Fields interpretable in rosy theories

Krzysztof Krupiński*

Abstract

We are working in a monster model $\mathfrak{C}$ of a rosy theory T. We prove the following theorems, generalizing the appropriate results from the finite Morley rank case and o-minimal structures. If R is a $\bigvee$-definable integral domain of positive, finite $\mathrm{U}^{\text{þ}}$-rank, then its field of fractions is interpretable in $\mathfrak{C}$. If A and M are infinite, definable, abelian groups such that A acts definably and faithfully on M as a group of automorphisms, M is A-minimal and $\mathrm{U}^{\text{þ}}(M)$ is finite, then there is an infinite field interpretable in $\mathfrak{C}$. If G is an infinite, solvable but non nilpotent-by-finite, definable group of finite $\mathrm{U}^{\text{þ}}$-rank and T has NIP, then there is an infinite field interpretable in $\langle G, \cdot \rangle$.

In the last part, we study infinite, superrosy, dependent fields. Using measures, we show that each such field K satisfies $K = K^n - K^n$ for every $n \geq 1$.

0 Introduction

An important goal in model theory is to obtain, in a definable way, classical algebraic structures in theories satisfying some general model theoretic or algebraic assumptions. There is a long history of results of this kind, e.g. different versions of the group configuration theorem (originally proved by E. Hrushovski, see [7, Chapters 5 and 7]), getting fields from definable actions of abelian groups in the finite Morley rank case [9, Chapter 3] or in o-minimal structures [6], or getting fields from solvable, non nilpotent-by-finite groups in the finite Morley rank case [9, Corollary 3.20] and from any non abelian-by-finite groups in o-minimal structures [6, Corollary 5.1].

The main goal of this paper is to generalize some of the results about the existence of an infinite field to a general context of rosy theories. In fact, in Section 2 we give very general proofs that work in the presence of any nice notion of dimension, without rosiness. It is worth noticing that our results are new even in the stable case (Theorem 1 below seems to be new even in the finite Morley rank context). In the last section, we try to understand the structure of superrosy fields satisfying NIP.

Let $\mathfrak{C}$ be a monster model of a rosy theory T.

At the beginning of Section 2, we generalize [6, Lemma 4.1]. Namely, we prove

*Research supported by the Polish Government grant N201 032 32/2231 and by the NSF grant DMS-0300639

[0]2000 Mathematics Subject Classification: 03C45, 03C60

[0]Key words and phrases: rosy theory, interpretable field, non independence property

Theorem 1 *Let R be a $\bigvee$-definable integral domain of positive, finite $U^{þ}$-rank. Then the field of fractions of R, call it F, is interpretable in $\mathfrak{C}$. Moreover, there is a $\bigvee$-definable ring embedding of R onto a subring of F with the same $U^{þ}$-rank as F.*

In [6], this was proved in the o-minimal context using essentially certain topology on $\bigvee$-definable rings, which was defined by means of the standard o-minimal topology. In our case, we give a very general proof, which just uses some properties of $\mathrm{U}^{þ}$-rank, and that is why it works in any situation where we have a nice notion of dimension, for example in stable or simple theories (the dimension there is SU-rank).

In the further part of Section 2, we prove some variants of [9, Theorem 3.7] and [6, Lemma 4.2] in our general rosy context. For example, we prove

Theorem 2 *Let A and M be infinite, definable, abelian groups such that A acts faithfully and definably on M as a group of automorphisms, M is A-minimal and $U^{þ}(M)$ is finite. Then there is an infinite field interpretable in $\mathfrak{C}$.*

The proofs of these results use Zilber's Indecomposability Theorem and chain conditions in the finite Morley rank case, and the topology in the o-minimal case. Neither of these tools are present in our situation. Our proofs rely on Theorem 1 and some tricks eliminating applications of topology or chain conditions on intersections of uniformly definable groups. Once again, the results are true without rosiness whenever we have a nice notion of dimension.

In Section 3, we prove the main result of this paper.

Theorem 3 *Let G be a group of finite $U^{þ}$-rank definable in $\mathfrak{C}$ and suppose that T has NIP. Assume that G is solvable-by-finite but not nilpotent-by-finite. Then there is an infinite field interpretable in $\langle G, \cdot \rangle$.*

Similarly to the finite Morley rank case, in the proof of Theorem 3 we use Theorem 2. But the proof in our context is different.

In the last section, we prove some partial results concerning the following conjecture formulated in [1].

Conjecture 4 *Each infinite, superrosy field with NIP is either a real closed or an algebraically closed field.*

In particular, we show that each such field has the property $K = K^n - K^n$ for every $n \geq 1$. In our proofs we use absolute connected components and Kiesler measures.

I would like to thank Clifton Ealy, Anand Pillay and Thomas Scanlon for interesting discussions and suggestions.

1 Preliminaries

Let $\mathfrak{C}$ be a monster model of a theory T. We work in $\mathfrak{C}^{eq}$. Our general assumption in this paper is that T is rosy, but often, we only need a nice notion of dimension. In the rosy (or rather superrosy) context such a notion of dimension is $\mathrm{U}^{\text{þ}}$-rank.

The most interesting examples of rosy theories are simple theories and o-minimal theories.

We are not going to repeat the basic definitions of rosiness, þ-forking and $\mathrm{U}^{\text{þ}}$-rank. The fundamental paper about these notions is [5]. For the basic theory of rosy groups, the reader is referred to [1] where a short exposition of rosy theories and þ-forking is also given. Let us only recall here that if T is rosy, then þ-independence, denoted by $\mathop{\smile\!\!\!\!\!\!\mid}^{\text{þ}}$, is a ternary relation on subsets of $\mathfrak{C}^{eq}$, which satisfies all the properties of forking independence in simple theories except for the Independence Theorem. Using $\mathop{\smile\!\!\!\!\!\!\mid}^{\text{þ}}$, we define $\mathrm{U}^{\text{þ}}$-rank in the same way as U-rank is defined in stable theories by means of $\mathop{\smile\!\!\!\!\!\!\mid}$. $\mathrm{U}^{\text{þ}}$-rank has most of the nice properties that U-rank has in stable theories, e.g. it satisfies Lascar Inequalities. If D is an A-definable set, then $\mathrm{U}^{\text{þ}}(D) := sup\{\mathrm{U}^{\text{þ}}(d/A) : d \in D\}$. Of course, if this supremum is finite, then it is just the maximum. It turns out that if D is a definable group and T is rosy, then the supremum is also attained [1, Remark 1.20].

A theory is superrosy if every type has an ordinal $\mathrm{U}^{\text{þ}}$-rank. Notice that a theory of a model interpretable in a (super) rosy theory is still (super) rosy.

Let us recall a few properties of $\mathrm{U}^{\text{þ}}$-rank in rosy theories that are useful in this paper.

1. For a definable set D we have $\mathrm{U}^{\text{þ}}(D) > 0$ iff D is infinite.

2. For definable sets $D_1, \dots, D_n$ and $D = D_1 \cup \dots \cup D_n$ we have $\mathrm{U}^{\text{þ}}(D) = \max\{\mathrm{U}^{\text{þ}}(D_i) : i \leq n\}$. In particular, if $X \subseteq Y$ are definable, then $\mathrm{U}^{\text{þ}}(X) \leq \mathrm{U}^{\text{þ}}(Y)$.

3. $\mathrm{U}^{\text{þ}}$-rank is preserved under definable bijections, i.e. if X and Y are definable sets and $f : X \to Y$ is a definable bijection, then $\mathrm{U}^{\text{þ}}(X) = \mathrm{U}^{\text{þ}}(Y)$.

4. For a definable set D with $\mathrm{U}^{\text{þ}}(D) < \omega$ and $n \in \omega$ we have $\mathrm{U}^{\text{þ}}(D^n) = n\mathrm{U}^{\text{þ}}(D)$.

5. If $E(x, y)$ is a definable equivalence relation on a definable set D with infinitely many classes and $\mathrm{U}^{\text{þ}}(D) < \omega$, then at least one of the classes of E is of smaller $\mathrm{U}^{\text{þ}}$-rank than D. In a group context, it is particularly useful when E is the relation of being in the same conjugacy class.

6. (Lascar Inequalities for groups) For definable groups $H < G$ we have $\mathrm{U}^{\text{þ}}(H) + \mathrm{U}^{\text{þ}}(G/H) \leq \mathrm{U}^{\text{þ}}(G) \leq \mathrm{U}^{\text{þ}}(H) \oplus \mathrm{U}^{\text{þ}}(G/H)$.

The following fact follows from a standard application of Lascar Inequalities.

Fact 1.1 *If K is a definable field and T is superrosy, then for every $n > 0$, $[K^* : (K^*)^n] < \omega$ and, if $char(K) = p \neq 0$, then the image of the function $f : K \to K$ defined by $f(x) = x^p - x$ is a subgroup of finite index in K^+.*

Another property of T that we sometimes assume is the non independence property (NIP), also called T being dependent.

Definition 1.2 *We say that T has the NIP if there is no formula $\varphi(x, y)$ and sequence $\langle a_i \rangle_{i<\omega}$ such that for every $w \subseteq \omega$ there is b_w such that $\models \varphi(a_i, b_w)$ iff $i \in w$.*

We will need the following consequence of NIP proved by Shelah (for the proof see e.g. [2, Theorem 6.1]).

Fact 1.3 *If G is a (type-)definable group and T has NIP, then G^{00} (the smallest type-definable subgroup of bounded index) exists.*

Recall that we say that a definable group G has icc (the uniform chain condition on intersections of uniformly definable subgroups) if for every formula φ there is $n_\varphi \in \omega$ such that any chain of intersections of φ-definable subgroups of G has length at most n_φ.

Another important for this paper consequence of NIP and rosiness is (see [1, Proposition 1.7]):

Fact 1.4 *Suppose T is rosy and has NIP. Then any definable group G has icc.*

We also have the following easy observation [1, Proposition 4.1].

Fact 1.5 *Let K be an infinite, definable field. If $(K^+)^{00}$ exists, then $(K^+)^{00} = K$. In particular, if T has NIP, then $(K^+)^{00} = K$.*

Recall that a (global) Kiesler measure on a definable set D is a finitely additive probability measure on definable subsets of D. We say that a Kiesler measure μ on D is definable (over A) if for each formula $\varphi(x, y)$ and closed subset C of $[0, 1]$, $\{b \in \mathfrak{C} : \mu(\varphi(D, b)) \in C\}$ is type-definable (over A). We say that a definable group G is definably amenable if there is a left invariant Kiesler measure on G.

In the last section, we will need the following fact [3, Lemma 5.8].

Fact 1.6 *Assume that T has NIP and G is a definably amenable, definable group. Then there is a left invariant, definable Kiesler measure on G.*

Now let us recall the definition of a $\bigvee$-definable ring from [6].

Definition 1.7 *We say that a ring $\langle R, \cdot, + \rangle$ is a $\bigvee$-definable (or rather $\bigvee$-interpretable) ring if $R = \bigcup_{i \in I} X_i$ where X_i's are A-definable subsets of some sort of $\mathfrak{C}^{eq}$ for some set A, for every $i, j \in I$ there is $k \in I$ such that $X_i \cup X_j \subseteq X_k$, and the restrictions of addition and multiplication to $X_i \times X_j$ are definable functions.*

The assumption that for every $i, j \in I$ there is $k \in I$ such that $X_i \cup X_j \subseteq X_k$ is purely cosmetic, because we can always extend the family $\{X_i : i \in I\}$ by adding all unions of finitely many X_i's.

By the compactness theorem, if D is a definable subset of X, then it is covered by finitely many X_i's (so by one X_i) and hence addition and multiplication restricted to D are definable.

We define similarly $\bigvee$-definable groups or just sets. $\bigvee$-definable groups and rings occur naturally as subgroups and subrings generated by definable subsets of definable groups and rings.

If $X = \bigcup_{i \in I} X_i$ is a $\bigvee$-definable set, then $\mathrm{U}^{\text{þ}}(X)$ is defined as the supremum of $\mathrm{U}^{\text{þ}}$-ranks of the X_i's. If this supremum is finite, then $\mathrm{U}^{\text{þ}}$-ranks of all X_i's are finite and $\mathrm{U}^{\text{þ}}(X)$ is just the maximum of $\mathrm{U}^{\text{þ}}$-ranks of the X_i's. Notice that in a rosy theory, by the compactness theorem, $\mathrm{U}^{\text{þ}}(X)$ is the supremum of $\mathrm{U}^{\text{þ}}$-ranks of all definable subsets of X.

Definition 1.8 *If $R_1 = \bigcup_{i \in I} X_i$ and $R_2 = \bigcup_{j \in J} Y_j$ are $\bigvee$-definable rings, then a homomorphism $f : R_1 \to R_2$ is called $\bigvee$-definable if its restriction to each X_i is definable.*

Another notion needed is G-minimality. If G is a definable group acting definably on another definable group H by automorphisms, then we say that H is G-minimal if H is infinite and does not have infinite, proper, definable subgroups invariant under the action of G.

2 Getting fields from $\bigvee$-definable rings and definable actions of abelian groups

In this section, we generalize some results from Section 4 of [6] and [9, Theorem 3.7]. The main obstacle here in comparison with the o-minimal case is that we do not have a nice topology, and with the finite Morley rank case, that we do not have Zilber's Indecomposability Theorem.

We work in a monster model $\mathfrak{C}$ of a rosy theory T.

The following theorem was proved in the o-minimal context [6, Lemma 4.1] using certain topology on $\bigvee$-definable rings. Here we give a very general proof, which works in any context in which we have a nice notion of dimension, e.g. in simple and in o-minimal structures. More precisely, in order to get the first part of the theorem (i.e. the existence of an infinite interpretable field), instead of assuming rosiness and using $\mathrm{U}^{\text{þ}}$-rank, it is enough to assume the existence of an abstract function

$$\dim : \text{definable subsets of } \mathfrak{C}^{eq} \to Ord \cup \{\infty\},$$

taking the role of $\mathrm{U}^{\text{þ}}$-rank and satisfying the following properties:

(i) if $X \subseteq Y$ are definable, then $\dim(X) \leq \dim(Y)$,

(ii) the function dim is preserved under definable bijections, i.e. if X and Y are definable sets and $f : X \to Y$ is a definable bijection, then $\dim(X) = \dim(Y)$,

(iii) if $\dim(D) < \omega$, then $\dim(D \times D) \geq 2\dim(D)$.

Theorem 2.1 *Let R be a $\bigvee$-definable integral domain of positive, finite $U^{\text{þ}}$-rank. Then the field of fractions of R, call it F, is interpretable in $\mathfrak{C}$. Moreover, there is a $\bigvee$-definable ring embedding of R onto a subring of F with the same $U^{\text{þ}}$-rank as F.*

Proof. Since R is $\bigvee$-definable, we have that $R = \bigcup_{i \in I} X_i$ where all X_i's are sets definable over some set A, for any $i, j \in I$ there is $k \in I$ such that $X_i \cup X_j \subseteq X_k$, and the restrictions of addition and multiplication to any $X_i \times X_j$ are definable. So for every $r \in R$ the map $f_r : R \to R$ given by $f_r(x) = rx$ restricted to any X_i is definable.

Now let $D := X_i$ be such that $\mathrm{U}^{\text{þ}}(D) = \mathrm{U}^{\text{þ}}(R)$.

Claim 1 *For any $a, b \in R \setminus \{0\}$, $(Da - Da) \cap (Db - Db) \neq \{0\}$.*

Proof of Claim 1. Consider the function $f : D \times D \to R$ defined by

$$f(r_1, r_2) = r_1 a + r_2 b.$$

As $f_a \restriction D$, $f_b \restriction D$ and $+$ restricted to any definable subset of R are definable, so is f. But $\mathrm{U}^{\text{þ}}(D \times D) = 2\mathrm{U}^{\text{þ}}(D) > \mathrm{U}^{\text{þ}}(D) = \mathrm{U}^{\text{þ}}(R)$, hence by Lascar Inequalities (or, more generally, by properties (i) and (ii) of dim), we easily get that f is not injective. Thus, there are two distinct pairs $(r_1, r_2), (r_1', r_2') \in D \times D$ such that $r_1 a + r_2 b = r_1' a + r_2' b$. So $(r_1 - r_1')a = r_1 a - r_1' a = r_2' b - r_2 b = (r_2' - r_2)b$. We know that $a, b \neq 0$ and at least one of the elements $r_1 - r_1'$ and $r_2' - r_2$ is nonzero. Hence the element $r_1 a - r_1' a$ is nonzero and, of course, it belongs to $(Da - Da) \cap (Db - Db)$. □

Choose any $a \in R \setminus \{0\}$ and put $X = Da - Da$.

Claim 2 *For any $r_1, r_2 \in R \setminus \{0\}$, $r_1 X \cap r_2 X \neq \{0\}$.*

Proof of Claim 2. Since R is commutative, $r_i X = r_i Da - r_i Da = D(r_i a) - D(r_i a)$. As r_1, r_2 and a are nonzero, we see that $r_1 a, r_2 a \in R \setminus \{0\}$. So by Claim 1, we get $r_1 X \cap r_2 X = (D(r_1 a) - D(r_1 a)) \cap (D(r_2 a) - D(r_2 a)) \neq \{0\}$. □

The rest of the proof that the field of fractions of R is interpretable is the same as in the proof of [6, Lemma 4.1]. Namely, the fraction field F equals $(R \times (R \setminus \{0\}))/{\sim}$ where $(r_1, s_1) \sim (r_2, s_2) \iff r_1 s_2 = r_2 s_1$. By Claim 2, F can and will further be identified with $(X \times (X \setminus \{0\}))/{\sim_X}$ where $\sim_X$ is the restriction of $\sim$ to $X \times (X \setminus \{0\})$, and obviously $\sim_X$ is definable. Since addition and multiplication in R restricted to any definable subset are definable, we easily get that addition and multiplication in F are definable. Hence F is an interpretable field.

The map $r \mapsto (x, y)/{\sim_X}$ where $(x, y) \in X \times (X \setminus \{0\})$ and $(x, y) \sim (r, 1)$ is a $\bigvee$-definable embedding of R into F.

The fact that $\mathrm{U}^{\flat}(F) = \mathrm{U}^{\flat}(R)$ requires an extra explanation. Since R is $\bigvee$-definably embeddable in F, we easily get that $\mathrm{U}^{\flat}(R) \leq \mathrm{U}^{\flat}(F)$. Let B be a set containing $A \cup \{a\}$ and such that addition and multiplication restricted to any X_i are definable over B. Then F is interpretable over B. Let us add the names for the elements of B to the language.

In the following two paragraphs, classes of $\sim_X$ or $\sim_{X_j}$ are regarded as elements of $\mathfrak{C}^{eq}$, unless it is explicitly stated that we treat them as sets.

Consider any $(r_1, r_2) \in X \times (X \setminus \{0\})$. We need to show that $\mathrm{U}^{\flat}([(r_1, r_2)]_{\sim X}) \leq \mathrm{U}^{\flat}(R)$.

Since the function $r \mapsto (r_1 r, r_2 r)$ from $R \setminus \{0\}$ to $R \times (R \setminus \{0\})$ is a $\bigvee$-definable injection and its image is contained in the $\sim$-class of (r_1, r_2) (treated as a subset of $R \times (R \setminus \{0\})$), there is $j \in I$ such that $X \subseteq X_j$ and $\mathrm{U}^{\flat}$-rank of the $\sim_{X_j}$-class of (r_1, r_2) treated as a subset of $X_j \times (X_j \setminus \{0\})$ is at least $\mathrm{U}^{\flat}(R)$. Since $(X \times (X \setminus \{0\}))/{\sim_X}$ can be definably identified with $(X_j \times (X_j \setminus \{0\}))/{\sim_{X_j}}$, we can work in $(X_j \times (X_j \setminus \{0\}))/{\sim_{X_j}}$, i.e. if we put $d = [(r_1, r_2)]_{\sim X_j} \in (X_j \times (X_j \setminus \{0\}))/{\sim_{X_j}}$, then $\mathrm{U}^{\flat}(d) = \mathrm{U}^{\flat}([(r_1, r_2)]_{\sim X})$. By choice of j, there is $(r_1', r_2') \sim_{X_j} (r_1, r_2)$ in $X_j \times (X_j \setminus \{0\})$ such that $\mathrm{U}^{\flat}((r_1', r_2')/d) \geq \mathrm{U}^{\flat}(R)$. Since $d \in dcl(r_1', r_2')$, by Lascar Inequalities, we get $2\mathrm{U}^{\flat}(R) \geq \mathrm{U}^{\flat}((r_1', r_2')) = \mathrm{U}^{\flat}((r_1', r_2'), d) \geq \mathrm{U}^{\flat}((r_1', r_2')/d) + \mathrm{U}^{\flat}(d) \geq \mathrm{U}^{\flat}(R) + \mathrm{U}^{\flat}(d)$. So $\mathrm{U}^{\flat}(d) \leq \mathrm{U}^{\flat}(R)$. ■

Notice that in the above theorem the assumption that R is of positive $\mathrm{U}^{\flat}$-rank is necessary. Indeed, if $\mathfrak{C}$ is a real closed field, then $\mathbb{Q}$ is a $\bigvee$-definable field of $\mathrm{U}^{\flat}$-rank 0 and it is not interpretable in $\mathfrak{C}$.

Now we are going to generalize some classical results about getting fields from definable actions of abelian groups [9, Theorem 3.1], [6, Lemma 4.2]. As in the o-minimal case, we cannot apply the method from the finite Morley rank case because we do not have Zilber's Indecomposability Theorem. But, as in [6], we can apply Theorem 2.1. Once again, we give here general proofs which omit any applications of o-minimal topology or chain conditions. As before, instead of rosiness and working with $\mathrm{U}^{\flat}$-rank, it is enough to assume the existence of an abstract notion of dimension, taking the role of $\mathrm{U}^{\flat}$-rank and satisfying conditions (i)-(iii) listed above Theorem 2.1 together with the following three extra properties:

(iv) for a definable set D we have $\dim(D) > 0$ iff D is infinite,

(v) for a definable set D with $\dim(D) < \omega$ and $n \in \omega$ we have $\dim(D^n) < \omega$.

(vi) for any definable sets X and Y and a definable surjection $f : X \to Y$ we have $\dim(X) \geq \dim(Y)$ (even the weaker conclusion that $\dim(X) < \omega$ implies $\dim(Y) < \omega$ is enough).

Of course, (ii) follows from (vi). As to the applications of the above three properties, the proof of Theorem 2.2 does not require (vi), the proof of Proposition 2.3 does not require (v) and (vi), and in the proof of Corollary 2.4 (which has a stronger conclusion than Theorem 2.2) we do not use (v).

Theorem 2.2 *Let A and M be infinite, definable, abelian groups such that A acts faithfully and definably on M as a group of automorphisms, M is A-minimal and $U^{\flat}(M)$ is finite. Then there is an infinite field interpretable in $\mathfrak{C}$.*

Proof. For $a \in A$ we define $Fix(a) = \{m \in M : am = m\}$ and for $m \in M$ we put $Stab(m) = \{a \in A : am = m\}$. Of course, $Fix(a)$ and $Stab(m)$ are definable subgroups of M and A, respectively.

Claim 1 *There are $m_1, \dots, m_n \in M$ such that $Stab(m_1) \cap \dots \cap Stab(m_n) = \{e\}$.*

Proof of Claim 1. For every $a \in A \setminus \{e\}$, $Fix(a)$ is a proper, definable subgroup of M invariant under the action of A. So by A-minimality of M, $Fix(a)$ is finite. Hence for any infinite subset S of M, $\bigcap_{m \in S} Stab(m) = \{e\}$. Thus, by the compactness theorem, there are $m_1, \dots, m_n \in M$ such that $Stab(m_1) \cap \dots \cap Stab(m_n) = \{e\}$. $\square$

Let R be the ring of endomorphisms of M generated by A. Then R is commutative.

Notice that every $r \in R$ is determined by $(r(m_1), \dots, r(m_n))$. If not, then there is $r \in R \setminus \{0\}$ such that $r(m_1) = \dots = r(m_n) = 0$. Since R is commutative, we get that $ker(r)$ is a proper, definable and invariant under the action of A subgroup of M containing $\{m_1, \dots, m_n\}$. So $Am_1 + \dots + Am_n \subseteq ker(r)$. On the other hand, by choice of $m_1, \dots, m_n$, we get that the function $a \mapsto (am_1, \dots, am_n)$ is an injection from A to M^n. So there is i such that Am_i is infinite, and hence $ker(r)$ is infinite. This contradicts the assumption that M is A-minimal.

Having made the above observation, we get the following in a rather standard way.

Claim 2 *Up to isomorphism, the ring R is $\bigvee$-definable so that it is contained in M^n with the addition inherited from M^n, and $0 < U^{\flat}(R) < \omega$.*

Proof of Claim 2. Let $H = \langle A(m_1, \dots, m_n) \rangle$, i.e. H is the subgroup of M^n generated by $\{(am_1, \dots, am_n) : a \in A\}$. By the above observation, the function $f : R \to H \subseteq M^n$ defined by $f(r) = (r(m_1), \dots, r(m_n))$ is a bijection. Of course, $H := \bigcup_{i<\omega} X_i$ where $X_i = \pm A(m_1, \dots, m_n) \pm \dots \pm A(m_1, \dots, m_n)$ (i-many times). So H is a $\bigvee$-definable subgroup of M^n.

By the definition of f, we see that for any $r_1, r_2 \in R$ we have $f(r_1 + r_2) = f(r_1) + f(r_2)$ ($+$ on the left hand side is the addition in R and $+$ on the right hand side is the addition in M^n).

Now we define multiplication, $*$, on H to make f a ring isomorphism, i.e. $f(r_1) * f(r_2) := f(r_1 r_2)$ for all $r_1, r_2 \in R$. We leave as an easy exercise to check that $* : H \times H \to H$ is $\bigvee$-definable, i.e. for any $i, j < \omega$, $* : X_i \times X_j \to M^n$ is definable.

As we noticed before, there is $i \leq n$ such that Am_i is infinite. So X_1 is infinite. Thus, using basic properties of $\mathrm{U}^{\flat}$ (or, more generally, of dim), we easily get $0 < \mathrm{U}^{\flat}(H) \leq \mathrm{U}^{\flat}(M^n) < \omega$. $\square$

The next claim has the same proof as in the finite Morley rank case.

Claim 3 *R is an integral domain.*

Proof of Claim 3. Take any $r_1, r_2 \in R$ such that $r_1 r_2 = 0$. If $r_2 \neq 0$, then $ker(r_2)$ is a proper, definable subgroup of M invariant under the action of A. So by A-minimality of M, $ker(r_2)$ is finite. So $im(r_2)$ is an infinite, definable subgroup of M invariant under the action of A. Thus $im(r_2) = M$ which implies $r_1 = 0$. $\square$

By Claims 2, 3 and Theorem 2.1, we get an infinite field interpretable in $\mathfrak{C}$. $\blacksquare$

Assuming that M does not have nontrivial, proper, definable subgroups invariant under the action of A, we get even more specific information about our interpretable field.

Proposition 2.3 *Let A and M be infinite, definable, abelian groups such that A acts faithfully and definably on M as a group of automorphisms, M does not have any nontrivial, proper, definable subgroups invariant under the action of A and $U^{þ}(M)$ is finite. Then for every nonzero $m \in M$ there is a field K definable in $\mathfrak{C}$ whose underlying additive group is $\langle M, + \rangle$, and $\langle A, \cdot \rangle$ is definably embeddable in K^* by sending $a \in A$ to am. After the embedding, the action of A on M becomes the scalar multiplication.*

Proof. Let R be the ring of endomorphisms of M generated by A. We easily see that every nonzero $r \in R$ is an automorphism of M. Indeed, since $ker(r)$ is a proper, definable subgroup of M invariant under the action of A, it must be trivial. So $im(r)$ is an infinite, definable subgroup of M invariant under the action of A, and hence it is equal to M.

Choose a nonzero $m \in M$. We conclude that every element $r \in R$ is determined by $r(m)$. So by the proof of Theorem 2.2, R is $\bigvee$-definable (after the indentification of every $r \in R$ with $r(m) \in M$), contained in M with addition inherited from M, and the field of fractions, F, of R is interpretable in $\mathfrak{C}$. More precisely, $F = (X \times (X \setminus \{0\}))/{\sim}$ where X is a definable subset of R.

The rest is the same as in the last paragraph of the proof of [6, Lemma 4.2]. Every element $(\alpha, \beta)/{\sim} \in F$ can be identified with the automorphism $\alpha\beta^{-1}$ of M. So F is a field of automorphisms of M. We easily see that the action of F on M is definable. We see that every element $k \in F$ is determined by $k(m)$. Hence F can be definably embedded into M by sending $k \in F$ to $k(m)$. The image of this map is a definable field, say K, whose additive group is a subgroup of M invariant under A, so it must be M. Of course, A is definably embeddable in K^* by sending $a \in A$ to am. The fact that after this embedding the action of A on M coincides with field multiplication is trivial. $\blacksquare$

Using Proposition 2.3, we obtain the following strengthening of Theorem 2.2.

Corollary 2.4 *Let A and M be infinite, definable, abelian groups such that A acts faithfully and definably on M as a group of automorphisms, M is A-minimal and*

$U^þ(M)$ is finite. Then there is an infinite field K interpretable in $\mathfrak{C}$ whose underlying additive group is M/M_0 for some finite subgroup M_0 of M invariant under A, and A/A_0 is definably embeddable in K^* for some finite subgroup A_0 of A. In fact, the action of A on M induces a faithful and definable action of A/A_0 on M/M_0 by automorphisms, and after the embedding this action becomes the scalar multiplication.

Proof. By Proposition 2.3, in order to prove the corollary, it is enough to find a finite subgroup M_0 of M which is invariant under the action of A, and a finite subgroup A_0 of A such that A/A_0 acts faithfully and definably on M/M_0 as a group of automorphisms and M/M_0 does not have nontrivial, proper, definable subgroups invariant under A/A_0.

Define $M_0 = \{m \in M : [A : Stab(m)] < \omega\}$. Of course, M_0 is a subgroup of M invariant under A. We claim that M_0 is finite (and hence definable). If not, there is an infinite, countable set S contained in M_0. Then $\bigcap_{m\in S} Stab(m)$ is a nontrivial (in fact, of bounded index) subgroup of A. So there is a nontrivial $a \in \bigcap_{m\in S} Stab(m)$, which means that $S \subseteq Fix(a)$, a contradiction with the fact that $Fix(a)$ is finite.

Since M_0 is invariant under A, the action of A on M induces an action of A on M/M_0. It is easy to see that A acts on M/M_0 by automorphisms.

Define A_0 as the set of those $a \in A$ which act as the trivial automorphism on M/M_0. Then A_0 is a subgroup of A. We claim that it is finite. Indeed, by Claim 1 in the proof of Theorem 2.2, there are $m_1, \dots, m_n \in M$ such that $Stab(m_1) \cap \dots \cap Stab(m_n) = \{e\}$. So every $a \in A$ is determined by $(am_1, \dots, am_n)$. On the other hand, if a induces the trivial automorphism of M/M_0, then $am_1 \in m_1 + M_0, \dots, am_n \in m_n + M_0$. Since M_0 is finite, we get only finitely many possibilities for $a \in A$ inducing the trivial automorphism of M/M_0, i.e. A_0 is finite.

Summarizing, we get that A/A_0 acts faithfully and definably on M/M_0 as a group of automorphisms. It remains to check that M/M_0 does not have nontrivial, proper, definable subgroups invariant under A/A_0. Consider any definable subgroup G of M/M_0 invariant under A/A_0 and let $M_1 < M$ be the preimage of G under the quotient map. We see that M_1 is a definable subgroup of M invariant under A. So either $M_1 = M$, and then $G = M/M_0$, or M_1 is finite. In the second case, for any $m \in M_1$ the orbit $Am \subseteq M_1$ is finite so $[A : Stab(m)] < \omega$, i.e. $m \in M_0$; hence $M_0 = M_1$, which means that G is trivial. ■

3 Getting fields in solvable non-nilpotent groups

In this section we prove the main result of the paper.

Theorem 3.1 *Let G be a group of finite $U^þ$-rank definable in a monster model of a rosy theory satisfying NIP. Assume that G is solvable-by-finite but not nilpotent-by-finite. Then there is an infinite field interpretable in $\langle G, \cdot \rangle$.*

Before we start to prove the theorem, let us show the following general lemma and a standard remark.

Lemma 3.2 *Suppose P and Q are infinite abelian groups, P acts on Q by automorphisms and for every $p \in P \setminus \{e_P\}$ and $q \in Q \setminus \{e_Q\}$, $p \cdot q \neq q$. Then $Q \rtimes P$ is solvable but not nilpotent-by-finite.*

Proof. Solvability is obvious. Suppose for a contradiction that $Q \rtimes P$ is nilpotent-by-finite. Let N be a nilpotent subgroup of finite index in $Q \rtimes P$.

We can identify Q with $Q \times \{e_P\} < Q \rtimes P$ and P with $\{e_Q\} \times P < Q \rtimes P$. After this identification $Q \rtimes P = QP$. Put $P_1 = P \cap N$ and $Q_1 = Q \cap N$. Then P_1 and Q_1 are subgroups of finite index in P and Q, respectively, such that the restriction of the action of P on Q gives us an action by automorphisms of P_1 on Q_1 satisfying the property $(\forall p \in P_1 \setminus \{e_P\})(\forall q \in Q_1 \setminus \{e_Q\})(p \cdot q \neq q)$, and moreover $Q_1 \rtimes P_1$ is nilpotent. So without loss of generality $Q \rtimes P$ is nilpotent. To get a contradiction, it is enough to show that $Z(Q \rtimes P)$ is trivial.

Let e be the neutral element of $Q \rtimes P$. By assumption, for all $p \in P \setminus \{e\}$ and $q \in Q \setminus \{e\}$ we have $pqp^{-1}q^{-1} = (p \cdot q)q^{-1} \neq qq^{-1} = e$.

Take any $qp \in Z(Q \rtimes P)$ where $p \in P$ and $q \in Q$. Then $qpq(qp)^{-1}q^{-1} = e$ so $pqp^{-1}q^{-1} = e$. By the last paragraph, we get $p = e$ or $q = e$. But once again using the last paragraph, we also see that $P \cap Z(Q \rtimes P) = Q \cap Z(Q \rtimes P) = \{e\}$. So $p = e$ and $q = e$. Thus we have proved that $Z(Q \rtimes P) = \{e\}$. ■

Remark 3.3 *(i) Let G be a group such that all definable quotients of definable subgroups of G have icc on centralizers. Assume that G is solvable-by-finite. Then G has a definable, solvable subgroup H of finite index, and H has a normal sequence $\{e\} = H_0 \lhd H_1 \lhd \cdots \lhd H_n = H$ such that each quotient H_{i+1}/H_i is abelian and each H_i is definable.*
(ii) Let G be a group such that all definable quotients of definable subgroups of G have icc on centralizers. Assume that N is a nilpotent subgroup of G. Then G has a definable nilpotent subgroup H containing N. Thus the upper central series of H consists of definable subgroups of G.

Proof. (i) By a standard trick, there is a normal, solvable subgroup L of finite index in G. Then the derivative sequence of L, call it $\{e\} = L_0 \lhd L_1 \lhd \cdots \lhd L_n = L$, consists of normal subgroups of G. Now we define a sequence $\{e\} = H_0 \lhd H_1 \lhd \cdots \lhd H_n$ of definable, normal subgroups of G with abelian quotients, and such that $H_i > L_i$ for every $0 \leq i \leq n$.

H_0 is defined as $\{e\}$. Suppose $H_0, \ldots, H_i$ satisfying all the above assumptions have been constructed. Then we define $H_{i+1} = \pi_i^{-1}[Z(C(L_{i+1}H_i/H_i))]$ where $\pi_i : G \to G/H_i$ is the natural quotient map. Using icc on centralizers, one can easily check that this construction works.

Now $H := H_n$ together with the sequence $\{e\} = H_0 \lhd H_1 \lhd \cdots \lhd H_n = H$ have the desired properties.
(ii) The proof is the same as in the stable case [9, Theorem 3.17], by induction on the nilpotent class. If N is abelian, then $H = Z(C(N))$ works. For the induction step, let $Z = Z(C(Z(N)))$. By icc on centralizers, Z is definable, abelian, it is centralized by N and $Z(N) < Z$. Hence $NZ/Z \cong N/(N \cap Z)$ is a nilpotent subgroup of

$C(Z)/Z$ of a smaller class of nilpotency than N so by induction hypothesis, there is a definable, nilpotent subgroup of $C(Z)/Z$ containing NZ/Z. Then its preimage under the natural quotient map is a definable, nilpotent subgroup of G containing N. ■

Recall that the centralizer connected component of a definable group G is the intersection of all centralizers $C(g)$, $g \in G$, of finite index in G. The group G is said to be centralizer connected if it is equal to its centralizer connected component. If H is a definable subgroup of G, then the centralizer connected component of H in G is the intersection of H and all centralizers $C(g)$, $g \in G$, intersecting H on a finite index subgroup. Finally, H is centralizer connected in G if it is equal to its centralizer connected component in G.

Notice that if we have icc, then the centralizer connected components are definable. We will also use an easy observation that if G is centralizer connected and $Z(G)$ is finite, then $G/Z(G)$ is centerless. Another useful remark is that if G is centralizer connected and G_0 is a subgroup of finite index, then G_0 is also centralizer connected, and if G was additionally centerless, then so is G_0.

Proof of Theorem 3.1. By Fact 1.4 and Remark 3.3, we can assume that G is solvable and it has a normal sequence consisting of definable subgroups with abelian quotients. It is also clear that we can assume that $\langle G, \cdot \rangle$ is our monster model.

The proof is by induction on $\mathrm{U}^{\flat}(G)$. In fact, in the paragraph below we will show that our assumptions on G imply that $\mathrm{U}^{\flat}(G) \geq 2$. The fact that G is infinite follows immediately from the assumption that G is not nilpotent-by-finite.

Assume that the theorem is true for groups of $\mathrm{U}^{\flat}$-rank smaller that $\mathrm{U}^{\flat}(G)$. By icc on centralizers, we can assume that G is centralizer connected. If $Z(G)$ is infinite, then since G is not nilpotent-by-finite, $G/Z(G)$ is a non nilpotent-by-finite, solvable group of $\mathrm{U}^{\flat}$-rank smaller than $\mathrm{U}^{\flat}(G)$. So by induction hypothesis, we get an infinite interpretable field (notice that if $\mathrm{U}^{\flat}(G) = 1$, then $[G : Z(G)] < \omega$, a contradiction). So we need to consider the case when $Z(G)$ is finite. Dividing out by $Z(G)$, we can assume that G is also centerless. This implies that G does not have nontrivial, finite, normal subgroups. Indeed, if $H \lhd G$ is finite, then $[G : C(h)] < \omega$ for every $h \in H$. But G is centralizer connected so $H \subseteq Z(G) = \{e\}$. So by solvability of G, there is an infinite, definable, abelian, normal subgroup H of G (notice that if $\mathrm{U}^{\flat}(G) = 1$, then $[G : H] < \omega$, a contradiction; so we have proved that $\mathrm{U}^{\flat}(G) \geq 2$).

Now choose a definable subgroup G_0 of finite index in G and an infinite, definable, abelian, normal subgroup H of G_0 with minimal possible $\mathrm{U}^{\flat}$-rank (ranging over all such pairs (G_0, H)). Without loss of generality $G = G_0$. We can also assume that H is centralizer connected in G.

Since G is centralizer connected and centerless, we have that $G/C(H)$ is infinite. As $\mathrm{U}^{\flat}(G/C(H)) < \mathrm{U}^{\flat}(G)$, by induction hypothesis, we can assume that $G/C(H)$ is nilpotent-by-finite. Using icc on centralizers and Remark 3.3, we can replace G by a definable subgroup of finite index so that $G/C(H)$ becomes nilpotent and centralizer connected. This implies that $Z(G/C(H))$ is infinite.

Put $A = Z(G/C(H))$ and $A_0 = \pi^{-1}[A]$ where $\pi : G \to G/C(H)$ is the natural quotient map. Then A_0 is a definable, normal subgroup of G and $A = A_0/C(H)$ is an infinite, abelian group interpretable in $\langle G, \cdot \rangle$. Moreover, A acts faithfully and definably on H by automorphisms: $aC(H) * h = h^a$ for $aC(H) \in A$ and $h \in H$.

Claim 1 *For every $a \in A_0 \setminus C(H)$, $C(a) \cap H = \{e\}$.*

Proof of Claim 1. Let $B = C(a) \cap H$. It is enough to show that $B \lhd G$ (because H was chosen to have minimal possible positive $\mathrm{U}^\flat$-rank, H is centralizer connected in G and G does not have nontrivial, finite, normal subgroups). Take any $g \in G$. We need to show $B^g = B$. Of course, $B^g = C(a^g) \cap H$. Since $A = Z(G/C(H))$, $a^g C(H) = (aC(H))^g = aC(H)$. So $a^g = ac$ for some $c \in C(H)$. Thus $B^g = C(a^g) \cap H = C(ac) \cap H = C(a) \cap H = B$. □

By Claim 1, $P := A$ and $Q := H$ satisfy the assumptions of Lemma 3.2 so we conclude that $R := H \rtimes A$ is an interpretable group which is solvable but not nilpotent-by-finite. We also have $\mathrm{U}^\flat(R) = \mathrm{U}^\flat(H) + \mathrm{U}^\flat(A)$. So if $\mathrm{U}^\flat(A) < \mathrm{U}^\flat(G/H)$, then $\mathrm{U}^\flat(R) < \mathrm{U}^\flat(H) + \mathrm{U}^\flat(G/H) = \mathrm{U}^\flat(G)$ and hence we are done by induction hypothesis. Therefore, we can assume that $\mathrm{U}^\flat(A) = \mathrm{U}^\flat(G/H)$. But $\mathrm{U}^\flat(A) = \mathrm{U}^\flat(A_0) - \mathrm{U}^\flat(C(H)) \leq \mathrm{U}^\flat(G) - \mathrm{U}^\flat(H) = \mathrm{U}^\flat(G/H)$ and equality holds iff $[G : A_0] < \omega$ and $[C(H) : H] < \omega$. So we get $[C(H) : H] < \omega$, and we can assume that $G = A_0$.

By NIP, H^{00} exists.

Claim 2 H^{00} is definable.

Proof. Take any $a \in G \setminus C(H)$. Notice that $C(a)$ is infinite. If not, then $\mathrm{U}^\flat(a^G) = \mathrm{U}^\flat(G)$ so $(aH)^G$ is an infinite subset of G/H. But since $G/C(H)$ is abelian and $[C(H) : H] < \omega$, we get a contradiction.

On the other hand, by Claim 1, $C(a) \cap H = \{e\}$. So if we put $G_1 = HC(a)$, then $[G_1 : H]$ is infinite.

By NIP, G_1^{00} exists. Notice that

$$(*) \qquad H^{00} = G_1^{00} \cap H.$$

The inclusion $(\subseteq)$ is obvious. To prove $(=)$, assume for a contradiction that $H^{00} \subsetneq G_1^{00} \cap H$. By the definition of G_1 and the fact that $C(a) \cap H = \{e\}$, we get that $H^{00}C(a)$ is a type-definable subgroup of G_1 of bounded index, not containing G_1^{00}, a contradiction.

Since $[G_1 : G_1 \cap C(H)] \geq \omega$, there is $b \in G_1^{00} \setminus C(H)$. By Claim 1, for every $c \in bH$ we have $\mathrm{U}^\flat(c^H) = \mathrm{U}^\flat(H) = \mathrm{U}^\flat(bH)$. As bH is closed under conjugations by elements of H, we get that $bH = c_1^H \cup \cdots \cup c_n^H$ for some $c_1, \ldots, c_n \in bH$. We also know that $G_1^{00} \lhd G_1$ so $bH \cap G_1^{00} = c_{i_1}^H \cup \cdots \cup c_{i_k}^H$ for some $1 \leq i_1 < \cdots < i_k \leq n$. Thus, $bH \cap G_1^{00}$ is definable. On the other hand, since $b \in G_1^{00}$, by $(*)$, we get $bH \cap G_1^{00} = b(H \cap G_1^{00}) = bH^{00}$. Therefore, H^{00} is definable. □

By Claim 2, replacing H by H^{00} (and repeating all arguments preceding Claim 2 for this new H), we can assume that $H = H^{00}$.

Claim 3 *H does not have nontrivial, proper, definable subgroups invariant under the action of A.*

Proof of Claim 3. Suppose H_1 is a definable subgroup of H invariant under A. Since $A = G/C(H)$, we get $H_1 \lhd G$. So, by minimality of $\mathrm{U}^{\text{þ}}(H)$, H_1 is either finite or of finite index in H. On the other hand, we know that G does not have nontrivial, finite, normal subgroups and $H = H^{00}$. Hence $H_1 = \{e\}$ or $H_1 = H$. □

By Claim 3, we see that $M := H$ and A satisfy the assumptions of Theorem 2.2 (or even Proposition 2.3) so an infinite, interpretable field exists. ■

In [1], Theorem 3.1 was proved in the case of $\mathrm{U}^{\text{þ}}(G) = 2$ but under a much stronger assumption that G has hereditarily fsg (finitely satisfiable generics). In fact, under this assumption there was proved even more, namely (see [1, Theorem 4.5]):

Fact 3.4 *Assume that G has NIP, hereditarily fsg, $U^{\text{þ}}(G)$=2 and G is not nilpotent-by-finite. Then, after possibly passing to a definable subgroup of finite index and quotienting by its finite center, G is (definably) the semidirect product of the additive and multiplicative groups of an algebraically closed field F interpretable in $\langle G, \cdot\rangle$, and moreover $G = G^{00}$.*

Analyzing carefully the proof of Theorem 3.1 and modifying it a little bit, we obtain the following strengthening of Theorem 3.1 in the $\mathrm{U}^{\text{þ}}$-rank 2 case.

Corollary 3.5 *Let G be a group of $U^{\text{þ}}$-rank 2 definable in a monster model of a rosy theory satisfying NIP. Assume that G is solvable-by-finite but not nilpotent-by-finite. Then, after possibly passing to a definable subgroup of finite index and quotienting by its finite center, G is (definably) the semidirect product of the additive group and a finite index subgroup of the multiplicative group of a field K interpretable in $\langle G, \cdot\rangle$.*

Proof. By the proof of Theorem 3.1, we know that there is no group of $\mathrm{U}^{\text{þ}}$-rank 0 or 1 satisfying the assumptions of Theorem 3.1. Therefore, under the assumption $\mathrm{U}^{\text{þ}}(G) = 2$, the proof of Theorem 3.1 necessarily leads us to the last paragraph and produces a field using Proposition 2.3. So for any nontrivial $h \in H$ we get an interpretable field, say K, whose additive group is $\langle H, \cdot\rangle$ and such that the map $f : G/C(H) \to K^*$ given by $f(gC(H)) = gC(H) * h = h^g$ is a definable embedding of $G/C(H)$ into K^*, and after this embedding the action of $G/C(H)$ on H coincides with the field multiplication. Since $\mathrm{U}^{\text{þ}}(K^*) = 1 = \mathrm{U}^{\text{þ}}(G/C(H))$, the image of $G/C(H)$ by f is a finite index subgroup of K^*, call it L.

Claim 1 *Without loss of generality we can assume that $G = HB$ where B is a definable, abelian group of $U^{\text{þ}}$-rank 1, $H \cap B = \{e\}$ and $C(H) = H$.*

Proof of Claim 1. Since $[C(H) : H]$ is finite, we can choose $a \in G \setminus C(H)$. By the first paragraph of the proof of Claim 2 and by Claim 1 in the proof of Theorem 3.1, we get that $C(a)$ is infinite and $C(a) \cap H = \{e\}$. Thus $\mathrm{U}^{\text{þ}}(C(a)) = 1$ and so $C(a)$

is nilpotent-by-finite. Using Remark 3.3 and considering the centralizer connected component of $C(a)$, we get that $C(a)$ has a definable abelian subgroup B of finite index. Since $\mathrm{U}^{\mathrm{þ}}(HB) = 2$, we can assume $G = HB$. In order to finish the proof of Claim 1, it is enough to show the following

Subclaim $C(H) = H$.

Proof of Subclaim. It is enough to show that $C(H) \cap B = \{e\}$. So we will be done if we show that for any $b \in B \setminus C(H)$, $C(b) \cap C(H) = \{e\}$. We proceed in the same way as in the proof of Claim 1 in the proof of Theorem 3.1.

Take any $b \in B \setminus C(H)$. Let $C = C(b) \cap C(H)$. It is enough to show that $C \lhd G$. Indeed, if $C \lhd G$, then since G does not have nontrivial, finite, normal subgroups, we get that either $C = \{e\}$ or C is infinite. In the second case, $[C(H) : C]$ is finite. This implies that $[H : H \cap C]$ is finite. Since $H = H^{00}$, we get $H \subseteq C$ and so $b \in C(H)$, a contradiction.

Take any $g \in G$. We need to show $C^g = C$. Of course, $C^g = C(b^g) \cap H$. Since $G/H \cong B$ is abelian, $b^g H = (bH)^g = bH$. So $b^g = bc$ for some $c \in H$. Thus $C^g = C(b^g) \cap C(H) = C(bc) \cap C(H) = C(b) \cap C(H) = C$. □

By Claim 1, $G/C(H) = G/H = BH/H \cong B$ and we see that B is definably isomorphic to L by sending $b \in B$ to h^b. We also easily see that the action of L on H by the field multiplication is the same as the action of B on H by conjugation. So $G = HB$ is definably isomorphic to $K^+ \rtimes L$. ■

4 Superrosy dependent fields

The main motivation in this section is Conjecture 4. To prove this conjecture, it is enough to show that each infinite, superrosy field K with NIP and containing $\sqrt{-1}$ is algebraically closed (see [4, Chapter XI, Proposition 2.4]). In fact, it suffices to show that for every natural number $n > 0$, $K^n = K$, and if p is the characteristic of K, then the function $f : K \to K$ defined by $f(x) = x^p - x$ is onto (because then we can apply the standard Macintyre's proof [9, Theorem 3.1]). The fact that f is onto follows from Facts 1.1 and 1.5.

In this paper we will prove a weaker condition than $K^n = K$, namely that for every natural number $n > 0$, $K = K^n - K^n$. In particular, if n is odd or if $\sqrt[2^k]{-1}$ exists in K where 2^k is the largest power of 2 dividing n, then $K = K^n + K^n$. We also prove other results of this kind. The main idea involved here is to apply definable measures.

Let us start from a general fact.

Proposition 4.1 *Let K be any field and $G = K \rtimes K^*$ (i.e. $(k_1, k_2) \cdot (k_1', k_2') = (k_1 + k_2 k_1', k_2 k_2')$). Then G is amenable and there is a finitely additive, probabilistic measure on K which is invariant under additive and non-zero multiplicative translations.*

Proof. Of course, G is solvable so it is amenable, i.e. there is a finitely additive, probabilistic, left (two-sided) invariant measure $\mathbf{m}$ on G.

Define a function $\mu : \mathcal{P}(K) \to [0,1]$ by

$$\mu(A) = \mathbf{m}(A \times K^*).$$

It is obvious that μ is a finitely additive, probabilistic measure on K. Now we will check that μ is additively and multiplicatively invariant. In the additive case, for every $A \subseteq K$ and $k \in K$ we have:

$$\mu(k + A) = \mathbf{m}((k + A) \times K^*) = \mathbf{m}((k, 1) \cdot (A \times K^*)) = \mathbf{m}(A \times K^*) = \mu(A).$$

In the multiplicative case, for every $A \subseteq K$ and $k \in K^*$ we have:

$$\mu(kA) = \mathbf{m}((kA) \times K^*) = \mathbf{m}((0, k) \cdot (A \times K^*)) = \mathbf{m}(A \times K^*) = \mu(A). \qquad \blacksquare$$

Note that if K is definable in a monster model $\mathfrak{C}$ of a theory satisfying NIP, then since $G := K \rtimes K^*$ is also definable in $\mathfrak{C}$ and G is amenable, by Fact 1.6, there is a definable, left invariant Keisler measure on G. Using this measure as $\mathbf{m}$ in the above proof we get:

Corollary 4.2 *Let K be any field definable in a monster model of a theory satisfying NIP. Then there is a definable Keisler measure on K invariant under additive and non-zero multiplicative translations.*

Definition 4.3 *Suppose G is a definably amenable group (with left invariant Keisler measure μ) definable in a monster model of any theory T. We say that a definable set $X \subseteq G$ is μ-generic if $\mu(X) > 0$. We say that a type (or its set of realizations) is μ-generic if the conjunction of any finitely many formulas in this type defines a set whose intersection with G is μ-generic.*

It is obvious that in every definably amenable group non-μ-generic sets form an ideal and hence every partial μ-generic type can be extended to a global μ-generic type. In particular, at least one global μ-generic type exists.

The following proposition (except for point (i)) is a variant of [2, Corollary 4.3] for μ-generics.

Proposition 4.4 *Let G be a definably amenable group definable in a monster model of a theory satisfying NIP. Then*
(i) there is a definable, left invariant Keisler measure μ on G,
(ii) there are only boundedly many global μ-generic types,
(iii) for every global type p, $Stab(p) \subseteq G^{00}$,
(iv) for every definable set $X \subseteq G$, $Stab_\mu(X) := \{g \in G : \mu(gX \triangle X) = 0\}$ is a type-definable subgroup of bounded index in G; in particular, $G^{00} \subseteq Stab_\mu(X)$,
(v) for every global μ-generic type p, $Stab(p) = G^{00}$.

Proof. (i) is just Fact 1.6.
(ii) is true for an arbitrary Keisler measure and it follows from [2, Corollary 3.4].

(iii) is true whenever G^{00} exists; it follows from the fact that some translate of G^{00} is defined by a partial type contained in p.
(iv). The fact that $Stab_\mu(X)$ is a subgroup follows from left invariance of μ. The fact that $Stab_\mu(X)$ is type-definable follows from definability of μ. Finally, the fact that the index of $Stab_\mu(X)$ in G is bounded follows from [2, Corollary 3.4] and the observation that for every $g, h \in G$ we have: $gStab_\mu(X) = hStab_\mu(X)$ iff $\mu(gX \triangle hX) = 0$.
(v) Since μ is left invariant, gp is μ-generic for every $g \in G$. So $\bigcap\{Stab_\mu(X) : X \in p\} \subseteq Stab(p)$. Hence we are done by (iii) and (iv). ■

From now on, let K be an infinite field definable in a monster model of a theory T satisfying NIP. By Corollary 4.2, we can find a definable Keisler measure invariant under additive and non-zero multiplicative translations; we denote it by μ.

Proposition 4.5 *If X is a definable (or type-definable) μ-generic subset of K, then*
(i) for every $k \in K$, $(k + X) \cap X$ is μ-generic and of the same measure as X,
(ii) $K = X - X$,
(iii) for every $k \in (K^*)^{00}$, $kX \cap X$ is μ-generic and of the same measure as X,
(iv) $(K^*)^{00} \subseteq XX^{-1}$.

Proof. Items (ii) and (iv) follow from (i) and (iii), respectively. Item (i) follows from Fact 1.5 and Proposition 4.4(iv). Item (iii) follows from Proposition 4.4(iv). ■

From now on, assume that T is additionally superrosy.

Corollary 4.6 *For every natural number $n > 0$ we have $K = K^n - K^n$. Hence, if n is odd or if $\sqrt[2^k]{-1}$ exists in K where 2^k is the largest power of 2 dividing n, then $K = K^n + K^n$.*

Proof. By Fact 1.1, K^* is a union of finitely many cosets of $(K^*)^n$. So at least one and hence all the cosets of $(K^*)^n$ are μ-generic (and of the same measure). So by Proposition 4.5(ii), we get $K = K^n - K^n$. From this the second part follows immediately. ■

In fact, we get even more, namely, for every $k \in K$, $(k + K^n) \cap K^n$ is μ-generic and of the same measure as K^n. We also get that every element of K can be written as a difference $x - y$ where $x, y \in (K^*)^{00} \subseteq \bigcap_{n>0} K^n$. Notice also that this property (and, in particular, the conclusion of Corollary 4.6) holds in every finite extension of K.

Conjecture 4.7 *Assume $\sqrt{-1}$ exists in K. Then for every natural number n and $a \in K^*$ we have that $K = K^n - aK^n$.*

If we proved the above conjecture (even only for prime numbers n), we could apply the proof of [8, Theorem 4.6] to get that the Brauer group of K is trivial (assuming that $\sqrt{-1} \in K$). Corollary 4.6 is a weaker result than Conjecture 4.7.

Now using our measure μ we will easily conclude that certain particular formulas have the order property.

Proposition 4.8 *For every natural number $n > 0$, if $K^n \neq K$, then the formula $(\exists z)(x - y = z^n)$ has the order property.*

Proof. We need to find sequences $(a_i)_{i\in\omega}$ and $(b_i)_{i\in\omega}$ such that $a_i - b_j \in K^n \iff i > j$.

For the base step it is enough to choose any $b_0 \in K$ and $a_0 \notin b_0 + K^n$. In order to do that, we need to know that $b_0 + K^n \neq K$. Take any $a \notin K^n$. Since by the comment right after Corollary 4.6, $\mu((b_0 + K^n) \setminus K^n) = 0$, we get $\mu((b_0 + K^n) \cap aK^n) = 0$. As $\mu(K^n) = \mu(aK^n) > 0$, it follows that $aK^n \setminus (b_0 + K^n) \neq \emptyset$.

Suppose we have chosen $(a_i)_{i\leq m}$ and $(b_i)_{i\leq m}$ satisfying the desired property. Then it is enough to choose any $a_{m+1} \in \bigcap_{i\leq m}(b_i + K^n)$ and $b_{m+1} \notin \bigcup_{i\leq m+1}(a_i - K^n)$. In order to do that, we need to show that $\bigcap_{i\leq m}(b_j + K^n) \neq \emptyset$ and $\bigcup_{i\leq m+1}(a_i - K^n) \neq K$. By the comment right after Corollary 4.6, we get that $\bigcap_{i\leq m}(b_i + K^n)$ is μ-generic so it is nonempty. To show $\bigcup_{i\leq m+1}(a_i - K^n) \neq K$, we use a similar argument as in the base induction step. ■

Proposition 4.9 *Assume $a \notin -K^n$. Then there is an indiscernible (over a) sequence $(a_i)_{i\in\omega}$ such that:*
(i) if $1 - a \in K^n$, then $a_i - aa_j \in K^n \iff i \geq j$,
(ii) if $1 - a \notin K^n$, then $a_i - aa_j \in K^n \iff i > j$.
In particular, the formula $(\exists z)(x - ay = z^n)$ has the order property.

Proof. By compactness and Ramsey's theorem, it is enough to construct a sequence satisfying (i) or (ii). For the base step we choose any $a_0 \in (K^*)^n$.

Suppose we have chosen $(a_i)_{i\leq m}$. Now it is enough to show that the set $A := ((K^*)^n \cap \bigcap_{i\leq m}(aa_i + K^n)) \setminus \bigcup_{i\leq m}(\frac{1}{a}a_i - \frac{1}{a}K^n)$ is non-empty and to choose any $a_{m+1} \in A$.

In order to see that $A \neq \emptyset$, notice that $(K^*)^n \cap \bigcap_{i\leq m}(aa_i + K^n)$ is μ-generic and is contained in K^n. On the other hand, $\mu(\bigcup_{i\leq m}(\frac{1}{a}a_i - \frac{1}{a}K^n) \setminus -\frac{1}{a}K^n) = 0$, so $\mu(K^n \cap \bigcup_{i\leq m}(\frac{1}{a}a_i - \frac{1}{a}K^n)) = 0$ (as $-\frac{1}{a}K^n \cap K^n = \{0\}$). Hence $A \neq \emptyset$. ■

Proposition 4.10 *If n is odd, then for every $a \in K$ the formula $\phi(x, y) := (\exists z)(x - ay = z^n)$ does not have the strict order property.*

Proof. We will show that there are no $k, l \in K$ such that $\phi(K, k)$ is a proper subset of $\phi(K, l)$. Suppose $\phi(K, k) \subseteq \phi(K, l)$, i.e. $ak + K^n \subseteq al + K^n$. Then $-ak - K^n \subseteq -al - K^n$, so $-ak + K^n \subseteq -al + K^n$, so $al + K^n \subseteq ak + K^n$. ■

It is well-known that if a formula ϕ has the order property and does not have the independence property, then a conjunction of finitely many instances (or their negations) of ϕ has the strict order property. In Propositions 4.8 and 4.9, we have found (assuming that $K \neq K^n$ for some n) some particular formulas with the order property about which, by Proposition 4.10, we know that they do not have the strict order property (if n is odd). Maybe a more complicated computation (involving somehow superrosiness) could show that also finite conjunctions of instances of those formulas do not have the strict order property, and then we would get that $K = K^n$ (at least for odd n's).

References

[1] C. Ealy, K. Krupiński, A. Pillay. *Superrosy dependent groups having finitely satisfiable generics*, Annals of Pure and Applied Logic 151, 1-21, 2008.

[2] E. Hrushovski, Y. Peterzil, A. Pillay. *Groups, measures and the NIP*, Journal of AMS 21, 563-596, 2008.

[3] E. Hrushovski, A. Pillay. *On NIP and invariant measures*, preprint.

[4] S. Lang. *Algebra*, Springer, 2002.

[5] A. Onshuus. *Properties and consequences of thorn-independence*, Journal of Symbolic Logic 71, no. 1, 1-21, 2006.

[6] Y. Peterzil, S. Starchenko. *Definable homomorphisms of abelian groups in o-minimal structures*, Annals of Pure and Applied Logic 101, 1-27, 2000.

[7] A. Pillay, *Geometric stability theory*, Clarendon Press, Oxford, 1996.

[8] A. Pillay, T. Scanlon, F. Wagner. *Supersimple fields and division rings*, Mathematical Research Letters 5, 473-483, 1998.

[9] B. Poizat. *Stable Groups*, Mathematical Surveys and Monographs, 87, American Mathematical Society, Providence, 2001.

Mathematical Institute, University of Wrocław
pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland.
e-mail: kkrup@math.uni.wroc.pl
and
Mathematics Department
University of Illinois at Urbana-Champaign
1409 W. Green Street, Urbana, IL 61801, USA.