On Podewski's conjecture

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Abstract

A long-standing conjecture of Podewski states that every minimal field is algebraically closed. It was proved by Wagner for fields of positive characteristic, but it remains wide open in the zero-characteristic case.

We reduce Podewski's conjecture to the case of fields having a definable (in the pure field structure), well partial order with an infinite chain, and we conjecture that such fields do not exist. Then we support this conjecture by showing that there is no minimal field interpreting a linear order in a specific way; in our terminology, there is no almost linear, minimal field.

On the other hand, we give an example of an almost linear, minimal group $(M, <, +, 0)$ of exponent 2, and we show that each almost linear, minimal group is torsion.

An infinite first-order structure is called *minimal* if every definable (with parameters) subset is either finite or co-finite (i.e. its complement is finite).

In this paper, we will deal mostly with minimal fields and groups. Minimal, pure groups were classified by Reineke in [5]. They are either (abelian) divisible with only finitely many elements of any given finite order, or elementary abelian of prime exponent. As to the fields, it is well-known that every algebraically closed, pure field is minimal, while the converse remains one of the oldest unproved conjectures in model theory [4].

Conjecture 1. *Each minimal field is algebraically closed.*

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In [8], Wagner proved this conjecture for fields of positive characteristic, but the zero-characteristic case is still open. In this paper, we reduce the problem to studying minimal fields with an ordering relation which, at first glance, has nothing to do with the field structure. More precisely, we prove that Podewski's conjecture follows from the following one.

Conjecture 2. *There is no minimal field K of characteristic zero in which there exists a definable partial order on K with infinite chains.*

The fact that Conjecture 2 implies Conjecture 1 follows from a dichotomy described in Proposition 1.1 and the following theorem proved in Section 3 (see Definition 1 for the notion of symmetric, minimal fields).

Theorem 1. *Each symmetric, minimal field is algebraically closed.*

We say that a structure $(M, <, \dots)$ is *ordered* if $<$ is a strict partial order with arbitrarily long finite chains. It is easy to see that minimal, ordered structures have infinite chains. Basic examples of minimal, ordered structures are $(\omega, <)$, $(\omega^*, <)$ and $(\omega + \omega^*, <)$ (where $(\omega^*, <)$ is the reversely ordered ω). These are linear orders. Most of the known non-linear, ordered, minimal structures derive from them by replacing elements by sufficiently large finite antichains and then adding a finite set arbitrarily. For example, consider $\{(n, m) \in \omega \times \omega \mid m \leq n\}$ and order it by $(n, m) < (n', m')$ iff $n < n'$. Here, the order is not far from being linear, namely the reflexive incomparability relation (defined by $x \sim y$ iff $\neg(x < y \vee y < x)$) is an equivalence relation, and, after factoring it out, we end up with $(\omega, <)$. The minimal, ordered structures in which there is a definable order such that the incomparability becomes an equivalence relation (necessarily yielding a linear order of type ω or ω^* or $\omega + \omega^*$) will be called *almost linear* (see Definition 2 and Remark 1.1). In Section 2, we will give an example of an almost linear, minimal group, whereas in Section 4, we will prove that there are no such fields, partially confirming Conjecture 2.

Theorem 2. *There is no almost linear, minimal field.*

Theorem 2 implies that a possible counterexample to Conjecture 2 would have to be a field which is not almost linear. It is hard to believe that such structures exist (all the known examples of minimal, ordered structures are almost linear).

Question 1. *Does there exist a minimal, ordered structure which is not almost linear?*

A negative answer to this question would imply the validity of Conjectures 2 and 1.

At the end notice that Theorem 2 makes a difference between minimal and quasi-minimal (uncountable structures whose definable subsets are countable or co-countable) fields: an almost linear, quasi-minimal field exists (see Example 5.1 in [3]).

1 Minimal structures with definable generic type

Let M be a minimal structure. A unique non-algebraic type $p \in S_1(M)$ will be called the *generic type* of M. In this section, we are interested in minimal structures whose generic type is definable. This class is interesting because of the next lemma, noticed by A. Pillay.

Lemma 1.1. *The generic type p of a minimal group G is its unique generic type in the sense of left [and right] translates, i.e. a formula $\phi(x)$ (with parameters from M) belongs to p if and only if finitely many left [right] translates of $\phi(G)$ cover G. In fact, $\phi(x) \in p$ if and only if two left [right] translates of $\phi(G)$ cover G. This characterization of p implies that p is definable over $\emptyset$.*

Proof. If finitely many translates of $\phi(G)$ cover G, then $\phi(G)$ is infinite, hence $\phi(x) \in p$. For the converse, suppose $\phi(x) \in p$. Then $G \setminus \phi(G)$ is a finite set $\{g_1, \dots, g_n\}$. Since the sets $g_1\phi(G)^{-1}, \dots, g_n\phi(G)^{-1}$ are co-finite, their intersection is non-empty, so it contains an element g. It is clear now that $G = \phi(G) \cup g\phi(G)$. □

Let M be a minimal structure whose generic type $p \in S_1(M)$ is definable over A. Let $\bar{M} \succ M$ be a monster model. Then p has a unique global heir $\overline{p} \in S_1(\bar{M})$, which is defined by the same defining scheme as p. For $C \subseteq \bar{M}$ containing A by a Morley sequence in p over C (or in $p|C$) we mean a sequence $(a_i : i \in \kappa)$ such that $a_i \models \overline{p}|Ca_{<i}$ for all $i \in \kappa$. Each Morley sequence in p over C is indiscernible over C, and the type over C of such a

sequence of a fixed length κ does not depend on its choice. Recall that for a formula $\varphi(x;\overline{y})$ by $(d_p x)\varphi(x,\overline{y})$ we denote a $\varphi(x;\overline{y})$-definition of p.

Now, we will prove a version of the dichotomy theorem for minimal structures from [7] in our context. The proof uses arguments from [3], where a similar result was proved for locally strongly regular types, and the description of minimal, ordered structures from [6].

Proposition 1.1. *Suppose that M is a minimal structure whose generic type $p \in S_1(M)$ is definable over A, and let (a,b) be a Morley sequence in p (over M). Then exactly one of the following two cases holds.*

1. (Symmetric) $\mathrm{tp}(a,b/M) = \mathrm{tp}(b,a/M)$. *In this case, a Morley sequence in p over any $C \subseteq \bar{M}$ containing A is totally indiscernible over C (i.e. indiscernible over C as a set).*

2. (Non-symmetric) $\mathrm{tp}(a,b/M) \neq \mathrm{tp}(b,a/M)$. *In this case, there is an M-definable, strict partial order on M such that $a < b$ and $M < a$. Moreover, $(M,\leq)$ is a directed, well partial order having infinite increasing chains and no such chain of order type $\omega+1$.*

Proof. 1. Suppose that (a',b') is a Morley sequence in p over C and that $\mathrm{tp}(a,b/M) = \mathrm{tp}(b,a/M)$ (recall that (a,b) is a Morley sequence in p). Symmetry implies that for any $\phi(x,y,\overline{z})$ without parameters and for any $\overline{m} \in M$ we have

$$\models (d_p x)\,(d_p y)(\phi(x,y,\overline{m}) \leftrightarrow \phi(y,x,\overline{m})).$$

Then the same formula is satisfied by any $\overline{c} \in \bar{M}$ in place of $\overline{m}$, so $\mathrm{tp}(a',b'/C) = \mathrm{tp}(b',a'/C)$. By induction, it follows easily that Morley sequences of any length are totally indiscernible.

2. Suppose $\mathrm{tp}(a,b/M) \neq \mathrm{tp}(b,a/M)$. We will find $\phi(x,y) \in \mathrm{tp}(a,b/M)$ such that:

(1) $\models (\forall x,y)(\phi(x,y) \rightarrow \neg\phi(y,x))$ ($\phi(x,y)$ is asymmetric);

(2) $\phi(M,b) = M$ (i.e. $\phi(c,y) \in p(y)$ for all $c \in M$);

(3) $\phi(a,y) \vdash p(y)$.

Since p is definable, it has a unique heir and a unique coheir in $S_1(Ma)$; $\mathrm{tp}(a,b/M) \neq \mathrm{tp}(b,a/M)$ implies that the two are distinct. Thus, $\mathrm{tp}(b/Ma)$ is not a coheir, so there is $\phi(a,y) \in \mathrm{tp}(b/Ma)$ which is satisfied by no element of M. The formula $\phi(x,y)$ satisfies condition (3), and, after slightly modifying it, we also get that (1) fulfilled. Since $\mathrm{tp}(a/Mb)$ is a coheir, $\phi(M,b)$ is infinite. Hence, by the definability of p, $\phi(M, b)$ is co-finite, and another slight modification guarantees that (2) is fulfilled as well.

We leave to the reader to verify that the formula

$$\phi(x, y) \wedge (\forall t)(\phi(y, t) \rightarrow \phi(x, t))$$

defines a strict partial order; denote it by $x < y$. Now, we prove that for any $c \in M$ we have $c < b$, i.e.

$$\models \phi(c, b) \wedge (\forall t)(\phi(b, t) \rightarrow \phi(c, t)).$$

$\models \phi(c, b)$ holds by (2). Let d be such that $\models \phi(b, d)$. Then, by (3), d realizes p, so $\phi(c, y) \in p(y)$ implies $\models \phi(c, d)$. Thus, $c < b$. Since this holds for all $c \in M$ and $\mathrm{tp}(a/bM)$ is finitely satisfiable in M, we conclude that $a < b$. We also noticed that $M < b$, which by virtue of the fact that $\mathrm{tp}(a/M) = \mathrm{tp}(b/M)$ implies $M < a$.

Now, the formula $x < b$ belonging to $\mathrm{tp}(a/bM)$ is satisfied in M, say by $c_1 \in M$. Then $c_1 < b$; in fact, since $M < a$, we also have $c_1 < a$. Hence, the formula $c_1 < x < b$ belongs to $\mathrm{tp}(a/bM)$, and so there is $c_2 \in M$ such that $c_1 < c_2 < b$; then also $c_2 < a$, so the formula $c_1 < c_2 < x < b$ belongs to $\mathrm{tp}(a/bM)$. Continuing in this way, we get an infinite increasing chain in M. We leave to the reader to verify that $\leq$ is a directed, well partial order with no chains of order type $\omega + 1$. □

The above proposition leads to the following definition.

Definition 1. Let M be a minimal structure whose generic type $p \in S_1(M)$ is definable over A. We say that M is *symmetric* if $\mathrm{tp}(a, b/M) = \mathrm{tp}(b, a/M)$ for each/some Morley sequence (a, b) in p over M; equivalently, if Morley sequences (of arbitrary length) in p over any set C containing A are totally indiscernible over C.

Now, fix for a while a minimal structure M, and let $p \in S_1(M)$ be its generic type. Recall that $\bar{M} \succ M$ is a monster model. From now on, whenever a definable order $<$ is clear from the context, $x \sim y$ will be defined as $\neg(x < y \vee y < x)$.

Definition 2. A definable order $<$ on M is *almost linear in M* if it has infinite chains and $\sim$ is an equivalence relation on $p(\bar{M})$. M is *almost linear* if such an order exists.

Remark 1.1. *Suppose that $<$ is a definable partial order on M with infinite chains. Then the following conditions are equivalent.*

(i) $<$ is almost linear in M.

(ii) After a modification of $<$ on a finite subset of M, $\sim$ becomes an equivalence relation on M (we allow here modifications of $<$ between elements of this finite set and all other elements, but in such a way that the resulting order is definable in M).

(iii) After a modification of $<$ on a finite subset of M, $\sim$ becomes an equivalence relation on M, and $M/\sim$ is totally ordered by the induced order $</\sim$ in one of the following order types: $(\omega, <)$, $(\omega^*, <)$ or $(\omega + \omega^*, <)$.

Proof. The implications $(iii) \to (ii) \to (i)$ are clear. In order to see that $(i) \to (ii)$, one should use compactness and the minimality of M. The fact that $(ii) \to (iii)$ follows from [6]. □

Remark 1.2. *Assume that a minimal, ordered structure $(M, <, \dots)$ contains an infinite increasing chain, and its generic type p is definable. Then $C := \{c \in M \mid c < x \in p(x)\}$ is a definable, co-finite subset of M, and hence $<$ is a well partial order and M is non-symmetric. After modifying $<$ so that the elements of $M \setminus C$ are below all the others, we get that $C = M$, and then $<$ is a directed, well partial order having infinite increasing chains and no such chain of order type $\omega + 1$. If, in addition, $<$ is almost linear in M, then after modifying $<$ on a finite set, we get that $M/\sim$ is ordered in the order type of ω.*

Of course, analogous observations are true when the structure $(M, <, \dots)$ contains an infinite decreasing chain. Proposition 1.1 together with Remark 1.2 yield the following observation.

Remark 1.3. *Let M be a minimal structure whose generic type is definable. Then M is non-symmetric iff there exists a definable partial order $<$ on M with an infinite chain (equivalently, there exists a definable partial order $<$ on M such that M expanded by $<$ is a minimal, ordered structure.)*

Lemma 1.2. *Suppose that the generic type of a minimal structure M is definable and that $<$ is an almost linear order in M with an infinite increasing chain. Then, whenever $(a_i \mid i \in \omega) \subseteq M$ is a sequence of pairwise distinct elements of M, we have $a_i < a_{i+1}$ for infinitely many i's.*

Proof. By Remark 1.2, we can assume that $M/\sim$ is ordered in the order type of ω. Let $(a_i \,|\, i \in \omega)$ be a sequence of pairwise distinct elements of M. Then the sequence $(a_i/\sim \,|\, i \in \omega)$ is infinite, because every antichain is finite. Since $M/\sim$ is ordered in the order type of ω, there exist infinitely many i's such that $a_i/\sim <_\sim a_{i+1}/\sim$. For those i's we have $a_i < a_{i+1}$. □

The next lemma will be used in Section 2 and, in consequence, in Section 4.

Lemma 1.3. *Assume the generic type of an almost linear, minimal structure M is definable. Let $f : M \to M$ be a definable function. Then there does not exist a sequence of pairwise distinct elements $(a_i \,|\, i \in \mathbb{Z})$ such that $f(a_i) = a_{i+1}$ for all $i \in \mathbb{Z}$.*

Proof. By almost linearity, there is an almost linear order $<$ in M with an infinite increasing chain. By Lemma 1.2, for infinitely many $i > 0$ we have $a_i < a_{i+1}$, and, similarly, for infinitely many $i < 0$ we have $a_i < a_{i-1}$. Since $a_{i+1} = f(a_i)$ for all $i \in \mathbb{Z}$, we conclude that the disjoint, definable sets $\{m \in M \,|\, f(m) > m\}$ and $\{m \in M \,|\, f(m) < m\}$ are infinite, which contradicts minimality. □

We finish this section with an example of a minimal, ordered structure from [1, Section 4.2]. Although the structure is almost linear, it has a definable order with infinite chains which is not almost linear.

Example 1. Let $M = \omega \times \{l, r\}$. Define an order $<$ on M by putting the natural orders on $\omega \times \{l\}$ and on $\omega \times \{r\}$, together with

$$(x, l) < (y, r) \iff x + 2 \leq y \quad \text{and} \quad (y, r) < (x, l) \iff y + 2 \leq x$$

for all natural numbers x and y.

We leave to the reader to verify that $(M, <)$ is minimal (or see [1] for a proof). For $n \in \omega$ define: $a_{2n} = (2n, l)$ and $a_{2n+1} = (2n + 1, r)$. Then a_n and a_{n+1} are incomparable for all n, and so, by Lemma 1.2, $<$ is not almost linear in M.

Now, we show that M interprets $(\omega, <)$. First note that 'y is maximal incomparable to x' is a definable function $f(x) = y$. Then for $x, y \in M$ define:

$$x <' y \quad \text{if and only if} \quad x < y \;\text{ or }\; y = f(x)$$

The $<'$-incomparability is an equivalence relation with 2-element classes $\{(n,r),(n,l)\}$, and, after factoring it out, we end up with $(\omega,<)$. By Remark 1.1, $<'$ is almost linear in M, so M is almost linear.

2 Almost linear, minimal groups

In this section, we give an example of an almost linear, minimal group of exponent 2. In particular, this is an example of a non-symmetric, minimal group. Then we notice that each almost linear, minimal group is torsion.

Example 2. Consider a structure $(M_0,+,<,0)$, where

- $(M_0,+,0)$ is the group of exponent 2 with neutral element 0 spanned freely over $\mathbb{Z}_2$ by $\{e_i \mid i\in\omega\}$, i.e. $M_0=\bigoplus_{i\in\omega}\mathbb{Z}_2\, e_i$.
- If $a=e_{n_1}+\cdots+e_{n_k}$ and $b=e_{m_1}+\cdots+e_{m_l}$ (where both tuples $(n_1,\ldots,n_k)$ and $(m_1,\ldots,m_l)$ consist of pairwise distinct numbers), then
$$a<b \iff \max\{n_1,\ldots,n_k\}<\max\{m_1,\ldots,m_l\}.$$
- $0<a$ holds for all $a\neq 0$.

In other words, M_0 is a subgroup of $\mathbb{Z}_2^\omega$ consisting of the elements with almost all coordinates equal to zero, while $a<b$ holds iff the largest non-zero coordinate of a is placed on the left of the largest non-zero coordinate of b. Thus, $0<e_0<e_1<\ldots$ and $\{e_i \mid i\in\omega\}$ is a *linearly ordered basis* (meaning that it is a basis of M_0 as a vector space over $\mathbb{Z}_2$ which is totally ordered by the restriction of $<$). Let $T=\mathrm{Th}(M_0,+,<,0)$ and let $\bar{M}\succ M_0$ be a monster model. Let $x\sim y$ denote $\neg(x<y\vee y<x)$. Note that $\sim$ defines an equivalence relation on M_0. We proceed with a sequence of easy claims, leaving most of the details to the reader.

(1) $T\models(\forall x\neq 0)(\forall y\neq 0)(x\sim y\leftrightarrow x+y<y)$

(2) $T\models(\forall x_1,\ldots,x_n)(x_1<\cdots<x_n\rightarrow x_1+\cdots+x_n\sim x_n)$

(3) Every chain $a_0<\cdots<a_n$ of non-zero elements of $\bar{M}$ is linearly independent, and the function $f:\{e_0,\ldots,e_n\}\to\{a_0,\ldots,a_n\}$ sending e_i to a_i (for each i) extends to an isomorphism between $(\mathrm{span}(e_0,\ldots,e_n),+,<,0)$ and $(\mathrm{span}(a_0,\ldots,a_n),+,<,0)$. To see this, note that no finite sum of a_i's equals 0, since, by (2), it is in the $\sim$-class of the largest a_i. Thus, a_i's are

linearly independent. In fact, by (2), we have that whenever both tuples $(n_1, \ldots, n_k)$ and $(m_1, \ldots, m_l)$ consist of pairwise distinct numbers, then

$$a_{n_1} + \cdots + a_{n_k} < a_{m_1} + \cdots + a_{m_l} \iff \max\{n_1, \ldots, n_k\} < \max\{m_1, \ldots, m_l\},$$

so f defined by $f(e_i) = a_i$ extends to an isomorphism of the ordered spans.

(4) For every $M \models T$, any linearly ordered basis of M contains a unique representative of each non-zero $\sim$-class. To prove this, let $a \in M \setminus \{0\}$ and let B be a linearly ordered basis of M. Then a is a finite sum of elements of B and, by (2), it is in the $\sim$-class of the largest element from the sum.

(5) If $M \models T$ is countable, then any finite, totally ordered subset of $M \setminus \{0\}$ is contained in a linearly ordered basis. To prove this, suppose that $\{a_0, \ldots, a_n\}$ is totally ordered by $<$. Then, by (3), it is linearly independent, so it can be extended to a basis $A = \{a_i \,|\, i \in \omega\}$ of M. Inductively we will define a new basis $\{b_i \,|\, i \in \omega\}$ satisfying our requirements. Let $b_i = a_i$ for $i \leq n$. Consider $a_{n+1} + \mathrm{span}(b_0, \ldots, b_n)$ and let b_{n+1} be its minimal element. Then b_{n+1} is $\sim$-equivalent with no $b \in \mathrm{span}(b_0, \ldots, b_n)$, as otherwise, by (1), we would have $b_{n+1} > b_{n+1} + b$, so $b_{n+1} + b \in a_{n+1} + \mathrm{span}(b_0, \ldots, b_n)$ would contradict the minimality of b_{n+1}. Thus, $(b_0, \ldots, b_{n+1})$ is totally ordered and $\mathrm{span}(a_0, \ldots, a_{n+1}) = \mathrm{span}(b_0, \ldots, b_{n+1})$. Continuing in this way, we get a totally ordered basis $\{b_i \,|\, i \in \omega\}$ containing $\{a_0, \ldots, a_n\}$.

(6) $M_0 = \mathrm{acl}(\emptyset)$, so $M_0 \prec M$ holds for any $M \models T$.

Now, let $M \models T$ be countable, and we will prove that all elements of $M \setminus M_0$ have the same type over M_0. First, by compactness, we easily find a countable $M_{\mathbb{Q}} \succ M$ such that $(M_{\mathbb{Q}}/\sim, <_\sim)$ is ordered in the order type $(\omega + \mathbb{Q} \times \mathbb{Z}, <)$. It suffices to prove that for $a, b \in M_{\mathbb{Q}} \setminus M_0$ there exists an M_0-automorphism of $M_{\mathbb{Q}}$ moving a to b. Assume $a > b$ (the case $a \sim b$ is left to the reader). Since $a > b$, by (5), there is a linearly ordered basis $A \supset \{a, b\}$ of $M_{\mathbb{Q}}$. By (4), A contains exactly one representative from each non-zero $\sim$-class, so it is $<$-ordered in the order type $(\omega + \mathbb{Q} \times \mathbb{Z}, <)$. Let $A = \{a_i \,|\, i \in \omega + \mathbb{Q} \times \mathbb{Z}\}$ be an enumeration of A such that the order of indices in $\omega + \mathbb{Q} \times \mathbb{Z}$ agrees with the order of the corresponding elements in $M_{\mathbb{Q}}$. Then $M_0 = \mathrm{span}(a_i \,|\, i \in \omega)$ and $a, b \in \{a_i \,|\, i \in \mathbb{Q} \times \mathbb{Z}\}$. There is an order preserving permutation f of A mapping a to b; note that it necessarily fixes $\{a_i \,|\, i \in \omega\}$ pointwise. By (3), f can be extended to an automorphism $\hat{f}$ of $M_{\mathbb{Q}}$. Then $\hat{f}$ fixes pointwise $M_0 = \mathrm{span}(a_i \,|\, i \in \omega)$ and maps a to b. We conclude that $\mathrm{tp}(a/M_0) = \mathrm{tp}(b/M_0)$, as desired.

We have just proved that for every M the elements in $M \setminus M_0$ have the same type over M_0. It follows that there is a unique non-algebraic 1-type in $S(M_0)$, and so M_0 is a minimal group. It is clearly ordered (so non-symmetric by Remark 1.3) and almost linear.

Proposition 2.1. *Each almost liner, minimal group G is is torsion.*

Proof. Suppose for a contradiction that there is $g_0 \in G$ of infinite order. Consider a definable function $f : G \to G$ defined by $f(x) = x + g_0$. Put $a_i = ig_0$ for $i \in \mathbb{Z}$, a sequence of pairwise distinct elements. Since $f(a_i) = a_{i+1}$ holds for all $i \in \mathbb{Z}$ and, by Lemma 1.1, the generic type of G is definable, we get a contradiction with Lemma 1.3. □

Notice that the above proposition together with [8] imply that almost linear, minimal fields are algebraically closed. However, in Section 4, we will show that such fields do not exist at all.

Question 2. *Does there exist an ordered, minimal structure which is a non-torsion group (i.e. a group with an element of infinite order)?*

If the answer to this question was negative, then Proposition 1.1 together with Theorem 1 (proved in the next section) and Wagner's theorem would imply Podewski's conjecture.

We conclude this section with a lemma which will be used in the last section. Note that it refers to *any* minimal ordered group (so $\sim$ may not be an equivalence relation and there may be elements of infinite order).

Lemma 2.1. *Let $(G, <, +, 0, \dots)$ be a a minimal, ordered structure, where $(G, +, 0)$ is a group, and let $g \in \bar{G} \setminus G$. Then $g + a \sim g$ for all $a \in G$.*

Proof. Since the generic type is definable and g is generic over G, the following subsets of G are definable.

$$A = \{x \in G \,|\, g+x > g\}, \;\; B = \{x \in G \,|\, g+x < g\}, \;\; C = \{x \in G \,|\, g+x \sim g\}.$$

Each of them is either finite or co-finite and they form a partition of G.

Now, we show that $x \mapsto -x$ defines a bijection between A and B. For this, suppose that $x \in A$. Note that the sets A, B and C do not depend on the choice of $g \in \bar{G} \setminus G$ (as all elements from $\bar{G} \setminus G$ realize the unique generic type over G). Hence, since $g + x \in \bar{G} \setminus G$ and $(g + x) + (-x) = g < g + x$, we see that $-x \in B$. Similarly, if $x \in B$, then $-x \in A$.

By minimality, we conclude that A and B are finite. Now, suppose that $a \in A$. Then, since $g + a$ is generic, $g < g + a < g + 2a < \dots$. Hence, the elements na (for $n \geq 1$) are pairwise distinct and they belong to A, which is not possible since A is finite. Thus $A = B = \emptyset$. $\square$

3 Minimal fields, the symmetric case

The main result of this section is Theorem 1 from the introduction. In order to prove it, first we make some preliminary observations about minimal fields in general.

In this section, K is a minimal field and $\mathcal{K} \succ K$ is a monster model. As usual, $p \in S_1(K)$ denotes the generic type of K and $\overline{p}$ its unique heir in $S_1(\mathcal{K})$ (see Lemma 1.1 and the paragraph below it). For $A \subseteq \mathcal{K}$ we say that $g \in \mathcal{K}$ is generic over A if $g \models \overline{p}|A$. Recall that from Lemma 1.1 we know that a formula $\varphi(x)$ belongs to p [or to $\overline{p}$] iff it is generic in the sense of translates iff two left translates of $\varphi(K)$ [or of $\varphi(\mathcal{K})$] cover the whole model.

Remark 3.1. *(i) $f[\mathcal{K}]$ is generic for every non-constant polynomial $f(x) \in \mathcal{K}[x]$.*

(ii) Suppose that $f(x) = c_n x^n + \dots + c_1 x + c_0 \in \mathcal{K}[x]$ is a non-constant polynomial and that c_0 is generic over $c_1, \dots, c_n$. Then f has a zero in $\mathcal{K}$.

(iii) $K^n = K$ for every $n > 0$. In particular, K is perfect.

Proof. (i) Since for any $c \in K$, $f(x) = c$ has only finitely many solutions, $f[K]$ is co-finite and hence generic, for every non-constant polynomial $f(x) \in K[x]$. The definability of p implies that $f[\mathcal{K}]$ is generic for every non-constant polynomial $f(x) \in \mathcal{K}[x]$.

(ii) Let $g(x) = f(x) - c_0$. Then $g(x)$ is non-constant and, by (i), $g[\mathcal{K}]$ is generic. Since $-c_0$ is generic over $c_1, \dots, c_n$, we get that $-c_0 \in g[\mathcal{K}]$, and so $f(x) = 0$ has a solution.

(iii) By (i), $(K^*)^n$ is a co-finite subgroup of K^*, so $(K^*)^n = K^*$ and $K^n = K$. $\square$

Lemma 3.1. *If g is generic over A and $g \in \operatorname{acl}(h, A)$, then h is generic over A.*

Proof. Suppose for a contradiction that h is not generic over A. Let $\varphi(x, h, \bar{a})$ witnesses $g \in \operatorname{acl}(h, A)$ and let $\psi(y, \bar{a})$ witnesses that that h is not generic over A (where $\bar{a} \subseteq A$):

(1) $\models \varphi(g,h,\bar{a}) \wedge \psi(h,\bar{a}) \wedge \neg(d_p u)\psi(u,\bar{a})$;

(2) $\varphi(\mathcal{K},h',\bar{a}') \subseteq \mathrm{acl}(h',\bar{a}')$ for all $h',\bar{a}' \in \mathcal{K}$; in particular, $\varphi(x,h',\bar{a}')$ is non-generic.

Thus, $(\exists y)\,(\varphi(x,y,\bar{a}) \wedge \psi(y,\bar{a}) \wedge \neg(d_p u)\psi(u,\bar{a})) \in \overline{p}|A(x)$, so there is $\bar{a}'$ in K such that

$$(\exists y)\,(\varphi(x,y,\bar{a}') \wedge \psi(y,\bar{a}') \wedge \neg(d_p u)\psi(u,\bar{a}')) \in p(x).$$

Since $\psi(y,\bar{a}') \wedge \neg(d_p u)\psi(u,\bar{a}')$ is non-generic, it defines a finite set, say $\{c_1,\dots,c_n\}$. Then, $\bigvee_{i=1}^{n} \varphi(x,c_i,\bar{a}')$ is generic, and so some $\varphi(x,c_i,\bar{a}')$ is also generic, which is in contradiction with (2). □

Lemma 3.2. *Assume K is symmetric. Let $(a_0,\dots,a_n)$ be a Morley sequence in p over K, and let $(b_0,\dots,b_n)$ be a sequence interalgebraic over K with $(a_0,\dots,a_n)$ in the field-theoretic sense. Then $(b_0,\dots,b_n)$ is also a Morley sequence in p over K.*

Proof. In this proof, algebraic dependence is considered only in the field-theoretic sense. By assumption, $\mathrm{trdeg}_K(b_0,\dots,b_n) = \mathrm{trdeg}_K(a_0,\dots,a_n) = n+1$, so $b_0,\dots,b_n$ are algebraically independent over K. Then there is a_{i_0} such that b_0 is interalgebraic with a_{i_0} over $K, a_{\neq i_0}$. By symmetry, a_{i_0} is generic over $K, a_{\neq i_0}$, and, since $a_{i_0} \in \mathrm{acl}(K, a_{\neq i_0}, b_0)$, we can apply Lemma 3.1 concluding that b_0 is generic over $K, a_{\neq i_0}$. It follows that $(a_{\neq i_0}, b_0)$ is a Morley sequence in p which is interalgebraic with $(b_0,\dots,b_n)$ over K. Now, we repeat this procedure with b_1 and $(a_{\neq i_0}, b_0)$ in place of b_0 and $(a_0,\dots,a_n)$. Since b_0 and b_1 are algebraically independent over K, there is $i_1 \neq i_0$ such that b_1 is interalgebraic with a_{i_1} over $K, a_{\neq i_0,i_1}, b_0$. Then, by symmetry and Lemma 3.1, $(a_{\neq i_0,i_1}, b_0, b_1)$ is a Morley sequence in p which is interalgebraic with $(b_0,\dots,b_n)$ over K. Continuing in this way, we finally obtain that $(b_0,\dots,b_n)$ is a Morley sequence in p. □

Having Lemmas 3.1 and 3.2, in order to prove the next theorem, we will argue as in the second part of the proof of Proposition 2.2 from [2].

Theorem 1. *Each symmetric, minimal field K is algebraically closed.*

Proof. Let F denote an algebraic closure of $\mathcal{K}$. Suppose for a contradiction that some $\alpha_1 \in F \setminus K$ is algebraic over K. Let $f(x) = x^n + a_{n-1}x^{n-1} + \dots + a_0 \in K[x]$ be the minimal polynomial of α_1 over K. Since K is perfect, f has n pairwise distinct roots $\alpha_1,\dots,\alpha_n$ in F. Let $(t_0,\dots,t_{n-1}) \subset \mathcal{K}$ be a Morley sequence in p over K, and define $r_i = t_0 + t_1\alpha_i + \cdots + t_{n-1}\alpha_i^{n-1}$ for $i = 1, \ldots, n$. Then

$$\begin{pmatrix} 1 & \alpha_1 & \ldots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \ldots & \alpha_2^{n-1} \\ \vdots & \vdots & \ldots & \vdots \\ 1 & \alpha_n & \ldots & \alpha_n^{n-1} \end{pmatrix} \begin{pmatrix} t_0 \\ t_1 \\ \vdots \\ t_{n-1} \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix}$$

and, since the matrix is invertible, $(r_1, \ldots, r_n)$ and $(t_0, \ldots, t_{n-1})$ are field-theoretically interalgebraic over K.

Let $c_0, \ldots, c_{n-1}$ be the symmetric functions of $r_1, \ldots, r_n$. Then, of course, $(c_0, \ldots, c_{n-1})$ and $(r_1, \ldots, r_n)$ are field-theoretically interalgebraic over K, so $(t_0, \ldots, t_{n-1})$ and $(c_0, \ldots, c_{n-1})$ are field-theoretically interalgebraic over K, too. Since $c_i, t_j \in \mathcal{K}$, we can apply Lemma 3.2 to conclude that $(c_0, \ldots, c_{n-1})$ is a Morley sequence in p over K. But then, by Remark 3.1(ii), the polynomial $x^n + c_{n-1}x^{n-1} + \cdots + c_0 \in K[x]$ has a zero in $\mathcal{K}$, so there is i such that $r_i = t_0 + t_1\alpha_i + \cdots + t_{n-1}\alpha_i^{n-1} \in \mathcal{K}$. This means that the degree of the minimal polynomial of α_i over $K(t_0, \ldots, t_{n-1})$ is smaller than n. Since $K(t_0, \ldots, t_{n-1}) \subseteq \mathcal{K}$, this implies that the degree of the minimal polynomial of α_i over $\mathcal{K}$ is also smaller than n. On the other hand, as $K \prec \mathcal{K}$ and $f(x) \in K[x]$ is the minimal polynomial of α_i over K (so it is irreducible in $K[x]$), we get that $f(x)$ is also irreducible in $\mathcal{K}[x]$, and so it is the minimal polynomial of α_i over $\mathcal{K}$. This is a contradiction, because $\deg(f) = n$. □

The following result is a corollary of Proposition 1.1 and Theorem 1.

Corollary 3.1. *Each minimal field whose theory does not have the strict order property is algebraically closed. In particular, each minimal field whose theory is simple is algebraically closed.*

4 Non-symmetric minimal fields

In Section 3, we proved that symmetric, minimal fields are algebraically closed. So, in order to prove Podewski's conjecture, it remains to show that non-symmetric, minimal fields are algebraically closed; we even conjecture that such fields do no exist at all. Recall that by Proposition 1.1, we know that non-symmetric, minimal fields are ordered structures with respect to some definable order. So, our goal (see Conjecture 2) is to show that there is no field whose expansion by some order is a minimal, ordered structure. In this section, we achieve this goal in the special case of almost linear structures. Namely, we will prove Theorem 2 from the introduction.

So, throughout, suppose for a contradiction that $(F, <, +, \cdot, 0, 1)$ is a minimal field in which the order $<$ is almost linear, and let $\bar{F} \succ F$ be a monster model. By Remark 1.2, we can assume that $\sim$ is an equivalence relation on F and that $F/\sim$ is ordered in the order type of ω. In particular, $(F, <)$ is a well partial order, so it satisfies:

(MIN) Every definable subset has a minimal element.

Since this is an elementary property, $\bar{F}$ also satisfies (MIN).

Claim 1. *F is locally finite.*

Proof. By Proposition 2.1, both F^+ and F^* are torsion groups, so F is locally finite. □

Fix $g \in \bar{F} \setminus F$; then g is generic over F. By Lemma 2.1, for all $a \in F$ and $a \neq 0$ we have:

$$g + a \sim g \quad \text{and} \quad g \cdot a \sim g. \tag{1}$$

Let $\bar{F}_<(a) = \{x \in \bar{F} \,|\, x < a\}$ for $a \in \bar{F}$.

Claim 2. *$\bar{F}_<(g)$ is a subfield of $\bar{F}$.*

Proof. First, we prove that $\bar{F}_<(g)$ is closed under addition. Otherwise

$$D(g) := \{x \in \bar{F}_<(g) \,|\, (\exists y)(y \in \bar{F}_<(g) \wedge x + y \notin F_<(g)\} \neq \emptyset.$$

By (MIN), $D(g)$ has a minimal element a. Let $b < g$ witness that $a \in D(g)$. Then $a + b \notin F_<(g)$ implies that either $a + b \sim g$ or $g < a + b$ holds. In both cases, since $F < g$, the element $a + b$ is generic over F and, by almost linearity, we get that $a < a + b$ and $b < a + b$ hold. Furthermore, since $a + b$ is generic, at least one of a and b, say a, has to be generic over F. By (1), we have $a \sim a + F$, so $a < a + b$ implies $b \notin F$. Thus, a and b are generic over F and they are strictly smaller then $a + b$. In particular, we conclude that every element which is generic over F can be written as a sum of two strictly smaller elements.

Since we have shown that a is generic over F, $a = a' + b'$ for some $a', b' < a$. Then $b' < a$, the minimality of a and the fact that $b \in \bar{F}_<(g)$ imply $b + b' \in \bar{F}_<(g)$. On the other hand, we have that $a' + (b' + b) = a + b \notin \bar{F}_<(g)$.

Since $a', b'+b \in \bar{F}_<(g)$ and $a' < a$, we get a contradiction with the minimality of a. This completes the proof of the fact that $\bar{F}_<(g)$ is closed under addition. A similar argument shows that $\bar{F}_<(g)$ is closed under multiplication.

It is easy to see that $-x \sim x$ and $x^{-1} \sim x$ hold for any element from $\bar{F} \setminus F$ (because each such element is generic). Since $\sim$ is an equivalence relation on $\bar{F}$ and $F < g$, we conclude that $\bar{F}_<(g)$ is closed under both additive and multiplicative inverses.

Summarizing, we get that $\bar{F}_<(g)$ is a subfield of $\bar{F}$. $\square$

The previous claim implies that for almost all $a \in F$, $\bar{F}_<(a)$ is a finite subfield of F. Now, consider $f : F \to F$ defined by $f(x) = x^p - x$. It is an additive endomorphism and $\mathrm{Ker}(f) = \mathbb{Z}_p$. f is surjective, since F has no proper, infinite, definable subgroups. On the other hand, if $\bar{F}_<(a)$ is a finite field, then $f[\bar{F}_<(a)]$ is a proper subset of $\bar{F}_<(a)$, and, since f is surjective, there is $b \notin \bar{F}_<(a)$ such that $f(b) \in \bar{F}_<(a)$; note that the fact that $\sim$ is an equivalence relation on F implies $f(b) < b$. Since $a \in F$ can be chosen arbitrarily large, we conclude that $f(x) < x$ is satisfied by infinitely many elements of F, so, by minimality, it is satisfied by almost all of them.

Finally, let $a \in F$ be large enough (i.e. a is chosen so that for all $x \not< a$, $|\bar{F}_<(x)|$ is a finite field with more than p elements and $f(x) < x$), and let b be its immediate successor. Then, $f[\bar{F}_<(b)] \subseteq \bar{F}_<(a)$ and $|f[\bar{F}_<(b)]| = |\bar{F}_<(b)|/p$ imply $p\,|\bar{F}_<(a)| \geq |\bar{F}_<(b)|$ which is not possible (for finite fields of size greater than p).

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