AROUND PODEWSKI'S CONJECTURE

KRZYSZTOF KRUPIŃSKI, PREDRAG TANOVIĆ, AND FRANK O. WAGNER

ABSTRACT. A long-standing conjecture of Podewski states that every minimal field is algebraically closed. Known in positive characteristic, it remains wide open in characteristic zero. We reduce Podewski's conjecture to the (partially) ordered case, and we conjecture that such fields do not exist. We prove the conjecture in case the incomparability relation is transitive (the almost linear case).

We also study minimal groups with a (partial) order, and give a complete classification of almost linear minimal groups as certain valued groups.

1. INTRODUCTION

Recall that an infinite first-order structure is *minimal* if every definable (with parameters) subset is either finite or co-finite (of finite complement). Minimal pure groups were classified by Reineke [8]; they are either abelian divisible with only finitely many elements of any given finite order, or elementary abelian of prime exponent. As for minimal fields, it is well-known that every algebraically closed pure field is minimal; the converse was predicted by Podewski [7] forty years ago:

Conjecture 1. *A minimal field is algebraically closed.*

It was shown by the third author in positive characteristic [11], but the characteristic zero case remains one of the oldest unsolved problems in algebraic model theory.

We shall say that a structure is *ordered* if it has a definable strict partial order on singletons which is not definable from equality. Otherwise the structure is *unordered*. Please note that in this paper ordered fields or groups are ordered structures in the above sense rather than in the usual algebraic sense.

2010 *Mathematics Subject Classification.* Primary 03C60, Secondary 12L12, 20A15, 03C45 .

Key words and phrases. Podewski's Conjecture; minimal field; minimal group; valued group.

Research supported by the Polish Government grant N N201 545938.

Supported by the Ministry of Education and Science of Serbia, grant ON174026.

Partially supported by the Agence Nationale de la Recherche under project ANR-09-BLAN-0047 Modig.

Lemma 1.1. *A minimal ordered structure M has an infinite chain.*

Proof. If there is a chain of order type ω^*, the reverse order of ω, we are done. So assume that there is no such chain; we shall construct inductively a chain of order type ω.

Suppose the set X of minimal elements is co-finite. Then the elements of X are incomparable. By minimality, for each $y \in M \setminus X$ the sets of elements less than y, bigger than y and incomparable to y are definable from equality. Hence the partial order is definable from equality, a contradiction.

Therefore the set of minimal elements is finite. If it is empty, there is a chain of order type ω^*, a contradiction. Hence, for some minimal element x_0 the set $X_0 = \{y : x_0 < y\}$ must be infinite, and hence co-finite by minimality. But if we can define the order on some co-finite set from equality, then, by minimality, we can define the order on M from equality. By induction, we obtain an infinite increasing chain. □

The minimal total (or linear, i.e. any two elements are comparable) orders are just $(\omega + n, <)$, $((\omega + n)^*, <)$ and $(\omega + \omega^*, <)$. Most of the known non-linear, ordered, minimal structures derive from them by replacing elements by sufficiently large finite antichains and then adding a finite set arbitrarily. For example, consider $\{(n, m) \in \omega \times \omega \,|\, m \leq n\}$ and order it by $(n, m) < (n', m')$ iff $n < n'$. Here, the order is not far from being linear, namely the incomparability relation (defined by $x \sim y$ iff $\neg(x < y \vee y < x)$) is transitive and hence an equivalence relation; after factoring it out we end up with $(\omega, <)$. The minimal structures in which there is a definable order on singletons with an infinite chain such that incomparability is transitive will be called *almost linear* (see Definition 2.5 and Remark 2.6). An equivalent way of describing almost linearity is: there exists a definable, total quasi-order with infinite strictly increasing chains.

In Section 3, we prove Conjecture 1 for unordered fields:

Theorem 1. *A minimal unordered field is algebraically closed.*

Thus, Podewski's conjecture is reduced to the ordered case:

Conjecture 2. *There is no minimal ordered field of characteristic zero.*

We study minimal ordered groups in Section 4 and show:

Theorem 2. *An almost linear minimal group G is either elementary abelian of exponent p or a finite sum of Prüfer p-groups for a fixed prime p. In particular, it is a torsion group.*

This implies immediately:

Theorem 3. *There is no almost linear minimal field.*

It is thus natural to ask:

Question 1. *Is every minimal ordered group a torsion group?*

Theorem 3 implies that a possible counter-example to Conjecture 2 would have to be a field which is not almost linear. It is hard to believe that such structures exist. In particular, all known examples of minimal ordered structures are almost linear.

Question 2. *Does there exist a minimal ordered structure [group] which is not almost linear?*

Note that the analogue of Theorem 3 for quasi-minimal fields (uncountable fields whose definable subsets are countable or co-countable) is false: There is an almost linear quasi-minimal field [6, Example 5.1].

Finally, in Section 5, we classify almost linear minimal groups as certain valued groups, showing in particular that all cases in the conclusion of Theorem 2 can be realized (so the analogue of Theorem 3 for groups is false).

2. Minimal structures with definable generic type

Let M be a minimal structure. The unique non-algebraic type $p \in S_1(M)$ will be called the *generic type* of M. So, p consists of all formulas over M defining co-finite subsets of M. In this section, we are interested in minimal structures whose generic type is definable. This class is interesting because of the next lemma, noticed by A. Pillay.

Lemma 2.1. *The generic type p of a minimal group G is its unique generic type in the sense of left [and right] translates, i.e. a formula $\phi(x)$ (with parameters from M) belongs to p if and only if finitely many left [right] translates of $\phi(G)$ cover G. In fact, $\phi(x) \in p$ if and only if two left [right] translates of $\phi(G)$ cover G. This characterization of p implies that p is definable over $\emptyset$.*

Proof. If finitely many translates of $\phi(G)$ cover G, then $\phi(G)$ is infinite, hence $\phi(x) \in p$. For the converse, suppose $\phi(x) \in p$. Then $G \setminus \phi(G)$ is a finite set $\{g_1, \dots, g_n\}$. Since the sets $g_1\phi(G)^{-1}, \dots, g_n\phi(G)^{-1}$ are co-finite, their intersection is non-empty, so it contains an element g. It is clear now that $G = \phi(G) \cup g\phi(G)$. □

For the remainder of this section M will be a minimal structure whose generic type $p \in S_1(M)$ is definable over $\emptyset$. Let $\bar{M} \succ M$ be a monster model. Then p has a unique global heir $\bar{p} \in S_1(\bar{M})$, which is defined by the same defining scheme as p. For $C \subseteq \bar{M}$ a *generic Morley sequence* is a sequence $(a_i : i \in \kappa)$ such that $a_i \models \bar{p}|Ca_{<i}$ for all $i \in \kappa$. Each generic Morley sequence over C is indiscernible over C, and the type over C of such a sequence of a fixed length κ does not depend on its choice. Recall that for a formula $\varphi(x;\bar{y})$ a $\varphi(x;\bar{y})$-definition of p is denoted by $d_p x\varphi(x,\bar{y})$.

Proposition 2.2. *For any $A \subseteq \bar{M}$ put*

$$\mathrm{cl}(A) = \{x \in \bar{M} : \mathrm{tp}(x/A) \textit{ is non-generic}\}.$$

Then cl *is a closure operator on $\bar{M}$. In particular, it is idempotent.*

Proof. Clearly cl has finite character, and $A \subseteq B$ implies $A \subseteq \mathrm{cl}(A) \subseteq \mathrm{cl}(B)$. Suppose $a \in \mathrm{cl}(\mathrm{cl}(A))$. Then there is a non-generic definable set $\phi(\bar{M}, \bar{b}, A)$ containing a, and for every $b_i \in \bar{b}$ a non-generic definable set $\phi_i(\bar{M}, A)$ containing b_i. If a were generic over A, then A would satisfy

$$d_p x \exists \bar{y}\, \{\phi(x,\bar{y},X) \wedge \neg d_p z \phi(z,\bar{y},X) \wedge \bigwedge_i [\phi_i(y_i,X) \wedge \neg d_p z \phi_i(z,X)]\}.$$

Hence, there is $A_0 \subseteq M$ satisfying this formula.

Since for each i the set

$$\{y_i \in M : M \models \phi_i(y_i, A_0) \wedge \neg d_p z \phi_i(z, A_0)\}$$

is finite, and for any fixed tuple $\bar{y}$ of elements of M the set

$$\{x \in M : M \models \phi(x, \bar{y}, A_0) \wedge \neg d_p z \phi(z, \bar{y}, A_0)\}$$

is also finite, we conclude that the formula

$$\exists \bar{y}\, \{\phi(x,\bar{y},A_0) \wedge \neg d_p z \phi(z,\bar{y},A_0) \wedge \bigwedge_i [\phi_i(y_i,A_0) \wedge \neg d_p z \phi_i(z,A_0)]\}$$

defines a finite set, which cannot be generic, a contradiction. Thus, $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$. □

We shall now prove a version of the dichotomy theorem for minimal structures from [10] in our context. The proof uses arguments from [6], where a similar result was proved for locally strongly regular types.

Recall that a *partial order* is a reflexive, antisymmetric and transitive binary relation, and a *strict partial order* is an irreflexive, asymmetric and transitive binary relation. A *well [strict] partial order* is a [strict] partial order without infinite decreasing chains and without infinite antichains. A partial order is *directed* if for any two elements a and b there is an element c such that $a \leq c$ and $b \leq c$.

Proposition 2.3. *Let (a, b) be a generic Morley sequence over M. Then exactly one of the following two cases holds.*

- **Symmetric:** $\mathrm{tp}(a, b/M) = \mathrm{tp}(b, a/M)$.
  In this case, a generic Morley sequence over any $C \subseteq \bar{M}$ is totally indiscernible over C (i.e. indiscernible over C as a set).
- **Asymmetric:** $\mathrm{tp}(a, b/M) \neq \mathrm{tp}(b, a/M)$.
  In this case, there is an M-definable strict partial order on M such that $M < a < b$. Moreover, $(M, \leq)$ is a directed, well partial order having infinite increasing chains and no such chain of order type $\omega + 1$.

Proof. Suppose that $\mathrm{tp}(a, b/M) = \mathrm{tp}(b, a/M)$. This implies that for any $\phi(x, y, \bar{z})$ without parameters and for any $\bar{m} \subseteq M$ we have

$$\models d_p x\, d_p y(\phi(x, y, \bar{m}) \leftrightarrow \phi(y, x, \bar{m})).$$

Then the same formula is satisfied by any $\bar{c} \subseteq C$ in place of $\bar{m}$, and so for every generic Morley sequence (a', b') over C we have $\mathrm{tp}(a', b'/C) = \mathrm{tp}(b', a'/C)$. By induction, it follows easily that generic Morley sequences of any length are totally indiscernible.

Now, suppose $\mathrm{tp}(a, b/M) \neq \mathrm{tp}(b, a/M)$. We will find $\phi(x, y) \in \mathrm{tp}(a, b/M)$ such that:

(1) $\models \forall x, y(\phi(x, y) \to \neg\phi(y, x))$ ($\phi(x, y)$ is asymmetric);
(2) $\phi(M, b) = M$ (i.e. $\phi(c, y) \in p(y)$ for all $c \in M$);
(3) $\phi(a, y) \vdash p(y)$.

Since p is definable, it has a unique heir and a unique coheir in $S_1(Ma)$; since $\mathrm{tp}(a, b/M) \neq \mathrm{tp}(b, a/M)$, the two must be distinct. Therefore, $\mathrm{tp}(b/Ma)$ is not a coheir, so there is $\phi'(a, y) \in \mathrm{tp}(b/Ma)$ which is satisfied by no element of M. Since $\mathrm{tp}(a/Mb)$ is a coheir, $\phi'(M, b)$ is infinite. Definability of p implies that $\phi'(M, b)$ is co-finite; modifying it slightly on M, we may assume $\phi'(M, b) = M$. The formula $\phi'(b, x)$ is satisfied by no element of M so $\phi'(b, x) \notin \mathrm{tp}(a/Mb)$ because $\mathrm{tp}(a/Mb)$ is a coheir. Therefore (a, b) satisfies

$$\phi(x, y) := \phi'(x, y) \wedge \neg\phi'(y, x).$$

Clearly $\phi(x, y)$ satisfies conditions (1)–(3).

We leave to the reader to verify that the formula

$$\phi(x, y) \wedge \forall t(\phi(y, t) \to \phi(x, t))$$

defines a strict partial order; denote it by $x < y$. Now, we prove that for any $c \in M$ we have $c < b$, i.e.

$$\models \phi(c,b) \wedge \forall t(\phi(b,t) \rightarrow \phi(c,t)).$$

Condition (2) implies $\models \phi(c,b)$. Let d be such that $\models \phi(b,d)$. Then, by (3), d realizes p, so $\phi(c,y) \in p(y)$ implies $\models \phi(c,d)$. Thus, $c < b$. Since this holds for all $c \in M$ and $\mathrm{tp}(a/bM)$ is finitely satisfiable in M, we conclude that $a < b$. As $\mathrm{tp}(a/M) = \mathrm{tp}(b/M)$ and $M < b$, we also get $M < a$.

Now, the formula $x < b$ belonging to $\mathrm{tp}(a/bM)$ is satisfied in M, say by $c_1 \in M$. Then $c_1 < b$; in fact, since $M < a$, we also have $c_1 < a$. Hence, the formula $c_1 < x < b$ belongs to $\mathrm{tp}(a/bM)$, and so there is $c_2 \in M$ such that $c_1 < c_2 < b$; then also $c_2 < a$, so the formula $c_1 < c_2 < x < b$ belongs to $\mathrm{tp}(a/bM)$. Continuing in this way, we get an infinite increasing chain in M.

To see that $\leq$ is directed, consider any $c, d \in M$. Since the formulas $c \leq x$ and $d \leq x$ belong to $p(x)$, they define co-finite subsets of M, and so we can find an element in their intersection.

To see that $\leq$ is a well partial order, it is enough to notice that otherwise there would be an element $c \in M$ for which the formula $\neg(c < x)$ belongs to $p(x)$ which contradicts the fact that $c < x$ belongs to $p(x)$. The same argument shows that there is no chain of order type $\omega + 1$. □

The above proposition leads to the following definition.

Definition 2.4. M is *symmetric* if $\mathrm{tp}(a, b/M) = \mathrm{tp}(b, a/M)$ for each/some generic Morley sequence (a, b) over M; equivalently, if generic Morley sequences (of arbitrary length) over any set C are totally indiscernible over C. Otherwise M is *asymmetric*.

From now on, whenever a definable strict partial order $<$ is clear from the context, $x \sim y$ will be defined as $\neg(x < y \vee y < x)$. Notice that $\sim$ is always reflexive and symmetric.

Definition 2.5. Let N be any minimal structure. A definable strict partial order $<$ on N with infinite chains is *almost linear* if $\sim$ is an equivalence relation on $p(\bar{N})$. We call N *almost linear* if such an order exists.

Remark 2.6. Suppose that N is minimal and that $<$ is a definable strict partial order with infinite chains on N. Then the following conditions are equivalent.

(1) $<$ is almost linear.
(2) After a definable modification of $<$ on a finite set, incomparability $\sim$ becomes an equivalence relation. More precisely, this modification

is obtained by putting finitely many elements below all others and ordering them linearly.

(3) After a definable modification of $<$ on a finite set, incomparability $\sim$ becomes an equivalence relation, and the set of equivalence classes has order type $(\omega, <)$, $(\omega^*, <)$ or $(\omega + \omega^*, <)$.

Proof. The implications (3) $\to$ (2) $\to$ (1) are clear. In order to see that (1) $\to$ (2), notice that since $\sim$ is an equivalence relation on $p(\bar{N})$, compactness implies that it is an equivalence relation on the set of realizations of some formula in p, and so on a co-finite subset of N. Finally, for (2) $\to$ (3) it suffices to notice that the order type of a minimal linear order is either one of the types listed in (iii), or $(\omega + n, <)$ or $((\omega + n)^*, <)$; clearly the latter two can be modified definably in order to get $(\omega, <)$ or $(\omega^*, <)$. $\square$

Remark 2.7. Assume that M is ordered by $<$ with an infinite increasing chain. Since the definability of p is assumed, $C := \{c \in M \mid c < x \in p(x)\}$ is a definable, co-finite subset of M, and hence $<$ is a well strict partial order and M is asymmetric. After modifying $<$ so that the elements of $M \setminus C$ are below all the others, we get that $C = M$, and then $<$ is a directed, well partial order having infinite increasing chains and no such chain of order type $\omega + 1$. If in addition $<$ is almost linear, then after modifying $<$ on a finite set we get that $M/{\sim}$ is ordered of order type ω.

Of course, analogous observations are true when M contains an infinite decreasing chain. Proposition 2.3 together with Remark 2.7 yield the following observation.

Remark 2.8. Let M be a minimal structure whose generic type is definable. Then M is asymmetric iff M is ordered.

We finish this section with an example of an ordered minimal structure which is due to Grzegorz Jagiella ([3]). Although the structure is almost linear, it has a definable order with infinite chains which is not almost linear.

Example 2.9. Let $M = \omega \times \{l, r\}$. Define an order $<$ on M by putting the natural orders on $\omega \times \{l\}$ and on $\omega \times \{r\}$, together with

$$(x, l) < (y, r) \iff x + 2 \leq y \quad \text{and} \quad (y, r) < (x, l) \iff y + 2 \leq x$$

for all natural numbers x and y.

We leave to the reader to verify that $(M, <)$ is minimal. For $n \in \omega$ define $a_{2n} = (2n, l)$ and $a_{2n+1} = (2n + 1, r)$. Then a_n and a_{n+1} are incomparable for all n, and so $<$ is not almost linear.

Now, we show that M interprets $(\omega, <)$. First, note that 'y is maximal incomparable to x' is a definable function $f(x) = y$. Then for $x, y \in M$ define:

$$x <' y \quad \text{if and only if} \quad x < y \ \text{ or } \ y = f(x).$$

Then $<'$-incomparability is an equivalence relation with 2-element classes $\{(n, r), (n, l)\}$, and, after factoring it out, we end up with $(\omega, <)$. By Remark 2.6, $<'$ is almost linear in M, so M is almost linear.

3. The symmetric case

In this section, we shall prove that symmetric minimal fields are algebraically closed. The following is a version of the corresponding result for locally strongly regular types from [6]. It is adapted to the context of minimal structures.

Proposition 3.1. *Let M be a symmetric minimal structure with definable (over $\emptyset$) generic type, $\bar{M}$ its monster model and* cl *the closure operator from Proposition 2.2. Then $(\bar{M}, \mathrm{cl})$ is an infinite dimensional pre-geometry, and $a_1, \dots, a_n$ is* cl*-independent over A if and only if it is a generic Morley sequence over A. In particular, if $(a_1, \dots, a_n)$ is a generic Morley sequence over A and $a_1, \dots, a_n \in \mathrm{cl}(A, b_1, \dots, b_n)$, then $(b_1, \dots, b_n)$ is also a generic Morley sequence over A.*

Proof. We only have to prove the exchange property. So, consider $a \in \mathrm{cl}(Ab) \setminus \mathrm{cl}(A)$. As $a \notin \mathrm{cl}(A)$, by definition $\mathrm{tp}(a/A)$ is generic. Suppose $b \notin \mathrm{cl}(Aa)$. Then $\mathrm{tp}(b/Aa)$ is generic, and (a, b) is a generic Morley sequence over A. By symmetry, (b, a) is a generic Morley sequence over A. In particular, $a \notin \mathrm{cl}(Ab)$, a contradiction. Thus, cl satisfies the exchange property. □

It follows that the generic type p is generically stable, and orthogonal to all non-generic types (see [6] for the definitions; we will not use this terminology in this paper).

Now, it is straightforward to deduce Theorem 1 from [2, Theorem 1.13], where it is proved that any field carrying a pre-geometry with certain homogeneity properties is algebraically closed; it is based on Macintyre's proof that ω_1-categorical fields are algebraically closed [4]. Here, we will give an alternative proof based on an argument of Wheeler [12, Theorem 2.1]; it was also used by Pillay [5, Proposition 5.2] to prove that ω-stable fields are algebraically closed.

Lemma 3.2. *If K is a minimal field, then $(K^*)^n = K$ for any $n > 0$. In particular K is perfect.*

Proof. $(K^*)^n$ is an infinite subgroup of K^*, so, by minimality, it is co-finite and $(K^*)^n = K^*$. □

Theorem 1. *A minimal unordered field is algebraically closed.*

Proof. By Proposition 2.3 it suffices to prove that a symmetric minimal field is algebraically closed. Let K be a symmetric minimal field, $\bar{K} \succ K$ a monster model, and p the generic type. Let F denote an algebraic closure of $\bar{K}$. Suppose for a contradiction that some $\alpha_1 \in F \setminus K$ is algebraic over K. Let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in K[x]$ be the minimal polynomial of α_1 over K. Since K is perfect, f has n pairwise distinct roots $\alpha_1, \ldots, \alpha_n$ in F. Let $(t_0, \ldots, t_{n-1}) \in \bar{K}^n$ be a Morley sequence in p over K, and define $r_i = t_0 + t_1\alpha_i + \cdots + t_{n-1}\alpha_i^{n-1}$ for $i = 1, \ldots, n$. Then

$$\begin{pmatrix} 1 & \alpha_1 & \ldots & \alpha_1^{n-1} \\ 1 & \alpha_2 & \ldots & \alpha_2^{n-1} \\ \vdots & \vdots & \ldots & \vdots \\ 1 & \alpha_n & \ldots & \alpha_n^{n-1} \end{pmatrix} \begin{pmatrix} t_0 \\ t_1 \\ \vdots \\ t_{n-1} \end{pmatrix} = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{pmatrix}$$

and, since the matrix is invertible, $(r_1, \ldots, r_n)$ and $(t_0, \ldots, t_{n-1})$ are interalgebraic over K.

Let $c_0, \ldots, c_{n-1}$ be the symmetric functions of $r_1, \ldots, r_n$. Then the sequences $(c_0, \ldots, c_{n-1})$ and $(r_1, \ldots, r_n)$ are interalgebraic over K, so the sequences $(t_0, \ldots, t_{n-1})$ and $(c_0, \ldots, c_{n-1})$ are interalgebraic over K, too. Since c_i, t_j are in $\bar{K}$, we can apply Proposition 3.1 to conclude that $(c_0, \ldots, c_{n-1})$ is a Morley sequence in p over K. But for generic x over $c_1, \ldots, c_{n-1}$ the element

$$c' = -(x^n + c_{n-1}x^{n-1} + \cdots + c_1 x)$$

is again generic by Proposition 3.1, so it has the same type over $c_1, \ldots, c_{n-1}$ as c_0, and there is an automorphism σ fixing $c_1, \ldots, c_{n-1}$ and moving c' to c_0. Then $\sigma(x)$ is a zero of the polynomial

$$z^n + c_{n-1}z^{n-1} + \cdots + c_1 z + c_0.$$

Thus there is i such that

$$\sigma(x) = r_i = t_0 + t_1\alpha_i + \cdots + t_{n-1}\alpha_i^{n-1} \in \bar{K}.$$

This means that the degree of the minimal polynomial of α_i over the field $K(t_0, \ldots, t_{n-1})$ is smaller than n. Since $K(t_0, \ldots, t_{n-1}) \subseteq \bar{K}$, this implies that the degree of the minimal polynomial of α_i over $\bar{K}$ is also smaller than n. On the other hand, as $K \prec \bar{K}$ and $f(x) \in K[x]$ is the minimal polynomial of α_i over K (so it is irreducible in $K[x]$), we get that $f(x)$ is also irreducible in $\bar{K}[x]$, and so it is the minimal polynomial of α_i over $\bar{K}$. This is a contradiction, because $\deg(f) = n$. □

Notice the following consequence of Proposition 2.3 and Theorem 1.

Corollary 3.3. *Each minimal field whose theory does not have the strict order property is algebraically closed. In particular, each minimal field whose theory is simple is algebraically closed.*

4. Asymmetric minimal groups

In this section, we shall show some general properties of asymmetric minimal groups. In particular, an almost linear minimal group is either elementary abelian or a finite sum of Prüfer p-groups for some prime p; it follows that there is no almost linear minimal field. Unfortunately, as far as asymmetric minimal groups in general are concerned, the following questions are still open; an affirmative answer would immediately imply Podewski's conjecture:

Question 3. *Is every asymmetric minimal group almost linear? Is it at least torsion?*

Given a minimal group G, a *generic element* is an element which is generic over G, i.e. a realization (in a monster model $\bar{G}$) of the unique generic type $p \in S_1(G)$. Notice that an element $g \in \bar{G}$ is generic if and only if $g \notin G$. By Proposition 2.3 and Remark 2.7, whenever we are working in an asymmetric minimal group $(G, <, +, 0, \dots)$, we can and do assume that $<$ is a directed, well, strict partial order with an infinite increasing chain and with no such chain of order type $\omega + 1$, and such that $G < g$ for any generic g. We would like to emphasize that in the results of this section we consider elements from the monster model $\bar{G} \succ G$.

Lemma 4.1. *Let $(G, <, +, 0, \dots)$ be an asymmetric minimal group, g generic, $g_1, \dots, g_k < g$, and $n_1, \dots, n_k$ integers. Then $n_1 g_1 + \dots + n_k g_k \neq g$.*

Proof. Suppose $n_1 g_1 + \dots + n_k g_k = g$. Then

$$\exists x_1, \dots, x_k \ \left(\bigwedge_{i=1}^{k} x_i < x \wedge \sum_{i=1}^{k} n_i x_i = x \right) \tag{4.1}$$

holds generically, and hence outside a finite set $X \subset G$. By Reineke's result, G is either elementary abelian, or abelian divisible. In either case, any finitely generated subgroup is a proper subgroup of G. Let H be the subgroup generated by X. Consider a minimal element a in $G \setminus H$. As (4.1) holds outside $X \subseteq H$, it is satisfied by a. But any $a_1, \dots, a_k < a$ are in H by the minimality of a, as is $\sum_{i=1}^k n_i a_i$. So $\sum_{i=1}^k n_i a_i \neq a$, a contradiction. $\square$

There is also a corresponding version for asymmetric minimal fields:

Lemma 4.2. *Let $(K, <, +, \cdot, 0, 1, \dots)$ be an asymmetric minimal field, g generic, and $g_1, \dots, g_k < g$. Let $f(x_1, \dots, x_k)$ be a rational function over K such that the tuple $(g_1, \dots, g_k)$ is in its domain. Then $f(g_1, \dots, g_k) \neq g$.*

Proof. Suppose $f(g_1, \dots, g_k) = g$. Then

$$\exists \bar{x} = (x_1, \dots, x_k) \ \left(\bigwedge_{i=1}^k x_i < x \wedge \bar{x} \in \mathrm{dom}(f) \wedge f(\bar{x}) = x \right) \tag{4.2}$$

holds generically, and hence outside a finite set $X \subset K$. Let F be the subfield of K generated by X. Since K is minimal, it is closed under n-th roots for all $n > 0$ by Lemma 3.2 and thus not finitely generated. Hence, F is a proper subfield; consider a minimal element a in $K \setminus F$. As (4.2) holds outside $X \subseteq F$, it is satisfied by a. Consider any $a_1, \dots, a_k < a$ such that $(a_1, \dots, a_k) \in \mathrm{dom}(f)$. Then $a_1, \dots, a_k \in F$ by the minimality of a, hence $f(a_1, \dots, a_k) \in F$, and so $f(a_1, \dots, a_k) \neq a$, a contradiction. $\square$

Recall that a divisible abelian group G splits as a direct sum of the torsion subgroup $Tor(G)$ and a direct sum of copies of $\mathbb{Q}$; furthermore, $Tor(G)$ is a direct sum of Prüfer p-groups (possibly with repetitions), where p ranges over prime numbers. Under the assumption that G is not a finite sum of Prüfer groups, we can strengthen the conclusion of Lemma 4.1:

Lemma 4.3. *Let $(G, <, +, 0, \dots)$ be an asymmetric minimal group. Let g be generic, $g_1, \dots, g_k < g$, and $n_1, \dots, n_k, n$ integers with $ng \neq 0$. Then $n_1 g_1 + \dots + n_k g_k \neq ng$, or G is a finite sum of Prüfer p-groups for some primes p dividing n (possibly with repetitions).*

Proof. The proof is very similar to the previous one. Suppose for a contradiction that G is not a finite sum of Prüfer p-groups with p dividing n and $n_1 g_1 + \dots + n_k g_k = ng$. Then we get that the set $X \subset G$ of realizations of the negation of the formula

$$\exists x_1, \dots, x_k \ \left(\bigwedge_{i=1}^k x_i < x \wedge \sum_{i=1}^k n_i x_i = nx \right) \tag{4.3}$$

is finite. Let H be the n-divisible hull of the subgroup generated by X, i.e. the collection of all elements $h \in G$ such that $n^m h \in \langle X \rangle$ for some $m \in \mathbb{N}$. If $H \neq G$, we finish as before by considering a minimal element $a \in G \setminus H$. So, it remains to show that H is a proper subgroup of G. By Reineke's result, either G is elementary abelian of prime exponent p and H is finite (note that $ng \neq 0$ implies that p does not divide n), or G is divisible with only finitely many elements of any given finite order. If it contains a copy Q of $\mathbb{Q}$, then $(n+1)^{-k}1_Q \notin H$ for sufficiently large k. Otherwise G contains a copy P of a Prüfer p-group for some p not dividing n, so we are done as $H \cap P$ must be finite. $\square$

Corollary 4.4. *Let $(G, <, +, 0, \dots)$ be an asymmetric minimal group. If g is generic and $h < g$, then $\pm g \pm h \sim g$.*

Proof. Fix any choice of $\pm$ in $\pm g \pm h$. If $\pm g \pm h > g$, we have that $g, h < \pm g \pm h$ and $\pm g \pm h$ is generic; if $\pm g \pm h < g$, we have $h, \pm g \pm h < g$; in either case, we contradict Lemma 4.1. $\square$

Corollary 4.5. *Let $(G, <, +, 0, \dots)$ be an asymmetric minimal group and let g be generic. Then $g \not< ng$ for any integer n. If $ng \neq 0$ and G is not a sum of finitely many Prüfer p-groups for primes p dividing n, then $ng \not< g$, whence $g \sim ng$. Moreover, $g \sim g'$ for any $g' \in \frac{1}{n}g$.*

Proof. If $ng > g$, then clearly ng is generic and it is a sum of strictly smaller elements, which contradicts Lemma 4.1. If $ng < g$, we take $g_1 = ng$ and $n_1 = 1$, contradicting Lemma 4.3. Thus $g \sim ng$. If $g' \in \frac{1}{n}g$, then g' is generic, so $g' \sim ng' = g$ by the first part of the proof. $\square$

Although the next lemma will be used in the last section of the paper, we include it here as it concerns general properties of generics in asymmetric minimal groups.

Lemma 4.6. *Let G be an asymmetric minimal group, not of exponent dividing n, and g generic. Then all elements of $\frac{1}{n}g$ are $\sim$-related.*

Proof. Since the set $X := \{g' \in \bar{G} : ng' = g\}$ is finite, it is enough to show that each minimal element g_0 of X is incomparable to each maximal element g_1 of X. Note that g_0 and g_1 are both generic. We see that

$$X = \{g' : ng' = ng_0\} = \{g' : ng' = ng_1\}$$

is invariant under an automorphism taking g_0 to g_1. Hence $g_0 \sim g_1$. $\square$

Theorem 2. *An almost linear minimal group G is either elementary abelian of exponent p or a finite sum of Prüfer p-groups for a fixed prime p. In particular, it is a torsion group.*

Proof. Suppose G is a counter-example. Then G is divisible, and at least one of the following cases holds:

1. G contains a copy P of a Prüfer p-group, but is not a sum of Prüfer p-groups,
2. G contains a copy P of $\mathbb{Q}$; in this case, put $p = 2$.

In any of these cases, by Corollary 4.5, the set

$$X := \{x \in G : x \not\sim y \text{ for some } y \in \tfrac{1}{p}x\}$$

is finite; let H be the subgroup generated by X. Then $H \cap P$ is a proper subgroup of P, so there is $a_0 \in P \setminus H$. Choose $a_k \in P$ with $pa_{k+1} = a_k$ for all $k < \omega$. Then all a_k's are outside H, so transitivity of $\sim$ implies that $\{a_k : k < \omega\}$ is a $\sim$-antichain. Since it must be finite, there is $j > i$ with $a_i = a_j$. Then $p^{j-i}a_0 = a_0$, a contradiction with the fact that the order of a_0 is a power of p in Case 1 and is infinite in Case 2. □

This implies immediately the non-existence of almost linear minimal fields.

Theorem 3. *There is no almost linear minimal field.*

Proof. The multiplicative group of an infinite field is neither elementary abelian, nor a finite sum of Prüfer p-groups for a fixed prime p. □

5. Almost linear minimal groups as valued groups

Recall that a *valued abelian group* is an abelian group G together with a surjective valuation $v : G \to \Gamma$, where Γ is a linearly ordered set with maximum ∞, such that:

(1) $v(x) = \infty$ if and only if $x = 0$.
(2) $v(x - y) \geq \min\{v(x), v(y)\}$.

Note that the axioms imply $v(-x) = v(x)$ and $v(x - y) = \min\{v(x), v(y)\}$ unless $v(x) = v(y)$. It follows that for every $\gamma \in \Gamma \setminus \{\infty\}$ the sets $\bar{B}(\gamma) := \{x \in G : v(x) \geq \gamma\}$ and $B^o(\gamma) := \{x \in G : v(x) > \gamma\}$ are subgroups of G. Valued abelian groups have been studied by Simonetta [9] and de Aldama [1], who consider the following conditions for all primes p:

(3) $\forall x, y\ [v(px) < v(py) \to v(x) < v(y)]$.
(4) $\forall x, y\ [v(x) < v(y) \to (v(px) < v(py) \vee px = 0)]$.
(5) $\forall x, y\ [v(x) < v(py) \vee \exists z\, pz = x]$.

Assume Axioms (1)–(5) (for all primes p). As v is surjective, for all $n \in \mathbb{N}^*$ the formula $f_n((v(x)) = v(nx)$ yields a well-defined function $f_n : \Gamma \to \Gamma$, which is increasing, strictly so on $\Gamma \setminus f_n^{-1}(\infty)$. In addition, for every $m \in \mathbb{N}^*$ we consider the unary relation R_m on Γ given by

$$R_m(x) \Leftrightarrow |\bar{B}(x)/B^o(x)| > m.$$

If $f_n(\gamma) \neq \infty$, then $R_m(\gamma) \Leftrightarrow R_m(f_n(\gamma))$. We put

$$\mathcal{L}_{vg} = \{+, 0, v, \leq, \infty\} \quad \text{and} \quad \mathcal{L}_v = \{\leq, R_n, f_n : n \in \mathbb{N}^*\}.$$

Simonetta shows that if G is a valued abelian group satisfying (1)–(5), then there is at most one prime p such that G is not p-divisible, and at most one prime q such that G has q-torsion. Moreover, he obtains the following relative quantifier elimination result:

Fact 5.1. [9, Theorem 3.3] *Every $\mathcal{L}_{vg}$-formula $\phi(\bar{x}, \bar{y})$ with variables $\bar{x}$ in the group sort and variables $\bar{y}$ in the value sort is equivalent in G to some formula $\phi_v(v(t_1(\bar{x})), \ldots, v(t_n(\bar{x})), \bar{y})$, where the $t_i(\bar{x})$ are group terms in $\bar{x}$ and ϕ_v is an $\mathcal{L}_v$-formula. Moreover, ϕ_v and $t_1, \ldots, t_n$ only depend on p and q.*

Clearly a valued group with infinite value set is almost linear, where we take the inverse order induced from the valuation. For minimal groups, we shall now prove the converse: An almost linear minimal group G carries interdefinably the structure of a valued group. Recall that by Remark 2.7, we can and do choose a definable order on G so that $G/\sim$ is ordered in type ω.

Let us start from the following easy but useful consequence of transitivity of $\sim$.

Remark 5.2. Suppose $<$ is a strict partial order on a set M for which $\sim$ is transitive. Then ($x \sim y$ and $x < z$) implies $y < z$, and similarly ($x \sim y$ and $z < x$) implies $z < y$.

Lemma 5.3. *Let G be an almost linear minimal group and put $H_g = \{x \in G : x \not> g\}$. Then H_g is a subgroup for co-finitely many $g \in G$. The collection of these subgroups is linearly ordered by inclusion in order type ω.*

Proof. Let g be generic and $x, y \not> g$. Suppose for a contradiction that $x \pm y > g$. By Remark 5.2, we easily get $x, y < x \pm y$, contradicting Lemma 4.1. Therefore H_g is a subgroup for generic g, and thus for co-finitely many g by minimality. Clearly $g < g'$ implies $H_g \subset H_{g'}$, and $g \sim g'$ implies $H_g = H_{g'}$ by Remark 5.2. Thus the set $\{H_g : g \in G\}$ has the same order type with respect to inclusion as $\{g/\sim : g \in G\}$ with respect to $<$, namely ω. □

Let G^* denote G with the opposite order, so $G^*/\sim$ is of order type ω^*.

Proposition 5.4. *Let G be an almost linear minimal group. After modifying the order on a finite set, the map $v : G \mapsto G^*/\sim$ endows G with the structure* *of a valued abelian group. In particular, G as almost linear group and G as valued group are interdefinable (with parameters).*

Proof. Let $X \subseteq G$ be the co-finite set of $g \in G$ such that H_g is a subgroup. We modify the order on G so that 0 is the unique minimal element. This can only increase X; in particular, $0 \in X$ and $H_0 = \{0\}$. Let g_0 be a minimal element greater than $G \setminus X$. We modify the order by making all elements of the finite set $H_{g_0} \setminus \{0\}$ incomparable. These modifications are clearly definable, the modified order on G is still almost linear, and H_g is now a subgroup of G for all $g \in G$. Of course, $g < g'$ still implies $H_g \subset H_{g'}$, and $g \sim g'$ yields $H_g = H_{g'}$.

Now, $H_0 = \{0\}$ implies Axiom (1); the fact that H_g is a subgroup for all $g \in G$ yields Axiom (2). As we have only modified an initial segment of the order on G, the set of equivalence classes $G^*/\sim$ still has order type ω^*. We have obtained the valuation v definably from the almost linear structure; inversely, as we only modified the order on finitely many elements, we can define the original almost linear structure from the valuation. □

We say that Axiom (3), (4) or (5) holds generically if for every prime number p there is a finite set $E_p \subset G$ such that the p-instance of this axiom holds for all arguments outside E_p.

Remark 5.5.
(1) If the p-instance of Axiom (3) for some prime number p holds generically but not everywhere, then G has only finitely many elements of order p.
(2) If Axiom (3) holds generically, then each function f_n is well-defined outside a finite set $D_n \subseteq \Gamma$.

Proof. (1) Choose $x, y \in G$ for which the p-instance of Axiom (3) does not hold, i.e. $v(px) < v(py)$ but $v(x) \not< v(y)$. Suppose for a contradiction that $G[p] := \{x \in G : px = 0\}$ is infinite. Since the p-instance of Axiom (3) holds generically, there are infinitely many elements $a_0, a_1, \dots$ in $G[p]$ for which $v(x + a_i) < v(y + a_i)$. As $v(x) \not< v(y)$, we easily get that $v(y) = v(a_i)$ and then $v(y + a_i) > v(y)$, for all i. This implies that $v(a_j - a_i) > v(y)$ for all i, j. But then $v(x + (a_j - a_i)) \not< v(y + (a_j - a_i))$ for all i, j. As

$$v(p(x + (a_j - a_i))) = v(px) < v(py) = v(p(y + (a_j - a_i))),$$

we get a contradiction with the assumption that the p-instance of Axiom (3) holds generically.
(2) By induction on n, we will show that there is a finite set $F_n \subset G$ such that for all $x, y \in G \setminus F_n$, if $v(x) = v(y)$, then $v(nx) = v(ny)$. Suppose it

holds for numbers less than n. If n is prime, it follows from the assumption putting $F_n := E_n$. So assume that $n = pm$, where p is a prime number and $m > 1$. If the p-instance of Axiom (3) holds everywhere, then $F_n := F_m$ satisfies our demands. If it does not hold everywhere, then by (1), the set $F := \{x \in G : px \in F_m\}$ is finite, and so $F_n := F \cup F_p$ satisfies our demands. $\square$

Whenever f_n is well-defined outside a finite set, we extend it to the whole of Γ by putting $f_n[D_n] = \{\infty\}$.

Let G be a minimal valued group. Since $B^o(v(g))$ is a proper definable subgroup of G and thus finite for all $g \in G$, it follows that Γ has order type ω^* or is finite. In the latter case, if $\gamma \in \Gamma$ is the minimal element, then $\bar{B}(\gamma) = G$, so the valuation is determined by the restriction of v to the finite group $B^o(\gamma)$ and hence definable in the pure group structure. Henceforth, we shall assume that Γ has order type ω^*, so G is almost linear; by Theorem 2, it is either elementary abelian of exponent p or a finite product of Prüfer p-groups, for some prime p. Note that for n coprime to p the function f_n is just the identity. Indeed, $f_n(v(g)) = v(ng) \geq v(g)$; as $kng = g$ for some k, we get

$$v(g) = v(kng) \geq v(ng),$$

whence $f_n(v(g)) = v(g)$. Clearly, if G has exponent p, then f_p maps Γ to ∞. By minimality, v has finite fibres. Recall that v induces an order $<$ on G (i.e. $g' < g$ if $v(g') > v(g)$), and the incomparability relation ($g \sim g'$ if $v(g) = v(g')$) is transitive.

Lemma 5.6. *If G is divisible and g generic, then $\bar{B}(v(g))/\bar{B}(v(pg))$ is finite and isomorphic to $\bar{B}(v(pg))/\bar{B}(v(p^2g))$ as valued groups (with the induced valuation), via the map induced by $x \mapsto px$. In particular, the interval $[v(g), v(pg)]$ is finite and $\mathcal{L}_v$-isomorphic to $[v(pg), v(p^2g)]$. Hence for all but finitely many γ the function f_p is well-defined and corresponds to a right shift by ℓ, where $[v(g), v(pg)]$ has length $\ell + 1$. It is thus definable from the order.*

Proof. Let us check first that if g is generic and $g \sim g'$, then $pg \sim pg'$. So, consider $X := \{pg' : g' \sim g\}$; it must have a minimal element g_0 and a maximal element g_1, both generic over G. By Lemma 4.6, all elements of $\frac{1}{p}g_0$ and $\frac{1}{p}g_1$ are $\sim$-related to g, so

$$X = \{pg' : g' \sim \tfrac{1}{p}g_0\} = \{pg' : g' \sim \tfrac{1}{p}g_1\}$$

is invariant under an automorphism taking g_0 to g_1. Hence $g_0 \sim g_1$.

Next, let us check that $pg \sim pg'$ implies $g \sim g'$ for generic g. So, consider a minimal element g_0 and a maximal element g_1 of the set $X := \{g' : pg' \sim pg\}$. Since

$$X = \{g' : pg' \sim pg_0\} = \{g' : pg' \sim pg_1\}$$

and the generic type is unique, we have $g_0 \sim g_1$.

It follows that the map $F\colon \bar{G} \to \bar{G}$ given by $F(x) = px$ maps $\bar{B}(v(g)) \setminus B^o(v(g))$ onto $\bar{B}(v(pg)) \setminus B^o(v(pg))$. As $\bar{B}(v(g))$ is generated by $\bar{B}(v(g)) \setminus B^o(v(g))$, the group $\bar{B}(v(pg))$ is generated by $\bar{B}(v(pg)) \setminus B^o(v(pg))$, and F is a group homomorphism, we get that F maps $\bar{B}(v(g))$ onto $\bar{B}(v(pg))$, and hence $B^o(v(g))$ onto $B^o(v(pg))$. Thus, $g' < g$ if and only if $pg' < pg$ for all g'. We conclude that the map induced by F is an isomorphism from $\bar{B}(v(g))/\bar{B}(v(pg))$ to $\bar{B}(v(pg))/\bar{B}(v(p^2g))$ as valued groups, and that f_p yields an $\mathcal{L}_v$-isomorphism between $[v(g), v(pg)]$ and $[v(pg), v(p^2g)]$. By minimality, all of this holds for all g outside some finite set Y.

It remains to show that $\bar{B}(v(g))/\bar{B}(v(pg))$ is finite. Let $g_0 \in G$ be such that $G[p] \cup Y \subseteq \bar{B}(v(g_0))$ (where $G[p] := \{x \in G : px = 0\}$). Then for all $h > g_0$ the map $x \mapsto px$ maps the finite group $\bar{B}(v(h))$ onto the subgroup $\bar{B}(v(ph))$ and has finite kernel $G[p]$ independent of h. So $|\bar{B}(v(h))/\bar{B}(v(ph))| = |G[p]|$ for all $h > g_0$. $\square$

An inspection of Simonetta's proof of Fact 5.1 shows that it yields the following proposition, as finite sets of exceptions can be dealt with definably.

Proposition 5.7. *Let G be a valued abelian group whose valuation v has finite fibres and whose value set Γ has order type ω^*. Assume that G is either elementary abelian of exponent p or a finite product of Prüfer p-groups for some prime p, and Axioms (3) and (4) hold generically. Then every $\mathcal{L}_{vg}$-formula $\phi(\bar{x}, \bar{y})$ with variables $\bar{x}$ in the group sort and variables $\bar{y}$ in the value sort is equivalent to a formula $\phi_v(v(t_1(\bar{x})), \ldots, v(t_n(\bar{x})), \bar{y})$, where the $t_i(\bar{x})$ are group terms in $\bar{x}$ (possibly with extra parameters) and ϕ_v is an $\mathcal{L}_v$-formula.*

Notice that if G is either elementary abelian or divisible, then Axiom (5) is clearly satisfied. Notice also that by Lemma 5.6 and the discussion preceding it, the assumptions of Proposition 5.7 are satisfied for any valued minimal group with infinite value set Γ.

Theorem 4. *A valued abelian group G with infinite value set Γ is minimal if and only if the induced $\mathcal{L}_v$-theory on Γ is minimal of order type ω^*, the map v has finite fibres, and either G is elementary abelian, or a finite product* *of Prüfer p-groups for some prime p and f_p is eventually a well-defined $\mathcal{L}_v$-isomorphism acting by right shift.*

Proof. If G is minimal as a valued group, then Γ with the induced structure must be minimal, since an infinite co-infinite subset X of Γ has an infinite co-infinite pre-image $v^{-1}(X)$. We have seen above that Γ has order type ω^*, the group is either elementary abelian or a finite product of Prüfer p-groups, and v has finite fibres. The fact that f_p is eventually an $\mathcal{L}_v$-isomorphism acting by right shift follows from Lemma 5.6.

Conversely, suppose that the $\mathcal{L}_v$-structure Γ is minimal of order type ω^*, all fibres of v are finite, and G is elementary abelian of exponent p, or a finite product of Prüfer p-groups and f_p is eventually an $\mathcal{L}_v$-isomorphism acting by right shift. This implies that f_n is the identity for n coprime to p, and $f_p(\Gamma) = \infty$ if G has exponent p. All of this implies that Axioms (3)-(5) hold generically.

Consider a formula $\phi(x, \bar{g})$, where $\bar{g}$ are parameters in G. (Clearly, we can replace any parameter $\gamma \in \Gamma$ by some element of $v^{-1}(\gamma)$.) By Proposition 5.7 (enlarging the tuple $\bar{g}$ of parameters if necessary), this formula is equivalent to a formula $\phi_v(v(t_1(x, \bar{g})), \ldots, v(t_k(x, \bar{g})))$, where ϕ_v is an $\mathcal{L}_v$-formula and $t_1, \ldots, t_k$ are group terms. As group terms are just $\mathbb{Z}$-linear combinations, there are integers $n_i \in \mathbb{Z}$ and $h_i \in \langle g_1, \ldots, g_k \rangle < G$ such that $t_i(x, \bar{g}) = n_i x + h_i$; if G has exponent p, we may choose $0 \leq n_i < p$. Since v has finite fibres, Γ has order type ω^* and G is elementary abelian or has finite n-torsion for all n, the set

$$X = \{g \in G : v(n_i g) \geq v(h_i) \text{ for some } i \text{ with } n_i \neq 0\}$$

is finite. Let Y be a finite subset of G such that all the f_{n_i} for $n_i > 0$ are well-defined outside $v(Y)$. On $G \backslash (X \cup Y)$ the formula $\phi_v(v(t_1(x, \bar{g})), \ldots, v(t_k(x, \bar{g})))$ is equivalent to

$$\phi_v(f_{n_1}(v(x)), \ldots, f_{n_k}(v(x))) = \phi'_v(v(x), \bar{\gamma}),$$

where we have put $f_{n_i}(v(x)) = v(h_i) = \gamma_i \in \Gamma$ whenever $n_i = 0$. Since $\phi'_v(y, \bar{\gamma})$ defines a finite or co-finite subset of Γ and the fibres of v are finite, $\phi'_v(v(x), \bar{\gamma})$ defines a finite or co-finite subset of G. It follows that $\phi(x, \bar{g})$ defines a finite or co-finite subset of G. Thus G is minimal as a valued group. $\square$

This yields a classification of valued minimal groups:

Theorem 5. *A valued group G with infinite value set Γ is minimal if and only if*

(1) Γ *has order type* ω^**,*
(2) v *has finite fibres,*
(3) either G *is elementary abelian of exponent* p *and*
  (a) either there is $n_0 < \omega$ *such that* $R_{p^{n_0}}(\Gamma) \cap \neg R_{p^{n_0+1}}(\Gamma)$ *is co-finite,*
  (b) or $R_{p^n}(\Gamma) \cap \neg R_{p^{n+1}}(\Gamma)$ *is finite for all* $n < \omega$*,*
(4) or G *is a finite product of Prüfer* p*-groups,* f_p *is eventually an* $\mathcal{L}_v$*-isomorphism acting by right shift, and there is* $n_0 < \omega$ *such that* $R_{p^{n_0}}(\Gamma) \cap \neg R_{p^{n_0+1}}(\Gamma)$ *is co-finite.*

Proof. Clearly the conditions are necessary for Γ to be minimal as an $\mathcal{L}_v$-structure; it is easy to see that they are also sufficient. Note that the analogue of option (3)(b) cannot occur in case (4), as $|\bar{B}(v(g))/\bar{B}(v(pg))|$ remains bounded by Lemma 5.6. □

Below we give examples showing that all possibilities in Theorem 5 can be realized as a valued abelian group, which hence must be almost linear minimal. In particular, we obtain examples of almost linear minimal groups which are elementary abelian of exponent p (for any prime p) as well as examples of almost linear minimal torsion groups of infinite exponent, namely finite products of Prüfer p-groups (for any fixed prime p). This shows that the conclusion of Theorem 2 is strongest possible.

Example 5.8. Let G be the elementary abelian group of exponent p spanned freely over $\mathbb{Z}_p$ by $\{e_i : i \in \omega\}$, i.e. $G = \bigoplus_{i\in\omega} \mathbb{Z}_p e_i$. Denote by π_i the projection from G to the i-th coordinate. Let $\Gamma = \omega^*$. We define a valuation $v\colon G \to \Gamma$ by

$$v(g) = \begin{cases} \max\{i+1 : \pi_i(g) \neq 0\} & \text{if } g \neq 0, \\ 0 = \infty & \text{if } g = 0, \end{cases}$$

where max is computed in ω. It is clear that Conditions (1), (2) and (3) from Theorem 5 are satisfied. Since $R_1(\Gamma) \cap \neg R_p(\Gamma) = \Gamma \setminus \{\infty\}$, we see that (3)(a) holds, too.

Example 5.9. Take G, Γ and the π_i as in the previous example. We define $v\colon G \to \Gamma$ by

$$v(g) = \begin{cases} \max\{i+1 : (\exists j \geq 2^i - 1)(\pi_j(g) \neq 0)\} & \text{if } g \neq 0, \\ 0 = \infty & \text{if } g = 0. \end{cases}$$

Then, $v(g) = i+1 \iff g \in \mathrm{Lin}(e_{<2^{i+1}-1}) \setminus \mathrm{Lin}(e_{<2^i-1})$ for all $i \in \omega$ (where $\mathrm{Lin}(\emptyset) := \{0\}$). This easily shows that $\neg R_{p^n}(\Gamma)$ is finite for every $n \in \omega$, and so (1), (2), (3) and (3)(b) from Theorem 5 are satisfied.

Example 5.10. Let $G = \mathbb{Z}[p^\infty]$ be the Prüfer p-group, and $\Gamma = \omega^*$. We define $v \colon G \to \Gamma$ by

$$v(g) = \min\{i : p^i g = 0\}.$$

Then, for all $n \in \Gamma \setminus \{\infty\}$, $f_p(n) = n - 1$ (so f_p is the right shift by 1 on $\Gamma \setminus \{\infty\}$), and $R_1(\Gamma) \cap \neg R_p(\Gamma) = \Gamma \setminus \{\infty\}$. Thus, (1), (2) and (4) from Theorem 5 hold.

One can easily generalize this example by taking as G any finite product $\mathbb{Z}[p^\infty]^k$ and as v the valuation defined in the same way as above.

References

[1] R. de Aldama, *Chaînes et dépendance*, PhD Thesis, Lyon 2009.

[2] T. Hyttinen, O. Lessmann, S. Shelah, *Interpreting groups and fields in some nonelementary classes*, J. Math. Logic 5 (2005), 1–47. Shelah [HLSh:821].

[3] G. Jagiella, *On ordered minimal structures*, Master's thesis, Wrocław, 2011.

[4] A. Macintyre, *On ω_1-categorical theories of fields*, Fund. Math. 71 (1971), 1–25.

[5] A. Pillay, *Model theory of algebraically closed fields*, in: Model Theory and Algebraic Geometry. An introduction to E. Hrushovski's proof of the geometric Mordell-Lang conjecture. E. Bouscaren (ed.), Springer, Berlin, 1998.

[6] A. Pillay, P. Tanović, *Generic stability, regularity, and quasiminimality*, in: Models, Logics and Higher-Dimensional Categories (A Tribute to the Work of Mihály Makkai), B. Hart et al.(eds), CRM Proceedings and Lecture Notes 53, American Mathematical Society 2011, 189–211. arXiv:0912.1115v1

[7] K. P. Podewski, *Minimale Ringe*, Math. Phys. Semesterber. 22 (1973), 193–197.

[8] J. Reineke, *Minimale Gruppen*, Zeitsch. Math. Logik Grundl. Math., vol. 21 (1975), 357–359.

[9] P. Simonetta, *Abelian C-minimal groups*, Ann. Pure Applied Logic 110 (2001), 1–22.

[10] P. Tanović, *Minimal first-order structures*, Annals of Pure and Applied Logic 162 (2011), 948–957.

[11] F. Wagner, *Minimal fields*, J. Symbolic Logic 65 (2000), 1833–1835.

[12] W. H. Wheeler *Amalgamation and elimination of quantifiers for theories of fields* Proc. Amer. Math. Soc. 77 (1979), 243–250.

Instytut Matematyczny, Uniwersytet Wrocławski, pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland

E-mail address: kkrup@math.uni.wroc.pl

Mathematical Institute SANU, Belgrade, Serbia

E-mail address: tane@mi.sanu.ac.rs

Université de Lyon; CNRS; Université Lyon 1; Institut Camille Jordan UMR5208, 43 bd du 11 novembre 1918, 69622 Villeurbanne Cedex, France

E-mail address: wagner@math.univ-lyon1.fr