible) Boolean algebra which does not refer to model theory; then ultrafilters on this algebra would be used in place of complete external types. One should be able to realize this suggestion by using Newelski's work on d-closed G-algebras [New14]. Namely, Newelski showed that whenever $\mathcal{A}$ is a d-closed G-algebra of subsets of G, then there is an explicitly given semigroup operation $*$ on the Stone space $S(\mathcal{A})$ which extends the action of G and is left continuous. Now, in our situation of an approximate subgroup X and $G := \langle X \rangle$, take $\mathcal{A}$ to be the d-closure (in the sense of [New14]) of the G-algebra generated by all left translates of the sets $X, X^2, X^3, \dots$. Let $S_G(\mathcal{A})$ be the subflow of $S(\mathcal{A})$ which consists of the ultrafilters containing one of the X^n's (for varying n). Then the above semigroup operation $*$ on $S(\mathcal{A})$ restricts to a left continuous semigroup operation on $S_G(\mathcal{A})$, and $S_G(\mathcal{A})$ is locally compact. One should be able to adapt the theory developed in this paper for $S_{G,M}(N) = S_{G,\text{ext}}(M)$ to $S_G(\mathcal{A})$; in particular, to state and prove a suitable variant of Theorem 3.30 yielding a generalized locally compact model of X which satisfies a version of definability (i.e. item (3) of Definition 3.1) in which separation by a definable set is replaced by separation by a set from the G-algebra $\mathcal{A}$ (or even from the Boolean algebra of subsets of G generated by right translates of $X, X^2, \dots$).

4. Universality

We will prove that the generalized definable locally compact model from Theorem 3.30 is an initial object in a certain category. In particular, this will explain what it means to be a generalized definable locally compact model in terms of factorization through $u\mathcal{M}/H(u\mathcal{M})$ (with the notation from Section 3).

As in Section 3, take the situation and notation as at the end of Subsection 2.1. We introduce the notion of good quasi-homomorphism which will be used to define morphisms in our category.

Definition 4.1. Let H be a locally compact group and S a compact, normal, symmetric subset of H. A *good quasi-homomorphism for* (H, S) is a quasi-homomorphism $h\colon H \to L : T$ for some compact, normal, symmetric subset T of a locally compact group L such that:

(1) for every compact $Y \subseteq L$, $h^{-1}[Y]$ is relatively compact in H;
(2) for every compact $V \subseteq H$, $h[V]$ is relatively compact in L;
(3) $h[S] \subseteq T^n$ for some $n \in \mathbb{N}$;
(4) there is $m \in \mathbb{N}$ such that for any compact $Y, Z \subseteq L$ with $T^m Y \cap T^m Z = \varnothing$, $S\,\mathrm{cl}(h^{-1}[Y]) \cap S\,\mathrm{cl}(h^{-1}[Z]) = \varnothing$.

Remark 4.2. Let (H, S) be as in Definition 4.1 and let $h\colon H \to L : T$ be a good quasi-homomorphism for (H, S). Then:

(1) for every $m \in \mathbb{N}$ there is $n_m \in \mathbb{N}$ with $h[S^m] \subseteq T^{n_m}$;
(2) for every $n \in \mathbb{N}$ there exists $m_n \in \mathbb{N}$ such that for any compact $Y, Z \subseteq L$ with $T^{m_n} Y \cap T^{m_n} Z = \varnothing$ we have $S^n\,\mathrm{cl}(h^{-1}[Y]) \cap S^n\,\mathrm{cl}(h^{-1}[Z]) = \varnothing$.

Proof. (1) follows by an easy induction from item (3) of Definition 4.1 and the assumption that T is an error set of h.

(2) By (1) and the assumption that T is an error set of h, we get that $h[S^{n-1}h^{-1}[Y]] \subseteq T^{k_n} Y$ and $h[S^{n-1}h^{-1}[Z]] \subseteq T^{k_n} Z$ for some k_n. We will show that $m_n := k_n + m$ works for any m satisfying the conclusion of item (4) of Definition 4.1. For that assume that $T^{m_n} Y \cap T^{m_n} Z = \varnothing$. Then $T^m(T^{k_n} Y) \cap T^m(T^{k_n} Z) = \varnothing$ and $T^{k_n} Y$ and $T^{k_n} Z$ are compact. Hence, $S\,\mathrm{cl}(h^{-1}[T^{k_n} Y]) \cap S\,\mathrm{cl}(h^{-1}[T^{k_n} Z]) = \varnothing$ by the choice of m. As the last intersection contains $S^n\,\mathrm{cl}(h^{-1}[Y]) \cap S^n\,\mathrm{cl}(h^{-1}[Z])$ (which follows from the choice of k_n and the obvious inclusion $A\,\mathrm{cl}(B) \subseteq \mathrm{cl}(AB)$), we get that $S^n\,\mathrm{cl}(h^{-1}[Y]) \cap S^n\,\mathrm{cl}(h^{-1}[Z]) = \varnothing$. □

Definition 4.3. Let $f\colon G \to H : S$ and $h\colon G \to L : T$ be definable generalized locally compact models of X. A *morphism* from f to h is a function $\rho\colon H \to L$ which is a good quasi-homomorphism $\rho\colon H \to L : T^k$ for (H, S) and $k \in \mathbb{N}$ is such that $\rho(f(g)) \in h(g)T^k$ for all $g \in G$. The set of morphisms from f to h will be denoted by $\mathrm{Mor}(f, h)$.

Remark 4.4. Morphisms are closed under composition, and so the class of generalized definable locally compact models of X with morphisms forms a category.

Proof. Let $f_1\colon G \to H_1 : S_1$, $f_2\colon G \to H_2 : S_2$, and $f_3\colon G \to H_3 : S_3$ be generalized definable locally compact models, and $\rho \in \mathrm{Mor}(f_1, f_2)$, $\delta \in \mathrm{Mor}(f_2, f_3)$. The goal is to show that there is k such that $\mathrm{error}_r(\delta\rho) = \{\delta(\rho(y))^{-1}\delta(\rho(x))^{-1}\delta(\rho(xy)) : x, y \in G\} \subseteq S_3^k$ and $\delta(\rho(f_1(g))) \in f_3(g)S_3^k$ for all $g \in G$. Indeed, once we prove it, the fact that $\delta\rho\colon H_1 \to H_3 : S_3^k$ is a good quasi-homomorphism for (H_1, S_1) easily follows using Remark 4.2 which we leave as an exercise.

Let k_1 and k_2 be numbers witnessing that ρ and δ are morphisms, respectively, and let n_{k_1} be the number from Remark 4.2(1) applied to the good quasi-homomorphism $\delta\colon H_2 \to H_3 : S_3^{k_2}$ for (H_2, S_2).

Regarding the first part of our goal, we have

$$\delta(\rho(y))^{-1}\delta(\rho(x))^{-1}\delta(\rho(xy)) \in S_3^{k_2}\delta(\rho(x)\rho(y))^{-1}\delta(\rho(xy)) \subseteq S_3^{k_2+2k_2}\delta((\rho(x)\rho(y))^{-1})\delta(\rho(xy)) \subseteq S_3^{k_2+2k_2+k_2}\delta(\rho(y)^{-1}\rho(x)^{-1}\rho(xy)) \subseteq S_3^{4k_2}\delta[S_2^{k_1}] \subseteq S_3^{4k_2+k_2n_{k_1}}.$$

Regarding the second part of our goal, we have

$$\delta(\rho(f_1(g))) \in \delta[f_2(g)S_2^{k_1}] \subseteq \delta(f_2(g))\delta[S_2^{k_1}]S_3^{k_2} \subseteq f_3(g)S_3^{k_2+k_2n_{k_1}+k_2} = f_3(g)S_3^{2k_2+k_2n_{k_1}}.$$

We conclude that $k := 4k_2 + k_2n_{k_1}$ works. □

The obtained category will be later modified to get that the generalized definable locally compact model from Theorem 3.30 is an initial object, as for the above category we will only obtain existence of a morphism and "approximate uniqueness". Before going to these main issues, let us make one more basic observation.

Proposition 4.5. *Let $f\colon G \to H : S$ be a generalized definable locally compact model of X and let $h\colon H \to L : T$ be a good quasi-homomorphism for (H, S). Then there is $n \in \mathbb{N}$ such that $h \circ f\colon G \to L : T^n$ is a generalized definable locally compact model of X and $h \in \mathrm{Mor}(f, h \circ f)$.*

Proof. The fact $h \circ f\colon G \to L : T^n$ is a quasi-homomorphism for some n follows from:

$$h(f(y))^{-1}h(f(x))^{-1}h(f(xy)) \in Th(f(x)f(y))^{-1}h(f(xy)) \subseteq T^3h((f(x)f(y))^{-1})h(f(xy)) \subseteq T^4h(f(y)^{-1}f(x)^{-1}f(xy)) \subseteq T^4h[S] \subseteq T^{4+n},$$

where n is a number witnessing item (3) of Definition 4.1 applied to the good quasi-homomorphism h.

To check item (1) of Definition 3.1 for $h \circ f$, consider any compact $V \subseteq L$. Then $h^{-1}[V]$ is relatively compact, so $(h \circ f)^{-1}[V] = f^{-1}[h^{-1}[V]] \subseteq X^i$ for some i.

To see item (2) of Definition 3.1, note that $\mathrm{cl}(f[X])$ being compact implies that $h[\mathrm{cl}(f[X])]$ is relatively compact, and so $\mathrm{cl}(h[f[X]])$ is compact.

To see that item (3) of Definition 3.1 holds for $h \circ f$, choose l witnessing item (3) of Definition 3.1 for f. Next, choose n so big that T^n is an error set of $h \circ f$ (the existence of such an n was justified at the beginning of the proof) and for any compact $Y, Z \subseteq L$ with $T^nY \cap T^nZ = \emptyset$, $S^l\,\mathrm{cl}(h^{-1}[Y]) \cap S^l\,\mathrm{cl}(h^{-1}[Z]) = \emptyset$ (the existence of such an n is guaranteed by Remark 4.2(2)). Then for any compact $Y, Z \subseteq L$ with $T^nY \cap T^nZ = \emptyset$ we have that $(h \circ f)^{-1}[Y]$ and $(h \circ f)^{-1}[Z]$ can be separated by a definable set.

The fact that $h \in \mathrm{Mor}(f, h \circ f)$ is trivial. □

Theorem 4.6. *(Universality of $f\colon G \to u\mathcal{M}/H(u\mathcal{M})$: existence) Let $f\colon G \to u\mathcal{M}/H(u\mathcal{M}) : C$ be the generalized definable locally compact model of X from Theorem 3.30. Let $h\colon G \to H : S$ be an arbitrary generalized definable locally compact model of X. Then there exists a morphism $\tilde{h} \in \mathrm{Mor}(f, h)$.*

More precisely, define $h_M\colon S_G(M) \to H$ by picking $h_M(p)$ arbitrarily from the set $\bigcap_{\varphi(x)\in p} \mathrm{cl}(h[\varphi(G)])$. Next, define $h^*\colon \bar{G} \to H$ by $h^*(a) := h_M(\mathrm{tp}(a/M))$, $\bar{h}\colon S_{G,M}(N) \to H$ by $\bar{h}(p) := h_M(p|_M)$, and finally $\tilde{h}\colon u\mathcal{M}/H(u\mathcal{M}) \to H$ by picking $\tilde{h}(p/H(u\mathcal{M}))$ arbitrarily *from the set* $\bar{h}[pH(u\mathcal{M})]$*. Then* $h^*, \bar{h}, \widetilde{h}$ *are quasi-homomorphisms with the distinguished error set being* S^n *for some* $n \in \mathbb{N}$ *depending only on* l *from item (3) of Definition 3.1 applied for* h*, and* $\widetilde{h} \in \mathrm{Mor}(f, h)$*.*

Proof. The proof is divided into parts. Items (1), (2), (6) below show that $h^*, \bar{h}, \widetilde{h}$ are quasi-homomorphisms with suitable error sets. Items (6)-(11) show that $\widetilde{h} \in \mathrm{Mor}(f, h)$.

Take $l \in \mathbb{N}$ witnessing item (3) of Definition 3.1 for h.

(1) $h^* : \bar{G} \to H : S^{4l+1}$.

We have

$$h^*(ab) \in \bigcap_{\varphi(x) \in \mathrm{tp}(a/M), \psi(x) \in \mathrm{tp}(b/M)} \overline{h[\varphi(G) \cdot \psi(G)]} \subseteq \bigcap_{\varphi(x), \psi(x)} \overline{h[\varphi(G)]h[\psi(G)]S} =$$

$$\bigcap_{\varphi(x), \psi(x)} \left(\overline{h[\varphi(G)]} \cdot \overline{h[\psi(G)]} \cdot S \right) = \bigcap_{\varphi(x)} \overline{h[\varphi(G)]} \cdot \bigcap_{\psi(x)} \overline{h[\psi(G)]} \cdot S,$$

where $\varphi(x)$ ranges over $\mathrm{tp}(a/M)$ and $\psi(x)$ over $\mathrm{tp}(b/M)$. (The last two equalities follow from compactness of $\overline{h[\varphi(G)]}$, $\overline{h[\psi(G)]}$, and S for sufficiently small $\varphi(G)$ and $\psi(G)$.) Therefore,

$$(*) \qquad h^*(ab) = \alpha\beta\gamma$$

for some $\alpha \in \bigcap_{\varphi(x) \in \mathrm{tp}(a/M)} \overline{h[\varphi(G)]}$, $\beta \in \bigcap_{\psi(x) \in \mathrm{tp}(b/M)} \overline{h[\psi(G)]}$, and $\gamma \in S$.

We claim that

$$(**) \qquad S^l \alpha \cap S^l h^*(a) \neq \varnothing.$$

Suppose not. By compactness of S and local compactness of H, there are compact neighborhoods F_1 of α and F_2 of $h^*(a)$ such that $S^l F_1 \cap S^l F_2 = \varnothing$. Then, by the choice of l, there is a formula $\theta(x) \in L_M$ such that $h^{-1}[F_1] \subseteq \theta(M)$ and $h^{-1}[F_2] \subseteq G \backslash \theta(M)$. If $\theta(x) \in \mathrm{tp}(a/M)$, then $h^*(a) \in \overline{h[\theta(M)]} \subseteq \overline{F_2^c}$ which contradicts the fact that F_2 is a neighborhood of $h^*(a)$. If $\neg\theta(x) \in \mathrm{tp}(a/M)$, then $\alpha \in \overline{h[(\neg\theta(M)]} \subseteq \overline{F_1^c}$ which contradicts the fact that F_1 is a neighborhood of α. So $(**)$ has been proved. Analogously, $S^l \beta \cap S^l h^*(b) \neq \varnothing$. From these two observations and $(*)$, we get $h^*(ab) \in S^{2l} h^*(a) S^{2l} h^*(b) S = S^{4l+1} h^*(a) h^*(b)$. Thus, (1) has been proved.

By (1) and the definition of the semigroup operation $*$ on $S_{G,M}(N)$, we immediately get

(2) $\bar{h} \colon S_{G,M}(N) \to H : S^{4l+1}$.

(3) $\bar{h}(ugu) \in h(g) S^{4(4l+1)}$.

Item (3) follows by the following computation

$$h(g)^{-1}\bar{h}(ugu) \in h(g)^{-1}\bar{h}(u)h(g)\bar{h}(u)S^{2(4l+1)} \subseteq h(g)^{-1}S^{4l+1}h(g)S^{4l+1}S^{2(4l+1)} = S^{4(4l+1)},$$

which uses (2), the fact that $\bar{h} \restriction_G = h$ (which follows from the formulas for $\bar{h}$ and h_M), and equation $u^2 = u$.

(4) *For every compact* $V \subseteq H$*,* $\bar{h}^{-1}[V] \subseteq S_{X^i,M}(N)$ *for some* i*, and so* $\bar{h}^{-1}[V] \cap u\mathcal{M}$ *is relatively quasi-compact in the* τ*-topology.*

Choose a compact neighborhood U of V. We have $h^{-1}[U] \subseteq X^i$ for some i, and we check that i is good. If not, there is $p \in S_{G,M}(N) \backslash S_{X^i,M}(N)$ with $\bar{h}(p) \in V$. Then $\bar{h}(p) \in \overline{h[G \backslash X^i]} \subseteq \overline{U^c}$ which is disjoint from V as U is a neighborhood of V, a contradiction. This implies that $\bar{h}^{-1}[V] \cap u\mathcal{M}$ is relatively quasi-compact in the τ-topology by Lemmas 3.23 and 3.22.

(5) $\bar{h}[S_{X^i,M}(N)]$ *is relatively compact for every* $i \in \mathbb{N}$*.*

This is immediate from the fact that $\bar{h}[S_{X^i,M}(N)] \subseteq \overline{h[X^i]}$ and $\overline{h[X^i]}$ is compact.

In order to show that $\widetilde{h} \in \mathrm{Mor}(f, h)$, we will use the above observations and the following claims.

Claim 1. *(i)* $h^*(a^{-1}) \in h^*(a)^{-1} S^{2(4l+1)}$*.*

(ii) $h^*(ab^{-1}) \in S^{3(4l+1)}$ *for every* $a \equiv_M b$*.*

(iii) $h^*[F_n] \subseteq S^{(4n-1)(4l+1)}$.

Proof. (i) follows from (1). The computation in (ii) is as follows: $h^*(ab^{-1}) \in h^*(a)h^*(b^{-1})S^{4l+1} \subseteq h^*(a)h^*(b)^{-1}S^{3(4l+1)} = h^*(a)h^*(a)^{-1}S^{3(4l+1)} = S^{3(4l+1)}$, where we used (1), (i), and the definition of h^*. Finally, (iii) follows from (ii) and (1). □(claim)

Claim 2. *$\tilde{h}(p/H(u\mathcal{M})) \in \bar{h}(p)S^{12(4l+1)}$ for all $p \in u\mathcal{M}$.*

Proof. By the definition of $\tilde{h}$, $\tilde{h}(p/H(u\mathcal{M})) = \bar{h}(q)$ for some $q \in u\mathcal{M}$ and $r \in H(u\mathcal{M})$ such that $q = pr$. By Lemma 3.35, $H(u\mathcal{M}) \subseteq \tilde{F}_3 \cap u\mathcal{M}$, so, by Claim 1(iii), $\bar{h}[H(u\mathcal{M})] \subseteq S^{11(4l+1)}$. Therefore, by (2), $\tilde{h}(p/H(u\mathcal{M})) = \bar{h}(q) \in \bar{h}(p)\bar{h}(r)S^{4l+1} \subseteq \bar{h}(p)S^{12(4l+1)}$. □(claim)

(6) $\tilde{h} \colon u\mathcal{M}/H(u\mathcal{M}) \to H : S^{37(4l+1)}$

This follows from (2) and Claim 2. Namely, we have: $\tilde{h}(p/H(u\mathcal{M}) \cdot q/H(u\mathcal{M})) = \tilde{h}(pq/H(u\mathcal{M})) \in \bar{h}(pq)S^{12(4l+1)} \subseteq \bar{h}(p)\bar{h}(q)S^{13(4l+1)} \subseteq \tilde{h}(p/H(u\mathcal{M}))\tilde{h}(q/H(u\mathcal{M}))S^{37(4l+1)}$.

(7) $\tilde{h}(f(g)) \in h(g)S^{16(4l+1)}$.

This follows from (3) and Claim 2. Namely: $\tilde{h}(f(g)) = \tilde{h}(ugu/H(u\mathcal{M})) \in \bar{h}(ugu)S^{12(4l+1)} \in h(g)S^{4(4l+1)}S^{12(4l+1)} = h(g)S^{16(4l+1)}$.

(8) *For every compact $V \subseteq H$, $\tilde{h}^{-1}[V]$ is relatively compact.*

This follows from (4). Namely, by the definition of $\tilde{h}$, $\tilde{h}^{-1}[V] \subseteq \pi[\bar{h}^{-1}[V] \cap u\mathcal{M}]$ (where $\pi \colon u\mathcal{M} \to u\mathcal{M}/H(u\mathcal{M})$ is the quotient map). By (4), $\mathrm{cl}_\tau(\bar{h}^{-1}[V] \cap u\mathcal{M})$ is quasi-compact, and so $\mathrm{cl}(\pi[\bar{h}^{-1}[V] \cap u\mathcal{M}]) = \pi[\mathrm{cl}_\tau(\bar{h}^{-1}[V] \cap u\mathcal{M})]$ is compact. Therefore, $\mathrm{cl}(\tilde{h}^{-1}[V])$ being a closed subset of $\mathrm{cl}(\pi[\bar{h}^{-1}[V] \cap u\mathcal{M}])$ is also compact.

(9) *For every compact $V \subseteq u\mathcal{M}/H(u\mathcal{M})$, $\tilde{h}[V]$ is relatively compact in H.*

By the definition of $\tilde{h}$, $\tilde{h}[V] \subseteq \bar{h}[\pi^{-1}[V]]$. By Lemma 3.36, the set $\pi^{-1}[V]$ is contained in some $S_{X^i,M}(N)$. Thus, by (5), $\bar{h}[\pi^{-1}[V]]$ is relatively compact, and so $\tilde{h}[V]$ is relatively compact, too.

(10) $\tilde{h}[C] \subseteq S^{51(4l+1)}$.

By Lemma 3.34(4), $C \subseteq (\tilde{F}_{10} \cap u\mathcal{M})/H(u\mathcal{M})$. Hence, using Claim 2, $\tilde{h}[C] \subseteq \bar{h}[\tilde{F}_{10}]S^{12(4l+1)}$. By Claim 1(iii), $\bar{h}[\tilde{F}_{10}] \subseteq S^{39(4l+1)}$. Therefore, $\tilde{h}[C] \subseteq S^{51(4l+1)}$.

(11) *There is $m \in \mathbb{N}$ such that for any compact $Y, Z \subseteq H$ with $S^mY \cap S^mZ = \varnothing$, $C\,\mathrm{cl}(\tilde{h}^{-1}[Y]) \cap C\,\mathrm{cl}(\tilde{h}^{-1}[Z]) = \varnothing$.*

By Lemma 3.34(4), $C \subseteq (\tilde{F}_{10} \cap u\mathcal{M})/H(u\mathcal{M})$, so it is enough to find m such that

$$(!) \qquad S^mY \cap S^mZ = \varnothing$$

implies

$$(\dagger) \qquad (\tilde{F}_{10} \cap u\mathcal{M})/H(u\mathcal{M})\,\mathrm{cl}(\tilde{h}^{-1}[Y]) \cap (\tilde{F}_{10} \cap u\mathcal{M})/H(u\mathcal{M})\,\mathrm{cl}(\tilde{h}^{-1}[Z]) = \varnothing.$$

Since $\tilde{h}^{-1}[Y] \subseteq \pi[\bar{h}^{-1}[Y] \cap u\mathcal{M}]$, $\tilde{h}^{-1}[Z] \subseteq \pi[\bar{h}^{-1}[Z] \cap u\mathcal{M}]$, and $\bar{h}^{-1}[Y] \cap u\mathcal{M}$ as well as $\bar{h}^{-1}[Z] \cap u\mathcal{M}$ are relatively quasi-compact by (4), we deduce that $\mathrm{cl}(\tilde{h}^{-1}[Y]) \subseteq \mathrm{cl}(\pi[\bar{h}^{-1}[Y] \cap u\mathcal{M}]) = \pi[\mathrm{cl}_\tau(\bar{h}^{-1}[Y] \cap u\mathcal{M})]$ and $\mathrm{cl}(\tilde{h}^{-1}[Z]) \subseteq \mathrm{cl}(\pi[\bar{h}^{-1}[Z] \cap u\mathcal{M}]) = \pi[\mathrm{cl}_\tau(\bar{h}^{-1}[Z] \cap u\mathcal{M})]$. We conclude that in order to show (†), it is enough to show that $(\tilde{F}_{10} \cap u\mathcal{M})H(u\mathcal{M})\,\mathrm{cl}_\tau(\bar{h}^{-1}[Y] \cap u\mathcal{M}) \cap (\tilde{F}_{10} \cap u\mathcal{M})H(u\mathcal{M})\,\mathrm{cl}_\tau(\bar{h}^{-1}[Z] \cap u\mathcal{M}) = \varnothing$. By virtue of Lemma 3.35, this is implied by

$$(\dagger\dagger) \qquad (\tilde{F}_{13} \cap u\mathcal{M})\,\mathrm{cl}_\tau(\bar{h}^{-1}[Y] \cap u\mathcal{M}) \cap (\tilde{F}_{13} \cap u\mathcal{M})\,\mathrm{cl}_\tau(\bar{h}^{-1}[Z] \cap u\mathcal{M}) = \varnothing.$$

We will show that $m := 56(4l+1) + 2l$ works. Suppose for a contradiction that (!) holds for this m, whereas (††) fails.

By (4), $\overline{\bar{h}^{-1}[Y]}$ and $\overline{\bar{h}^{-1}[Z]}$ are contained in some $S_{X^n,M}(N)$ which is closed in $S_{G,M}(N)$. Hence, $\overline{\bar{h}^{-1}[Y]} = [\pi_1(x)]$ and $\overline{\bar{h}^{-1}[Z]} = [\pi_2(x)]$ for some partial types $\pi_1(x)$, $\pi_2(x)$ over N, where $\overline{\bar{h}^{-1}[Y]}$ and $\overline{\bar{h}^{-1}[Z]}$ are closures computed in $S_{G,M}(N)$. Thus, by Lemma 3.21, we get that any element p in the intersection from (††) is of the form

$$\mathrm{tp}(a_1b_1^{-1}\dots a_{13}b_{13}^{-1}\alpha\beta/N) = \mathrm{tp}(a_1'b_1'^{-1}\dots a_{13}'b_{13}'^{-1}\alpha'\beta'/N)$$

for some $a_i, b_i, a_i', b_i', \alpha, \alpha', \beta, \beta' \in \bar{G}$ satisfying $a_i \equiv_M b_i$, $a_i' \equiv_M b_i'$, $\alpha \models u$, $\alpha' \models u$, $\mathrm{tp}(\beta/N) \in \overline{\bar{h}^{-1}[Y]}$, $\mathrm{tp}(\beta'/N) \in \overline{\bar{h}^{-1}[Z]}$. Pick such an element p. By Lemma 3.32(1), it equals

$$\mathrm{tp}(a_1b_1^{-1}\dots a_{14}b_{14}^{-1}\beta/N) = \mathrm{tp}(a_1'b_1'^{-1}\dots a_{14}'b_{14}'^{-1}\beta'/N)$$

for some $a_{14}, b_{14}, a_{14}', b_{14}' \in \bar{G}$ with $a_{14} \equiv_M b_{14}$ and $a_{14}' \equiv_M b_{14}'$.

Claim 3. *$h^*(\beta) \in S^{2l}Y$ and $h^*(\beta') \in S^{2l}Z$.*

Proof. Suppose for a contradiction that $h^*(\beta) \notin S^{2l}Y$. So $S^l h^*(\beta) \cap S^l Y = \emptyset$. Then $S^l U \cap S^l V = \emptyset$ for some compact neighborhoods U of $h^*(\beta)$ and V of Y. By the choice of l, there is a formula $\theta(x) \in L_M$ such that $h^{-1}[U] \subseteq \theta(G)$ and $h^{-1}[V] \subseteq G\backslash\theta(G)$.

Case 1. $\theta(x) \in \mathrm{tp}(\beta/M)$. Since $\mathrm{tp}(\beta/N) \in \overline{\bar{h}^{-1}[Y]}$, there is $q \in [\theta(x)] \cap \bar{h}^{-1}[Y]$. Then $\bar{h}(q) \in \overline{h[\theta(G)]} \cap Y$. On the other hand, $h[\theta(G)] \subseteq V^c$, which implies that $\overline{h[\theta(G)]} \cap Y = \emptyset$, because V is a neighborhood of Y. This is a contradiction.

Case 2. $\neg\theta(x) \in \mathrm{tp}(\beta/M)$. Then $h^*(\beta) \in \overline{h[G\backslash\theta(G)]}$. On the other hand, $h[G\backslash\theta(G)] \subseteq U^c$, which implies that $h^*(\beta) \notin \overline{h[G\backslash\theta(G)]}$, because U is a neighborhood of $h^*(\beta)$. This is a contradiction. □(claim)

Using (1), Claim 1(iii), and Claim 3, we get:

$$\bar{h}(p) = h^*\left(\prod_{i=1}^{14} a_ib_i^{-1}\beta\right) \in h^*\left(\prod_{i=1}^{14} a_ib_i^{-1}\right) h^*(\beta)S^{4l+1} \subseteq S^{55(4l+1)}S^{2l}YS^{4l+1} = S^{56(4l+1)+2l}Y.$$

Similarly, $\bar{h}(p) \in S^{56(4l+1)+2l}Z$. Hence, $S^{56(4l+1)+2l}Y \cap S^{56(4l+1)+2l}Z \neq \emptyset$, which contradicts (!) for $m := 56(4l+1)+2l$. □

The usual notion of definable map from a definable subset D of M to a compact space is explained in terms of a factorization of this map through the type space $S_D(M)$ via a continuous map. The notion of definability in item (3) of Definition 3.1 is less obvious. The next corollary explains it using a factorization through $u\mathcal{M}/H(u\mathcal{M})$.

Corollary 4.7. *A quasi-homomorphism $h\colon G \to H : S$ with a compact, normal, symmetric subset S of a locally compact group H is a generalized definable locally compact model of X if and only if there exists a good quasi-homomorphism $\tilde{h}\colon u\mathcal{M}/H(u\mathcal{M}) \to H : S^m$ for $(u\mathcal{M}/H(u\mathcal{M}), C)$, for some $m \in \mathbb{N}$ such that $\tilde{h}(f(g)) \in h(g)S^m$ for all $g \in G$ (where $f\colon G \to u\mathcal{M}/H(u\mathcal{M}) : C$ is the generalized definable locally compact model of X from Theorem 3.30).*

Proof. The implication (⇒) follows directly from Theorem 4.6.

(⇐) We check items (1), (2), (3) of Definition 3.1 applied to h.

(1) For any compact $V \subseteq H$ the set S^mV is also compact, and so $\tilde{h}^{-1}[S^mV]$ is relatively compact. Thus, $h^{-1}[V] \subseteq f^{-1}[\tilde{h}^{-1}[S^mV]]$ is contained in some X^i.

(2) We know that $f[X]$ is relatively compact, and so $\tilde{h}[f[X]]$ is relatively compact. Hence, $S^m\tilde{h}[f[X]]$ is relatively compact. Since $h[X] \subseteq S^m\tilde{h}[f[X]]$, we conclude that $h[X]$ is relatively compact, too.

(3) We will show that $l := m + m_2$ works, where m_2 is a number witnessing that Remark 4.2(2) holds for the good quasi-homomorphism $\tilde{h}$. For that take any compact $Y, Z \subseteq H$ with $S^lY \cap S^lZ = \emptyset$. Then $S^{m_2}(S^mY) \cap S^{m_2}(S^mZ) = \emptyset$ and S^mY, S^mZ are compact. So, by the choice of m_2,

$$C^2\,\mathrm{cl}(\tilde{h}^{-1}[S^mY]) \cap C^2\,\mathrm{cl}(\tilde{h}^{-1}[S^mZ]) = \emptyset.$$

Therefore, $f^{-1}[\tilde{h}^{-1}[S^mY]]$ and $f^{-1}[\tilde{h}^{-1}[S^mZ]]$ can be separated by a definable set. Since $h^{-1}[Y] \subseteq f^{-1}[\tilde{h}^{-1}[S^mY]]$ and $h^{-1}[Z] \subseteq f^{-1}[\tilde{h}^{-1}[S^mZ]]$, we conclude that $h^{-1}[Y]$ and $h^{-1}[Z]$ can be separated by a definable set. □

Theorem 4.8. *(Universality of $f\colon G \to u\mathcal{M}/H(u\mathcal{M})$: approximate uniqueness) Take $f\colon G \to u\mathcal{M}/H(u\mathcal{M}) : C$ from Theorem 3.30. Let $h\colon G \to H : S$ be an arbitrary generalized definable locally compact model of X, and $\rho \in \mathrm{Mor}(f, h)$ any morphism. Let $\tilde{h} \in \mathrm{Mor}(f, h)$ be a (non uniquely determined) morphism constructed in Theorem 4.6. Then there is $n \in \mathbb{N}$ (depending only on l in item (3) of Definition 3.1 applied to h, and on k in Definition 4.3 and m_2 in Remark 4.2(2) both applied to ρ) such that $\rho(p/H(u\mathcal{M})) \in \tilde{h}(p/H(u\mathcal{M}))S^n$ for all $p \in u\mathcal{M}$.*

Proof. We will show that $n := 4\max(m_2, k + 12(4l + 1))$ works. Suppose not, i.e. $\rho(p/H(u\mathcal{M})) \notin \tilde{h}(p/H(u\mathcal{M}))S^n$ for some $p \in u\mathcal{M}$. Then $S^{\frac{n}{2}}\rho(p/H(u\mathcal{M})) \cap S^{\frac{n}{2}}\tilde{h}(p/H(u\mathcal{M})) = \emptyset$. So we can find a compact neighborhood V of the neutral element in H such that

$$S^{\frac{n}{2}}\rho(p/H(u\mathcal{M}))V \cap S^{\frac{n}{2}}\tilde{h}(p/H(u\mathcal{M}))V = \emptyset.$$

Put $V' := VS^{\frac{n}{4}}$. Then

$$S^{\frac{n}{4}}\rho(p/H(u\mathcal{M}))V' \cap S^{\frac{n}{4}}\tilde{h}(p/H(u\mathcal{M}))V' = \emptyset,$$

and $\rho(p/H(u\mathcal{M}))V'$ and $\tilde{h}(p/H(u\mathcal{M}))V'$ are compact sets. Since $n/4 \geqslant m_2$, we get

$$(*) \qquad C^2\,\mathrm{cl}(\rho^{-1}[\rho(p/H(u\mathcal{M}))V']) \cap C^2\,\mathrm{cl}(\rho^{-1}[\tilde{h}(p/H(u\mathcal{M}))V']) = \emptyset.$$

Put $P := \mathrm{cl}(\rho^{-1}[\rho(p/H(u\mathcal{M}))V'])$ and $Q := \mathrm{cl}(\rho^{-1}[\tilde{h}(p/H(u\mathcal{M}))V'])$. By part (7) of the proof of Theorem 3.30, we conclude from $(*)$ that there exists $\theta(x) \in L_M$ such that

$$(**) \qquad \hat{f}^{-1}[P] \subseteq [\theta(x)] \text{ and } \hat{f}^{-1}[Q] \subseteq [\neg\theta(x)].$$

Since $p/H(u\mathcal{M}) \in P$ and $\hat{f}(p) = upu/H(u\mathcal{M}) = p/H(u\mathcal{M})$, we see that $p \in \hat{f}^{-1}[P]$, hence $\theta(x) \in p$, and so $\bar{h}(p) \in \overline{h[\theta(G)]}$ (where $\bar{h}$ is chosen as in Theorem 4.6). By Claim 2 from the proof of Theorem 4.6, we conclude that $\tilde{h}(p/H(u\mathcal{M})) \in \overline{h[\theta(G)]}S^{12(4l+1)}$. So

$$\tilde{h}(p/H(u\mathcal{M})) \in \overline{\rho[f[\theta(G)]]S^k}S^{12(4l+1)} = \overline{\rho[f[\theta(G)]]}S^{k+12(4l+1)} \subseteq \overline{\rho[\hat{f}[[\theta(x)]]]}S^{k+12(4l+1)} \subseteq \overline{\rho[Q^c]}S^{k+12(4l+1)} \subseteq \overline{(\tilde{h}(p/H(u\mathcal{M}))V')^c}S^{k+12(4l+1)} = \overline{(\tilde{h}(p/H(u\mathcal{M}))VS^{\frac{n}{4}})^c}S^{k+12(4l+1)} \subseteq \overline{(\tilde{h}(p/H(u\mathcal{M}))VS^{\frac{n}{4}})^c}S^{\frac{n}{4}},$$

where the first belonging is by the choice of k, the first equality by compacntess of S, the first inclusion is obvious, the second follows by $(**)$, the next one by the definition of Q, the next equality by the definition of V', and the last inclusion since $n/4 \geqslant k + 12(4l + 1)$. Thus, $\tilde{h}(p/H(u\mathcal{M})) \in \overline{(\tilde{h}(p/H(u\mathcal{M}))VS^{\frac{n}{4}})^c}S^{\frac{n}{4}}$, which is impossible, because it implies that $\tilde{h}(p/H(u\mathcal{M}))VS^{\frac{n}{4}} \cap \overline{(\tilde{h}(p/H(u\mathcal{M}))VS^{\frac{n}{4}})^c} \neq \emptyset$. □

To get full uniqueness (i.e. that f is the initial object) we have to modify the notion of morphism.

Definition 4.9. Let $f\colon G \to H : S$ and $h\colon G \to L : T$ be generalized definable locally compact models of X. Let $\rho_1, \rho_1' \in \mathrm{Mor}(f, h)$. We say that ρ_1 and ρ_1' are *equivalent* (symbolically, $\rho_1 \sim \rho_1'$) if for some $l \in \mathbb{N}$, for every $p \in H$ we have $\rho_1'(p) \in \rho_1(p)T^l$.

Remark 4.10. $\sim$ is an equivalence relation on $\mathrm{Mor}(f, h)$.

Proposition 4.11. *If $f_i\colon G \to H_i : S_i$ for $i \in \{1, 2, 3\}$ are generalized definable locally compact models of X and $\rho_1 \sim \rho_1'$ belong to $\mathrm{Mor}(f_1, f_2)$ and $\rho_2 \sim \rho_2'$ belong to $\mathrm{Mor}(f_2, f_3)$, then $\rho_2\rho_1 \sim \rho_2'\rho_1'$. Thus, all generalized definable locally compact models of X with morphisms modulo $\sim$ form a category.*

Proof. Let l_1 and l_2 be witnesses for $\rho_1 \sim \rho_1'$ and $\rho_2 \sim \rho_2'$, that is $\rho_1'(p) \in \rho_1(p)S_2^{l_1}$ and $\rho_2'(q) \in \rho_2(q)S_3^{l_2}$ for all $p \in H_1$ and $q \in H_2$. Let k_2' be a witness that $\rho_2' \in \mathrm{Mor}(f_2, f_3)$, that is $\rho_2' \colon H_2 \to H_3 : S_3^{k_2'}$ and $\rho_2'(f_2(g)) \in f_3(g)S_3^{k_2'}$, and let n_{l_1} be the number from Remark 4.2(1) obtained for ρ_2'. Then, for every $p \in H_1$ we have

$$\rho_2'(\rho_1'(p)) \in \rho_2'[\rho_1(p)S_2^{l_1}] \subseteq \rho_2'(\rho_1(p))\rho_2'[S_2^{l_1}]S_3^{k_2'} \subseteq \rho_2(\rho_1(p))S_3^{l_2}S_3^{k_2'n_{l_1}}S_3^{k_2'} = \rho_2(\rho_1(p))S_3^{k_2'n_{l_1}+l_2+k_2'}.$$

Thus, for $\rho_1 \in \mathrm{Mor}(f_1, f_2)$ and $\rho_2 \in \mathrm{Mor}(f_2, f_3)$ we have a well-defined

$$\rho_1/\!\sim \circ \rho_2/\!\sim := (\rho_1 \circ \rho_2)/\!\sim .$$

So it is clear that the all generalized definable locally compact models of X with morphisms modulo $\sim$ form a category. □

By Theorems 3.30, 4.6, and 4.8, we get the main result of this section.

Corollary 4.12. *The generalized definable locally compact model $f \colon G \to u\mathcal{M}/H(u\mathcal{M}) : C$ from Theorem 3.30 is the initial object in the category from the last proposition.*

We finish with some natural questions which arise in the special case of Theorem 4.6 when $h := f$, where $f \colon G \to u\mathcal{M}/H(u\mathcal{M}) : C$ is from Theorem 3.30. In this special case, the construction described in the second paragraph of Theorem 4.6 yields non uniquely determined functions $\bar{f} \colon S_{G,M}(N) \to u\mathcal{M}/H(u\mathcal{M})$ and $\tilde{f} \colon u\mathcal{M}/H(u\mathcal{M}) \to u\mathcal{M}/H(u\mathcal{M})$ such that $\tilde{f} \in \mathrm{Mor}(f, f)$. On the other hand, clearly $\mathrm{id} \in \mathrm{Mor}(f, f)$. This leads to

Question 4.13.
(1) Can we choose $\tilde{f}$ by the construction in Theorem 4.6 so that $\tilde{f} = \mathrm{id}$?
(2) Can we choose $\bar{f}$ by the construction in Theorem 4.6 so that $\bar{f}|_{u\mathcal{M}} \colon u\mathcal{M} \to u\mathcal{M}/H(u\mathcal{M})$ is the quotient map?
(3) Can we choose $\bar{f}$ by the construction in Theorem 4.6 so that $\bar{f}(p) = \hat{f}(p) := upu/H(u\mathcal{M})$ for all $p \in S_{G,M}(N)$?

By how $\tilde{f}$ is obtained from $\bar{f}$, we see that a positive answer to (2) implies a positive answer to (1). Since $\hat{f}(p) = p/H(u\mathcal{M})$ for all $p \in u\mathcal{M}$, we get that a positive answer to (3) implies a positive answer to (2).

In the next section, the example with X being a definable, generic, symmetric subset of the universal cover $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ of $\mathrm{SL}_2(\mathbb{R})$ will yield a negative answer to (3), but not to (2).

5. Compact case

In this section, we focus on the special case when the definable approximate subgroup X generates a group in finitely many steps. This is equivalent to $G := \langle X \rangle$ being a definable group in which X is a definable, generic, symmetric set (X being *generic* in G means that finitely many left translates of X cover G). Thus, we will consider just this case or, slightly more generally, the case of a definable generic subset X of a definable group G (notice that then $\langle X \rangle$ has finite index in G), which is fundamental in model theory.

In the case when $G = \langle X \rangle$, the group $u\mathcal{M}/H(u\mathcal{M})$ in the generalized definable locally compact model $f \colon G \to u\mathcal{M}/H(u\mathcal{M})$ from Theorem 3.30 is compact, which follows from Lemma 3.22, since $u\mathcal{M} \subseteq S_{X^n,M}(M)$ for some $n \in \mathbb{N}$; in fact, in this case, all the topological dynamics developed in Subsection 3.1 boils down to the classical topological dynamics of the compact flow $S_{G,M}(N)$, so $u\mathcal{M}/H(u\mathcal{M})$ is compact.

In the more general context of X being a definable, generic, symmetric subset of a definable group G, one can also use the compact group $u\mathcal{M}/H(u\mathcal{M})$ computed for the compact G-flow $S_{G,M}(N) := \{p \in S_G(N) : p \text{ finitely satisfiable in } M\}$ and adapting (and even simplifying some parts of) the arguments from Subsection 3.2, we conclude with

Theorem 5.1. *The function $f\colon G \to u\mathcal{M}/H(u\mathcal{M})$ given by $f(g) := ugu/H(u\mathcal{M})$ has the following properties.*

(1) f is a quasi-homomorphism with compact, normal, symmetric error set $C := \mathrm{cl}_\tau(\tilde{F}) \cup \mathrm{cl}_\tau(\tilde{F})^{-1}$, where:

$$F_n := \{x_1 y_1^{-1} \dots x_n y_n^{-1} : x_i, y_i \in \bar{G} \text{ and } x_i \equiv_M y_i \text{ for all } i \leqslant n\},$$
$$\tilde{F}_n := \{\mathrm{tp}(a/N) \in S_{G,M}(N) : a \in F_n\},$$
$$\tilde{F} := ((\tilde{F}_7 \cap u\mathcal{M})/H(u\mathcal{M}))^{u\mathcal{M}/H(u\mathcal{M})}.$$

Moreover, $(\tilde{F}_3 \cap u\mathcal{M})/H(u\mathcal{M})$ is an error set of f.

(2) $f^{-1}[C] \subseteq X^{30}$.

(3) There is a compact neighborhood U of the neutral element in $u\mathcal{M}/H(u\mathcal{M})$ such that $f^{-1}[U] \subseteq X^{14}$ and $f^{-1}[UC] \subseteq X^{34}$.

(4) For any closed $Z, Y \subseteq u\mathcal{M}/H(u\mathcal{M})$ with $C^2Y \cap C^2Z = \varnothing$ the preimages $f^{-1}[Y]$ and $f^{-1}[Z]$ can be separated by a definable set.

Note that if X is definable, generic, but not symmetric, then replacing it by XX^{-1}, we get a definable, generic, and symmetric set, and it is clear how to modify items (2) and (3) in this context. So the assumption that X is symmetric is rather minor.

The proof of Fact 3.4(1) adapts to

Remark 5.2. For every neighborhood U of $u/H(u\mathcal{M})$ the preimage $f^{-1}[UC]$ is generic in G, that is the preimage under f of any neighborhood of C is generic in G.

Theorem 5.1(3) and Remark 5.2 can be thought of as a structural result on definable generic subsets of an arbitrary definable group G. In concrete examples, this can lead to more precise information on generics.

In Subsection 5.1, we will illustrate it by the universal cover $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ of $\mathrm{SL}_2(\mathbb{R})$. Our analysis of $\widetilde{\mathrm{SL}_2(\mathbb{R})}$ also yields a negative answer to item (3) of Question 4.13, and a positive answer to item (2) in the special case of definable generics in $\widetilde{\mathrm{SL}_2(\mathbb{R})}$. Moreover, our analysis confirms a certain weakening of Newelski's conjecture (that we have had in mind for a while) in the special case of $\widetilde{\mathrm{SL}_2(\mathbb{R})}$. So we take the opportunity and state this weakened conjecture below.

Let G a group definable in a structure M. Let $N \succ M$ be $|M|^+$-saturated, and $\mathfrak{C} \succ N$ a monster model. By $\bar{G}$ we denote the interpretation of G in $\mathfrak{C}$. Let $u\mathcal{M}$ be the Ellis group of the flow $(G, S_{G,M}(N))$, and let $\bar{G}^{00}_M$ be the smallest type-definable over M subgroup of $\bar{G}$ which has bounded index. Newelski's conjecture says that the group epimorphism $\theta\colon u\mathcal{M} \to \bar{G}/\bar{G}^{00}_M$ given by $\theta(\mathrm{tp}(a/N)) := a/\bar{G}^{00}_M$ is an isomorphism under suitable assumptions on tameness of the ambient theory [New09]. In [GPP15], the conjecture was refuted for $G := \mathrm{SL}_2(\mathbb{R})$ treated as a group definable in $M := (\mathbb{R}, +, \cdot)$, where the Ellis group turned out to be $\mathbb{Z}_2$ while $\bar{G}/\bar{G}^{00}_M$ is trivial. On the other hand, the conjecture was confirmed in [CS18] for definably amenable groups definable in NIP theories. In [KP17], we refined Newelski's epimorphism θ obtaining a sequence of epimorphisms

$$u\mathcal{M} \to u\mathcal{M}/H(u\mathcal{M}) \to \bar{G}/\bar{G}^{000}_M \to \bar{G}/\bar{G}^{00}_M,$$

where $\bar{G}^{000}_M$ is the smallest bounded index subgroup of $\bar{G}$ which is invariant under $\mathrm{Aut}(\mathfrak{C}/M)$. This leads to many counter-examples to Newelski's conjecture. Namely, whenever $\bar{G}^{000}_M \neq \bar{G}^{00}_M$, then Newelski's conjecture fails; in fact, we proved that then even $u\mathcal{M}/H(u\mathcal{M}) \to \bar{G}/\bar{G}^{000}_M$ is not an isomorphism. The first example where $\bar{G}^{000}_M \neq \bar{G}^{00}_M$ was found in [CP12]: $G := \widetilde{\mathrm{SL}_2(\mathbb{R})}$ treated as a group definable in the two-sorted structure $M := ((\mathbb{R}, +, \cdot), (\mathbb{Z}, +))$ has this property. Many other examples were then found in [GK15], e.g. the non-abelian free groups equipped with the full structure. Another situation in which Newelski's conjecture fails is when $H(u\mathcal{M})$ is nontrivial, equivalently when $u\mathcal{M}$ is not Hausdorff in the τ-topology. While in general we are able to find examples in which $u\mathcal{M}$ is not Hausdorff, we have not found any such example with NIP. This leads to the following weakening of Newelski's conjecture.

Conjecture 5.3. *If M has NIP, then $u\mathcal{M}$ is Hausdorff.*

From the above discussion, this is true for definably amenable groups definable in NIP theories. It is also true whenever $u\mathcal{M}$ is finite, as the τ-topology is T_1 and so Hausdorff when $u\mathcal{M}$ is finite. In Subsection 5.1, we will confirm it for $G := \widetilde{\mathrm{SL}_2(\mathbb{R})}$ treated as a group definable in the two-sorted structure $M := ((\mathbb{R}, +, \cdot), (\mathbb{Z}, +))$ which clearly has NIP; more precisely, the Ellis group in this case will turn out to be topologically isomorphic to the profinite completion $\hat{\mathbb{Z}}$ of $\mathbb{Z}$.

5.1. **Case study of $\widetilde{\mathrm{SL}_2(\mathbb{R})}$.** From now on, M is the 2-sorted structure with the sorts $(\mathbb{R}, +, \cdot)$ and $(\mathbb{Z}, +)$, $G := \mathrm{SL}_2(\mathbb{R})$, and $\tilde{G} := \widetilde{\mathrm{SL}_2(\mathbb{R})}$. So now $\tilde{G}$ will play the role of G from the above discussion.

Recall from [Asa70] (in particular, see [Asa70, Theorem 2]) that $\tilde{G}$ can be written as $\mathrm{SL}_2(\mathbb{R}) \times \mathbb{Z}$ with the group operation given by $(x, m)(y, n) := (xy, m + n + h(x, y))$, where $h\colon G \times G \to \mathbb{Z}$ is the 2-cocycle defined as follows. For $c, d \in \mathbb{R}$ put

$$c(d) := \begin{cases} c, & \text{if } c \neq 0 \\ d, & \text{if } c = 0. \end{cases}$$

Then for any $\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$, writing $\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \cdot \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix}$, we have

$$h\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\right) := \begin{cases} 1, & \text{if } c_1(d_1) > 0, c_2(d_2) > 0, c_3(d_3) < 0, \\ -1, & \text{if } c_1(d_1) < 0, c_2(d_2) < 0, c_3(d_3) > 0, \\ 0, & \text{otherwise.} \end{cases}$$

From this formula, we see that h is definable in M, and so $\tilde{G}$ is definable in M. It is clear that each set of the form $G \times k\mathbb{Z}$ is a definable generic subset of $\tilde{G}$. Using Theorem 5.1, we will deduce a weak converse.

Proposition 5.4. *For every definable, generic, symmetric subset X of $\tilde{G}$ there exists a nonzero $k \in \mathbb{N}$ such that $G \times k\mathbb{Z} \subseteq X^{640}$.*

Besides Theorem 5.1, we will need a few other ingredients, some of which will be also used in the proof of Proposition 5.13 below. The proof of Proposition 5.4 is given after the proof of Lemma 5.12.

By [CP12, Theorem 3.2], we know that $\tilde{G}$ does not have any definable subgroups of finite index, so for every definable generic subset X of $\tilde{G}$ we have that $\langle X \rangle = \tilde{G}$. Hence, in this situation, Theorem 5.1 is a particular case of Theorem 3.30.

One of the ingredients will be *12-connectedness* of $\mathrm{SL}_2(\mathbb{R})$ which follows from [Gis24, Theorem 7.7], which we will briefly discuss. By a *thick* subset of a group H definable in a structure N we mean a definable symmetric subset Y of H for which there exists a positive $m \in \mathbb{N}$ such that for every $g_1, \dots, g_m \in H$ there are $i < j$ with $g_i^{-1} g_j \in Y$. Note that when Y is a definable generic, then $Y^{-1}Y$ is thick. We will say that H is *n-connected* if for every (definable) thick subset Y of H we have $Y^n = H$. This is equivalent to saying that for P being the intersection of all N-definable thick subsets of $\bar{H} := H(\mathfrak{C})$ (where $\mathfrak{C} \succ N$ is a monster model) we have $P^n = \bar{H}$. The next fact is a particular case of [Gis24, Theorem 7.7], because $\mathrm{SL}_2(\mathbb{R})$ is perfect and satisfies the assumptions of [Gis24, Theorem 7.7] (i.e., it is $\mathbb{R}$-split since the maximal torus consisting of the diagonal matrices is $\mathbb{R}$-isomorphic to the multiplicative group, and it is semisimple in the sense that it does not have nontrivial, connected, normal abelian subgroups).

Fact 5.5. $\mathrm{SL}_2(\mathbb{R})$ *is* 12*-connected in any structure in which* $\mathrm{SL}_2(\mathbb{R})$ *is definable, in particular in* M*.*

Everywhere below $\mathfrak{C} \succ M$ is a monster model, and bars are used to denote the interpretations of various objects in $\mathfrak{C}$.

Corollary 5.6. *Let* X *be a definable, generic, symmetric subset of* $\tilde{G}$*. Then for every* $g \in \bar{G}$ *there exists* $n \in \{-11, \dots, 0, \dots, 11\}$ *such that* $(g, n) \in \bar{X}^{24}$*.*

Proof. By Fact 5.5, $\bar{G} = P^{12}$, where P is the intersection of all M-definable thick subsets of $\bar{G}$. Hence, by [Gis24, Lemma 2.3(1)], $g = \prod_{i=1}^{12} a_i^{-1} b_i$ for some $a_i, b_i \in \bar{G}$ such that $a_i \Theta_M b_i$, meaning that (a_i, b_i) starts an infinite M-indiscernible sequence. Then clearly $((a_i, 0), (b_i, 0))$ starts an infinite M-indiscernible sequence of pairs, i.e.

$$(*) \qquad (a_i, 0)\Theta_M(b_i, 0).$$

The corresponding entries of matrices a_i and b_i have the same sign (because they have the same type), so, using the explicit definition of h recalled above and the formula $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ for matrices in $\mathrm{SL}_2(\mathbb{R})$, one easily checks that $h(a_i^{-1}, b_i) = h(a_i, a_i^{-1})$. Hence,

$$(a_i, 0)^{-1}(b_i, 0) = (a_i^{-1}, -h(a_i, a_i^{-1}))(b_i, 0) = (a_i^{-1} b_i, h(a_i^{-1}, b_i) - h(a_i, a_i^{-1})) = (a_i^{-1} b_i, 0).$$

As $\mathrm{Im}(h) = \{-1, 0, 1\}$, the last thing implies that

$$\prod_{i=1}^{12} (a_i, 0)^{-1}(b_i, 0) \in \left\{ \prod_{i=1}^{12} a_i^{-1} b_i \right\} \times \{-11, \dots, 0, \dots, 11\}.$$

On the other hand, by thickness of $\bar{X}^2$, $(*)$, and [Gis24, Lemma 2.3(1)], we get that $\prod_{i=1}^{12} (a_i, 0)^{-1}(b_i, 0) \in \bar{X}^{24}$. So there exists $n \in \{-11, \dots, 0, \dots, 11\}$ such that $(g, n) \in \bar{X}^{24}$. □

The topological dynamics of the G-flow $S_G(\mathbb{R})$ was worked out in [GPP15], including the computation of the Ellis group which turns out to be $\mathbb{Z}_2$. We will also need the topological dynamics of the $\tilde{G}$-flow $S_{\tilde{G}}(M)$ studied in [Jag15, Section 5].

First of all, it is well-known that all types in $S(M)$ are definable, because this is true for all types in $S(\mathbb{R})$ (first time stated in [Dri86, page 71]) and in $S(\mathbb{Z})$ (by stability of $(\mathbb{Z}, +)$) and there is no interaction between the two sorts of M. So $S_G(M)$ and $S_{\tilde{G}}(M)$ coincide with $S_{G,\mathrm{ext}}(M)$ and $S_{\tilde{G},\mathrm{ext}}(M)$, respectively, and hence the Ellis semigroup operation on these sets is given by $p * q = \mathrm{tp}(ab/M)$ for some/any $b \models q$ and $a \models p$ such that $\mathrm{tp}(a/M, b)$ is a coheir over M.

By [Jag15, Example 5.7] (which actually follows from [GPP15]), the Ellis group of the flow $S_G(M)$ consists of two types q_0, q_1, where $q_0 := \mathrm{tp}(A/M)$ and $q_1 := \mathrm{tp}(-A/M)$ for

$$A := \begin{pmatrix} (1-x)b & (1-x)c - yb^{-1} \\ yb & yc + (1-x)b^{-1} \end{pmatrix}$$

where $b > \mathbb{R}$, $c > \mathrm{dcl}(\mathbb{R}, b)$, x positive infinitesimal, y positive infinitesimal with $(1-x)^2 + y^2 = 1$, and $\mathrm{tp}(x, y/M, b, c)$ coheir over M (which implies that x, y are greater than all infinitesimals in $\mathrm{dcl}(\mathbb{R}, b, c)$). Then q_0 is the neutral element of the Ellis group $\{q_0, q_1\}$, so an idempotent in a minimal left ideal of $S_G(M)$, and hence we will denote q_0 by u_G.

By [Jag15, page 9], the space $S_{\tilde{G}}(M)$ is naturally homeomorphic with $S_G(\mathbb{R}) \times S_{\mathbb{Z}}(\mathbb{Z})$, and the induced semigroup operation is given by

$$(p, q) * (p', q') = (p * p', q + q' + h(p, p')),$$

where $h(p, p') := h(a, a')$ for some/any $a \models p$ and $a' \models p'$ such that $\mathrm{tp}(a/M, a')$ is a coheir over M, $+$ denotes the semigroup operation on $S_{\mathbb{Z}}(\mathbb{Z})$ (which is indeed commutative), and $h(a, a')$ is identified with $\mathrm{tp}(h(a, a')/\mathbb{Z})$. From now on, $(S_{\tilde{G}}(M), *)$ will be identified with $(S_G(\mathbb{R}) \times S_{\mathbb{Z}}(\mathbb{Z}), *)$. Since $*$ uses h, we will denote this semigroup as $S_G(\mathbb{R}) \times_h S_{\mathbb{Z}}(\mathbb{Z})$. As to the semigroup $S_{\mathbb{Z}}(\mathbb{Z})$, we will interchangeably use additive and multiplicative notation.

By [New09, Corollary 1.9] and the discussion in the penultimate paragraph on page 68 in [New09], since $(\mathbb{Z}, +)$ is stable, there is a unique minimal left ideal $\mathcal{M}_{\mathbb{Z}}$ and it consists of the generic types. There is also a unique idempotent $u_{\mathbb{Z}}$ in $\mathcal{M}_{\mathbb{Z}}$ which is the generic type concentrated on the component $\bar{\mathbb{Z}}^0$ (the intersection of all definable subgroups of $\bar{\mathbb{Z}}$ of finite index), and $u_{\mathbb{Z}}\mathcal{M}_{\mathbb{Z}} = \mathcal{M}_{\mathbb{Z}}$. In fact, by basic stable group theory, for each coset of $\bar{\mathbb{Z}}^0$ there is a unique generic type concentrated on it. Thus, there is a natural isomorphism between $u_{\mathbb{Z}}\mathcal{M}_{\mathbb{Z}}$ and $\bar{\mathbb{Z}}/\bar{Z}^0$ given by $\mathrm{tp}(a/\mathbb{Z}) \mapsto a/\bar{\mathbb{Z}}^0$.

By the explicit formula for h and the idempotency of u_G, we get $h(u_G, u_G) = 0$. So [Jag15, Proposition 5.6] yields

Fact 5.7. *Let $\mathcal{M}_G \ni u_G$ be a minimal left ideal of $S_G(\mathbb{R})$. Then:*
 (1) $\mathcal{M}_{\tilde{G}} := \mathcal{M}_G \times \mathcal{M}_{\mathbb{Z}}$ is a minimal left ideal of $S_{\tilde{G}}(M)$;
 (2) $u_{\tilde{G}} := (u_G, u_{\mathbb{Z}})$ is an idempotent in $\mathcal{M}_{\tilde{G}}$;
 (3) The Ellis group $u_{\tilde{G}}\mathcal{M}_{\tilde{G}}$ equals $u_G\mathcal{M}_G \times_h \mathcal{M}_{\mathbb{Z}} = u_G\mathcal{M}_G \times_h u_{\mathbb{Z}}\mathcal{M}_{\mathbb{Z}}$.

$f_1 \colon u_G\mathcal{M}_G \to \mathbb{Z}_2$ given by $q_i \mapsto i$ is clearly an isomorphism. By the discussion before Fact 5.7 (which uses stability of $(\mathbb{Z}, +)$), the natural map $f_2 \colon u_{\mathbb{Z}}\mathcal{M}_{\mathbb{Z}} \to \bar{\mathbb{Z}}/\bar{\mathbb{Z}}^0$ given by $\mathrm{tp}(a/\mathbb{Z}) \mapsto a/\bar{\mathbb{Z}}^0$ is an isomorphism. Note also that $\bar{\mathbb{Z}}/\bar{\mathbb{Z}}^0$ is topologically isomorphic to the profinite completion $\hat{\mathbb{Z}}$. Thus, the following corollary is deduced in [Jag15, Example 5.7].

Corollary 5.8. *The map $(f_1, f_2) \colon u_{\tilde{G}}\mathcal{M}_{\tilde{G}} \to \mathbb{Z}_2 \times \hat{\mathbb{Z}}$ is an isomorphism, with the group operation on $\mathbb{Z}_2 \times \hat{\mathbb{Z}}$ given by $(x, n)(x', n') := (x +_2 x', n + n' - xx')$. The target group is moreover topologically isomorphic to $\hat{\mathbb{Z}}$ via the map $(x, n) \mapsto x - 2n$.*

Our next goal is to show that the isomorphism in Corollary 5.8 is topological. This implies that $u_{\tilde{G}}\mathcal{M}_{\tilde{G}}$ is topologically isomorphic to $\hat{\mathbb{Z}}$, so Hausdorff which confirms Conjecture 5.3 for $\widetilde{\mathrm{SL}_2(\mathbb{R})}$.

As the τ-topology on $u_G\mathcal{M}_G$ is T_1, it is discrete, and so f_1 is a topological isomorphism. Since f_2 is an isomorphism which is continuous by [KP17, Theorem 0.1], we get that it is a topological isomorphism (with $u_{\mathbb{Z}}\mathcal{M}_{\mathbb{Z}}$ equipped with the τ-topology).

The fact that the isomorphism (f_1, f_2) from Corollary 5.8 is topological follows immediately from the above paragraph and the next proposition.

Proposition 5.9. *The τ-topology on $u_G\mathcal{M}_G \times_h u_{\mathbb{Z}}\mathcal{M}_{\mathbb{Z}}$ is the product of the τ-topologies on $u_G\mathcal{M}_G$ and $u_{\mathbb{Z}}\mathcal{M}_{\mathbb{Z}}$.*

Before the proof, let us show a few properties of h which will be used also in the proof of Proposition 5.13.

First, recall that h satisfies the 2-cocyle formula (see [Asa70, Lemma 2]): $h(g_1, g_2) + h(g_1g_2, g_3) = h(g_1, g_2g_3) + h(g_2, g_3)$ for all $g_1, g_2, g_3 \in \mathrm{SL}_2(\mathbb{R})$. So the same holds for $g_1, g_2, g_3 \in \mathrm{SL}_2(\bar{\mathbb{R}})$. This implies the 2-cocyle formula for types: $h(p_1, p_2) + h(p_1p_2, p_3) = h(p_1, p_2p_3) + h(p_2, p_3)$ for all $p_1, p_2, p_3 \in S_G(\mathbb{R})$. In order to see it, just consider $g_1 \models p_1, g_2 \models p_2, g_3 \models p_3$ such that $\mathrm{tp}(g_2/M, g_3)$ and $\mathrm{tp}(g_1/M, g_2, g_3)$ are both finitely satisfiable in M, and apply the 2-cocycle formula for g_1, g_2, g_3.

Lemma 5.10. *(1) $h(p, u_G) = 0$ for all $p \in S_G(M)$.*
 (2) $h(u_G, \mathrm{tp}(g/M)) = 0$ for all $g \in G$.
 (3) $h(u_G, gu_G) = 0$ for all $g \in G$.

Proof. (1) Present u_G as $\mathrm{tp}(A/M)$ for $A := \begin{pmatrix} (1-x)b & (1-x)c - yb^{-1} \\ yb & yc + (1-x)b^{-1} \end{pmatrix}$, where x, y, b, c are as above. Write p as $\mathrm{tp}(B/M)$ for $B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ so that $\mathrm{tp}(B/M, x, y, b, c)$ is a coheir over M. Then $pu_G = \mathrm{tp}(BA/M)$ and $BA = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}$ with $c_{21} := \gamma(1-x)b + \delta yb$. Since $yb > 0$, by the explicit formula for h, the only possibility for $h(p, u_G) \neq 0$ would be the case when $\gamma(\delta) > 0$ and $c_{21}(c_{22}) < 0$. We will show that it never happens, namely $c_{21} > 0$ whenever $\gamma(\delta) > 0$. So assume that $\gamma(\delta) > 0$.

If $\gamma = 0$, then $\delta > 0$, and so $c_{21} = \delta yb > 0$. So assume that $\gamma > 0$. Suppose for a contradiction that $c_{21} \leqslant 0$. Then $\frac{\delta}{\gamma} \leqslant \frac{x-1}{y} < \mathbb{R}$ which contradicts the assumption that $\mathrm{tp}(\gamma, \delta/\mathbb{R}, x, y)$ is finitely satisfiable in $\mathbb{R}$.

(2) The proof is similar and left to the reader.

(3) By the 2-cocycle formula discussed before Lemma 5.10, we have $h(u_G, g) + h(u_G g, u_G) = h(u_G, gu_G) + h(g, u_G)$. So the conclusion follows from (1) and (2). □

Proof of Proposition 5.9. Denote by $\mathcal{T}$ the product of the τ-topologies on $u_G\mathcal{M}_G$ and $u_{\mathbb{Z}}\mathcal{M}_{\mathbb{Z}}$. Our goal is to prove that $\tau = \mathcal{T}$.

($\supseteq$) It is enough to show that any subbasic closed set of the form $F \times u_{\mathbb{Z}}\mathcal{M}_{\mathbb{Z}}$ or $u_G\mathcal{M}_G \times E$ (where $F \subseteq u_G\mathcal{M}_G$ and $E \subseteq u_{\mathbb{Z}}\mathcal{M}_{\mathbb{Z}}$ are τ-closed) is closed in τ.

First, consider $F \times u_{\mathbb{Z}}\mathcal{M}_{\mathbb{Z}}$. Take any $a \in \mathrm{cl}_\tau(F \times u_{\mathbb{Z}}\mathcal{M}_{\mathbb{Z}})$. Then $a = \lim (g_i, n_i)(f_i, a_i)$, where $((g_i, n_i))_i$ is a net from $\tilde{G}$ converging to $u_{\tilde{G}} = (u_G, u_{\mathbb{Z}})$ and $(f_i, a_i) \in F \times u_{\mathbb{Z}}\mathcal{M}_{\mathbb{Z}}$. As $(g_i, n_i)(f_i, a_i) = (g_i f_i, n_i + a_i + h(g_i, f_i)) \in \{g_i f_i\} \times \mathcal{M}_{\mathbb{Z}}$, we get that $a \in \mathrm{cl}_\tau(F) \times \mathcal{M}_{\mathbb{Z}} = F \times u_{\mathbb{Z}}\mathcal{M}_{\mathbb{Z}}$, as required.

Now, consider $u_G\mathcal{M}_G \times E$. Take any $a \in \mathrm{cl}_\tau(u_G\mathcal{M}_G \times E)$. Then $a = \lim (g_i, n_i)(f_i, a_i)$, where $((g_i, n_i))_i$ is a net from $\tilde{G}$ converging to $u_{\tilde{G}} = (u_G, u_{\mathbb{Z}})$ and $(f_i, a_i) \in u_G\mathcal{M}_G \times E$.

Claim 1. *$h(g_i, f_i) = 0$ for i large enough.*

Proof. Since $\lim g_i = u_G$ is the type over M of a matrix with positive left bottom entry, there is i_0 such that the left bottom entry of g_i is positive for all $i > i_0$. Consider any $i > i_0$. If $f_i = u_G$, then $h(g_i, f_i) = 0$ by Lemma 5.10(1). If $f_i = q_1$, then the left bottom entry of any matrix realizing f_i is negative, so $h(g_i, f_i) = 0$ by the explicit formula for h. □(claim)

By this claim, $(g_i, n_i)(f_i, a_i) = (g_i f_i, n_i + a_i + h(g_i, f_i)) = (g_i f_i, n_i + a_i)$ for i large enough. So $a \in \mathrm{cl}_\tau(u_G\mathcal{M}_G) \times \mathrm{cl}_\tau(E) = u_G\mathcal{M}_G \times E$, as required.

($\subseteq$) Consider a τ-closed $A \subseteq u_{\tilde{G}}\mathcal{M}_{\tilde{G}}$. We need to show that it is closed in $\mathcal{T}$. So take any $a = (a_1, a_2) \in \mathrm{cl}_{\mathcal{T}}(A)$. There are are nets $(a_{1,i})_i \subseteq u_G\mathcal{M}_G$ and $(a_{2,i})_i \subseteq u_{\mathbb{Z}}\mathcal{M}_{\mathbb{Z}}$ τ-converging to a_1 and a_2, respectively, with $(a_{1,i}, a_{2,i}) \in A$ for all i. Passing to subnets, we can assume that the nets $(a_{1,i})_i$ and $(a_{2,i})_i$ converge in the usual topologies on $S_G(M)$ and $S_{\mathbb{Z}}(M)$ to some b_1 and b_2, respectively. By Lemma 3.20 and Hausdorffness of $u_G\mathcal{M}_G$ and $u_{\mathbb{Z}}\mathcal{M}_{\mathbb{Z}}$, we get $u_G b_1 = a_1$ and $u_{\mathbb{Z}} + b_2 = a_2$. Approximating u_G by elements of G and $u_{\mathbb{Z}}$ by elements of $\mathbb{Z}$, using left continuity of the semigroup operations and the fact that the actions of G on $S_G(M)$ and of $\mathbb{Z}$ on $S_{\mathbb{Z}}(M)$ are continuous, passing to subnets, we can assume that there are nets $(g_i)_i$ in G and $(n_i)_i$ in $\mathbb{Z}$ converging to u_G and $u_{\mathbb{Z}}$, respectively, and such that $\lim g_i a_{1,i} = a_1$ and $\lim n_i + a_{2,i} = a_2$ (in the usual topology on type spaces). Then $\lim(g_i, n_i) = (u_G, u_{\mathbb{Z}})$. On the other hand, by Claim 1, $h(g_i, a_{1,i}) = 0$ for sufficiently large i's, and so

$$(g_i, n_i)(a_{1,i}, a_{2,i}) = (g_i a_{1,i}, n_i + a_{2,i} + h(g_i, a_{1,i})) = (g_i a_{1,i}, n_i + a_{2,i})$$

for sufficiently large i's. Hence, $\lim(g_i, n_i)(a_{1,i}, a_{2,i}) = (a_1, a_2)$. Therefore, $a \in \mathrm{cl}_\tau(A) = A$. □

The next lemma follows by an elementary matrix computation.

Lemma 5.11. *For every* $\mathrm{tp}(B/M) \in S_G(M)$ *with* $B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, *writing* $u_G = \mathrm{tp}(A/M) = \mathrm{tp}(A'/M)$ *for* $A := \begin{pmatrix} (1-x)b & (1-x)c - yb^{-1} \\ yb & yc + (1-x)b^{-1} \end{pmatrix}$, $A' := \begin{pmatrix} (1-x')b' & (1-x')c' - y'b'^{-1} \\ y'b' & y'c' + (1-x')b'^{-1} \end{pmatrix}$ *with the elements satisfying the requirements described before and such that* $\mathrm{tp}(B/M, A)$ *and* $\mathrm{tp}(A'/M, B, A)$ *are coheirs over* M, *we have that* $u_G \mathrm{tp}(B/M) u_G = \mathrm{tp}(C/M)$ *with the left bottom entry of* C *equal to* $y'b'(\alpha(1-x)b + \beta yb) + (y'c' + (1-x')b'^{-1})(\gamma(1-x)b + \delta yb)$.

Lemma 5.12. *(1) For* $B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in G$ *we have that* $u_G \mathrm{tp}(B/M) u_G = u_G = q_0$ *if* $\gamma > 0$, *and* $u_G \mathrm{tp}(B/M) u_G = q_1$ *if* $\gamma < 0$.

(2) $u_G \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} u_G = u_G$.

(3) $u_G \mathrm{tp}\left(\begin{pmatrix} -1 & 0 \\ \gamma & -1 \end{pmatrix}/M\right) u_G = q_1$ *for all positive infinitesimals* γ.

Proof. First, let $B = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \bar{G}$ with $\gamma > 0$. Pick $x, y, b, c, x', y', b', c'$ and matrices A and A' as in Lemma 5.11, satisfying additionally that $\mathrm{tp}(B/M, x, y, b, c)$ and $\mathrm{tp}(x', y', b', c'/M, x, y, b, c, \alpha, \beta, \gamma, \delta)$ are coheirs over M. Observe that $\gamma(1-x)b + \delta yb > 0$. Indeed, it is clear if $\delta \geqslant 0$. If $\delta < 0$, then it is equivalent to $-\frac{\gamma}{\delta} > \frac{y}{1-x}$ which is true as $\frac{y}{1-x}$ is a positive infinitesimal, $-\frac{\gamma}{\delta} > 0$, and $\mathrm{tp}(\gamma, \delta/M, x, y)$ is a coheir over M. Let d be the left bottom entry of $A'BA$. Hence, by Lemma 5.11, we conclude that $d > 0$ if $\alpha(1-x)b + \beta yb \geqslant 0$. In the case when $\alpha(1-x)b + \beta yb < 0$, we have that $d > 0$ if and only if $\frac{b'}{c'} < (1 + \frac{1-x'}{y'b'c'})(-\frac{\gamma(1-x)+\delta y}{\alpha(1-x)+\beta y}) = (1 + \frac{1-x'}{y'b'c'})(-\frac{\gamma+\delta\frac{y}{1-x}}{\alpha+\beta\frac{y}{1-x}}) =: \zeta$.

(1) Since $u_G = \mathrm{tp}(A/M)$ and $q_1 = \mathrm{tp}(-A/M)$, replacing B by $-B$, we see that it is enough to consider the case when $\gamma > 0$ and to show that then $d > 0$. By the above consideration, this boils down to showing that $\frac{b'}{c'} < \zeta$ if $\alpha(1-x)b + \beta yb < 0$. By the assumption of (1), $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. Thus, $\alpha(1-x)b + \beta yb < 0$ implies that $\alpha < \beta\frac{y}{x-1}$ which is infinitesimal, so $\alpha \leqslant 0$. Now, if $\alpha = 0$, then $\zeta > \mathbb{R}$, so $\zeta > \frac{b'}{c'}$ (as $\frac{b'}{c'}$ is infinitesimal). If $\alpha < 0$, then $\zeta > -\frac{\gamma+\delta\frac{y}{1-x}}{\alpha+\beta\frac{y}{1-x}} > -\frac{\gamma}{2\alpha} > \frac{b'}{c'}$, where the first inequality follows from the fact that $\frac{1-x'}{y'b'c'} > 0$ and $-\frac{\gamma+\delta\frac{y}{1-x}}{\alpha+\beta\frac{y}{1-x}} > 0$, the last one from the fact that $-\frac{\gamma}{2\alpha}$ is a positive real number and b'/c' infinitesimal, and the middle one is easily seen to be equivalent to $-\gamma\alpha > (2\delta\alpha - \gamma\beta)\frac{y}{1-x}$ which is obviously true as $-\gamma\alpha$ is a positive real and $(2\delta\alpha - \gamma\beta)\frac{y}{1-x}$ is infinitesimal.

(2) is a particular case of (1).

(3) It is enough to see that $d \leqslant 0$; equivalently that $\zeta \leqslant \frac{b'}{c'}$. We have $\zeta = (1 + \frac{1-x'}{y'b'c'})(\gamma - \frac{y}{1-x}) < 2(\gamma - \frac{y}{1-x})$ which is a positive infinitesimal (as $\frac{1-x'}{y'b'c'}$, γ and $\frac{y}{1-x}$ are positive infinitesimals, and $\mathrm{tp}(\gamma/M, x, y)$ is a coheir over M). The conclusion follows from the fact that $\frac{b'}{c'}$ is a positive infinitesimal and $\mathrm{tp}(b', c'/M, x, y, \gamma)$ is a coheir over M. ☐

We have now all the tools to prove Proposition 5.4.

Proof of Proposition 5.4. Let $B := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Then $B^2 = -I$ and $B^4 = I$. So, by the explicit formula for the 2-cocycle h, we have $h(B, B) = 1$ and $h(B^2, B^2) = -1$. Hence, working in $\tilde{G}$, we get

$$(*) \qquad (B, 0)^4 = (I, 2h(B, B) + h(B^2, B^2)) = (I, 1).$$

As observed after Corollary 5.8, $u_{\tilde{G}}\mathcal{M}_{\tilde{G}} \cong \hat{\mathbb{Z}}$ is Hausdorff, so $H(u_{\tilde{G}}\mathcal{M}_{\tilde{G}})$ is trivial. By the explicit formula for f_2 and the fact that $f_2(u_{\mathbb{Z}}) = 0$, we also have that $f_2(u_{\mathbb{Z}} + \text{tp}(n/\mathbb{Z}) + u_{\mathbb{Z}}) = n/\bar{\mathbb{Z}}^0$ for $n \in \mathbb{Z}$. Take $f \colon \tilde{G} \to u_{\tilde{G}}\mathcal{M}_{\tilde{G}}$ from Theorem 5.1. Using Lemma 5.10,

$$f((g,n)) := (u_G, u_{\mathbb{Z}})(g,n)(u_G,u_{\mathbb{Z}}) = (u_G g u_G, u_{\mathbb{Z}} + \text{tp}(n/\mathbb{Z}) + u_{\mathbb{Z}} + h(g,u_G) + h(u_G, gu_G)) = (u_G g u_G, u_{\mathbb{Z}} + \text{tp}(n/\mathbb{Z}) + u_{\mathbb{Z}}).$$

Hence, using Theorem 5.1(3), Proposition 5.9 and the topological isomorphism f_2, we get a positive $k \in \mathbb{N}$ such that $\{g \in G : u_G g u_G = u_G\} \times k\mathbb{Z} \subseteq X^{14}$. Thus, by Lemma 5.12(2), $\{B\} \times k\mathbb{Z} \subseteq X^{14}$. Therefore, using $(*)$, we conclude that $\{I\} \times (1 + k\mathbb{Z}) \subseteq X^{14\cdot 4} = X^{56}$, and so, as X is symmetric, $\{I\} \times (\{-11, \dots, 0, \dots, 11\} + k\mathbb{Z}) \subseteq X^{56\cdot 11} = X^{616}$. Using Corollary 5.6, this implies that $G \times k\mathbb{Z} \subseteq X^{616+24} = X^{640}$. □

One could also prove more directly (without using topological dynamics) a version of Proposition 5.4 with a bigger number in place of 640, but we will not do that, as our point was to illustrate by a non-trivial example how Theorem 5.1 leads to a better understanding of generics in definable groups.

Finally, we give a negative answer to Question 4.13(3), and a positive answer to Question 4.13(2) in the particular case when X is a definable, generic, symmetric subset of $\tilde{G}$. First, let us describe the context. Let X be a definable, generic, symmetric subset of $\tilde{G}$. Then X is an approximate subgroup definable in M and, as discussed after Proposition 5.4, $\tilde{G} = \langle X \rangle$. By definability of types in $S(M)$, the flows $S_{\tilde{G},M}(N)$ and $S_{\tilde{G}}(M)$ are identified (as discussed above). We have already proved that $H(u_{\tilde{G}}\mathcal{M}_{\tilde{G}})$ is trivial. Let $f \colon \tilde{G} \to u_{\tilde{G}}\mathcal{M}_{\tilde{G}}$ be the generalized definable locally compact model from Theorem 3.30 and let $\bar{f} = f_M \colon S_{\tilde{G}}(M) \to u_{\tilde{G}}\mathcal{M}_{\tilde{G}}$ be as discussed before Question 4.13. Note that $\bar{f}$ extends f. On the other hand, we have the function $\hat{f} \colon S_{\tilde{G}}(M) \to u_{\tilde{G}}\mathcal{M}_{\tilde{G}}$ given by $\hat{f}(p) := u_{\tilde{G}} p u_{\tilde{G}}$ which also extends f.

Proposition 5.13. *The function $\bar{f}$ is uniquely determined by the construction from Theorem 4.6 and continuous, whereas $\hat{f}$ is not continuous, and so $\bar{f} \neq \hat{f}$. However, $\bar{f}|_{u_{\tilde{G}}\mathcal{M}_{\tilde{G}}} = \hat{f}|_{u_{\tilde{G}}\mathcal{M}_{\tilde{G}}} = \text{id}$.*

Proof. Identifying $u_{\mathbb{Z}}\mathcal{M}_{\mathbb{Z}}$ with $\hat{\mathbb{Z}}$ via f_2, the second displayed computation in the proof of Proposition 5.4 yields

$$f((g,n)) = (u_G g u_G, n).$$

It follows from Lemma 5.11 and definability of types in $S(M)$ that the sets $\{g \in G : u_G g u_G = u_G\}$ and $\{g \in G : u_G g u_G \neq u_G\}$ are both definable. On the other hand, the function $\mathbb{Z} \to \hat{\mathbb{Z}}$ given by $n \mapsto n$ is definable in the sense that the preimages of any two disjoint closed subsets of $\hat{\mathbb{Z}}$ can be separated by a definable set.

All of this together with Proposition 5.9 implies that $f \colon \tilde{G} \to u_{\tilde{G}}\mathcal{M}_{\tilde{G}}$ is a definable map. Therefore, by Lemma 3.2 of [GPP14] and its proof, $\bar{f} = f_M$ is uniquely determined by the construction from Theorem 4.6, and $\bar{f}$ is continuous.

Pick a positive infinitesimal γ. Let $B := \begin{pmatrix} -1 & 0 \\ \gamma & -1 \end{pmatrix}$. Choose any net $(g_i)_i$ of elements of G converging to $p := \text{tp}(B/M)$. Then the left bottom entries of the matrices g_i are positive for all $i > i_0$ for some i_0. So, by Lemma 5.12(1), $u_G g_i u_G = u_G$ for all $i > i_0$. On the other hand, by Lemma 5.12(3), $u_G p u_G = q_1$. Therefore, $\hat{f}((p,0)) \in \{u_G p u_G\} \times u_{\mathbb{Z}}\mathcal{M}_{\mathbb{Z}} = \{q_1\} \times u_{\mathbb{Z}}\mathcal{M}_{\mathbb{Z}}$ and $\hat{f}((g_i,0)) \in \{u_G g_i u_G\} \times u_{\mathbb{Z}}\mathcal{M}_{\mathbb{Z}} = \{u_G\} \times u_{\mathbb{Z}}\mathcal{M}_{\mathbb{Z}}$ for all $i > i_0$. Since the net $((g_i,0))_i$ tends to $(p,0)$ and $q_1 \neq u_G$, we conclude that $\hat{f}$ is not continuous at $(p,0)$. Hence, $\bar{f} \neq \hat{f}$ by continuity of $\bar{f}$. More precisely, since $\bar{f}|_{\tilde{G}} = \hat{f}|_{\tilde{G}}$, we get that $\bar{f}((p,0)) \neq \hat{f}((p,0))$.

It remains to show that $\bar{f}|_{u_{\tilde{G}}\mathcal{M}_{\tilde{G}}} = \mathrm{id}$, as directly from the definition of $\hat{f}$ we have $\hat{f}|_{u_{\tilde{G}}\mathcal{M}_{\tilde{G}}} = \mathrm{id}$. Consider any $n \in \hat{\mathbb{Z}}$. Our goal is to show that $\bar{f}((u_G, n)) = (u_G, n)$ and $\bar{f}((-u_G, n)) = (-u_G, n)$. We will prove the first equality; the second one can be proved analogously.

Choose any net $((g_i, n_i))_i$ from $\tilde{G}$ converging to (u_G, n). Then the left bottom entry of g_i is positive for all $i > i_0$ for some i_0. By Lemma 5.12(1), $u_G g_i u_G = u_G$ for all $i > i_0$. Therefore, using Lemma 5.10, we get

$$\bar{f}((g_i, n_i)) = (u_G g_i u_G, n_i + h(g_i, u_G) + h(u_G, g_i u_G)) = (u_G, n_i)$$

for all $i > i_0$, so it clearly tends to (u_G, n). Since the net $((g_i, n_i))_i$ tends to (u_G, n), by continuity of $\bar{f}$, we conclude that $\bar{f}((u_G, n)) = (u_G, n)$. □

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