Abelian profinite groups

Krzysztof Krupiński*

Abstract

We investigate abelian profinite groups as profinite structures in the sense of Newelski. We show that each abelian profinite group of finite exponent, being the inverse limit of a system indexed by ω and considered with the standard structural group, is small, m-normal and m-stable. We also obtain similar results for abelian profinite groups with some non-standard structural groups.

0 Introduction

We start from the general definition of profinite structure. A profinite space is an inverse limit of finite discrete spaces (so the topology is induced from the product of finite spaces of our inverse system). We assume that all inverse systems are countable.

Definition 0.1 *A profinite structure is a pair $(X, Aut^*(X))$ consisting of a profinite topological space X and a distinguished structural group $Aut^*(X)$ which is a closed subgroup of the group of all homeomorphisms of X respecting the inverse system defining X. Two profinite structures $(X, Aut^*(X))$ and $(Y, Aut^*(Y))$ are isomorphic if there is a homeomorphism $f : X \to Y$ such that the pullback function f^* maps $Aut^*(Y)$ onto $Aut^*(X)$.*

A profinite group in this context is an inverse limit of finite groups with a structural group preserving the group action. We say that a structural group of a profinite structure [group] X is standard if it is the group of all homeomorphisms [topological automorphisms] of X respecting the appropriate inverse system. When it is clear what the structural group is we just write X instead of $(X, Aut^*(X))$.

W.l.o.g. we can and do assume that all connecting maps of the inverse system defining X are surjections.

The simplest examples of profinite groups are products of countably many finite groups. Let $X = \prod_{i<\omega} X_i$ be such a product. We consider it as the inverse limit of finite groups $X \restriction n = \prod_{i<n} X_i$, $n > 0$, with the natural projections. The standard

*Research supported by KBN grant 2 P03A 016 24

Key words: small profinite groups, abelian profinite groups, abelian structures

structural group consists here of these automorphisms of X which induce automorphisms of each $X\restriction n$ for $n>0$. Any other structural group of X can be chosen as a closed subgroup of the standard one.

We say that X is small if for every natural number $n>0$, on the set $X^n = X\times\ldots\times X$ there are countably many orbits under the action of $Aut^*(X)$. Profinite structures and groups in this sense have been introduced in [N1] and [N2]. Small profinite groups occur naturally in model theory as profinite groups interpretable in small theories [N1]. Newelski has developed the model theory of small profinite structures. Many results from stable model theory have been proven in this context. m-normality and m-stability play the prominent role in all these considerations.

The only known explicit examples of small profinite groups are products of finite groups [K1]. More precisely, if X is a product of finitely many finite groups and countably many finite abelian groups of bounded exponent with the standard structural group, then X is small. In [K1] examples of small products of finite groups with some non-standard structural groups are also given.

The goal of this paper is to find new classes of examples of small profinite groups. We generalize here most of the results obtained for products of finite groups in [K1] to the case of abelian profinite groups. The main result is Theorem 1.9, which says that each abelian profinite group X of finite exponent, being the inverse limit of a system indexed by ω and considered with the standard structural group, is small, m-normal and m-stable. Similarly as for products of finite groups in [K1], to prove this result, first we find a description of orbits in X under the action of the standard structural group. However, we obtain such a description by means of model theory of abelian structures, which is the different approach from that in [K1].

Here is a good place to recall that each torsion abelian profinite group is isomorphic to a direct product of finite abelian groups (see [RZ], Corollary 4.3.9) and to explain why this result does not reduce the case of abelian profinite groups (regarded as profinite structures) to the case of products of finite groups considered in [K1]. The point is that the above result yields only an isomorphism of topological groups and not of profinite structures. More precisely, we are going to consider an abelian profinite group X of finite exponent as a profinite structure with the standard structural group $Aut^*(X)$ induced from the inverse system defining X. Corollary 4.3.9 from [RZ] tells us that $X\cong G=\prod_{i\in\omega}G_i$ for some finite abelian groups G_i but, of course, the inverse system defining X need not to be isomorphic to the system of all initial subproducts $G\restriction n=\prod_{i<n}G_i$, $n>0$, with the natural projections. So there is no reason for $(X, Aut^*(X))$ to be isomorphic to $(G, Aut^*(G))$, where $Aut^*(G)$ is the standard structural group of G (i.e. it is induced by the inverse system $G\restriction n$, $n>0$).

Now we present all necessary definitions and basic facts on profinite structures and groups in the sense of Newelski. For the proofs and more details about profinite structures and groups see [N1], [N2] and [Wa].

Let X be a profinite structure, e.g. a profinite group. Let $A\subseteq X$ be finite. By $Aut^*(X/A)$ we denote the set of elements of $Aut^*(X)$ fixing A pointwise. We say that $V\subseteq X$ is A-invariant if $f[V]=V$ for every $f\in Aut^*(X/A)$. If V is additionally closed, then we say that V is A-definable. For $a\in X^n$ and $A\subseteq X$ we define $o(a/A)=$

$\{f(a) : f \in Aut^*(X/A)\}$ (the orbit of a over A). Let $O_n(A) = \{o(a/A) : a \in X^n\}$. From now on $A, B, \ldots$ usually denote finite subsets of X and $a, b, \ldots$ denote elements or finite tuples of elements of X.

Definition 0.2 *We say that a profinite structure X is small if $|O_n(\emptyset)| \leq \omega$ for every natural number $n > 0$. Equivalently $O_1(A)$ is countable for every finite set $A \subseteq X$.*

Every profinite structure can be enlarged to X^{eq} by adding so called imaginary elements i.e. elements of the form a/E, where $a \in X^n$ and E is a $\emptyset$-definable equivalence relation on X^n. If X is small, then, for every such E, X^n/E is still a profinite structure, where the structural group is induced by $Aut^*(X)$ acting on X/E. For more details see [N2].

We say that a profinite structure X is interpretable in a profinite structure Y if there is a continous $1-1$ mapping f of X onto a set $f(X)$ definable in Y^{eq} over a finite set A, such that the pullback function maps $Aut^*(Y/A)$ onto a closed subgroup of $Aut^*(X)$. It is easy to see that any profinite structure interpretable in a small one is also small.

For a finite $A \subseteq X$, by $acl(A)$ (or $acl^{eq}(A)$) we denote the algebraic closure of A, i.e. the set of these elements of X^{eq} which have finitely many conjugates under $Aut^*(X/A)$. When A is infinite, we define $acl(A) = \bigcup\{acl(A') : A' \subseteq A \text{ is finite}\}$.

Definition 0.3 *The rank $\mathcal{M}$ is the function from the collection of orbits over finite sets to the ordinals together with ∞ satisfying*

$$\mathcal{M}(a/A) \geq \alpha + 1 \text{ iff there is a finite } B \supseteq A \text{ with } o(a/B) \text{ nowhere dense in } o(a/A) \text{ and } \mathcal{M}(a/B) \geq \alpha.$$

Definition 0.4 *A profinite structure X is m-stable if every orbit has an ordinal $\mathcal{M}$-rank. Equivalently there is no infinite sequence $A_1 \subseteq A_2 \subseteq \ldots$ of finite subsets of X and $a \in X$ such that $o(a/A_{i+1})$ is nowhere dense in $o(a/A_i)$ for every i.*

Definition 0.5 *A profinite structure X is m-normal if for every finite $a, A \subseteq X$, there is a clopen $U \ni a$, such that $U \cap o(a/A)$ has finitely many conjugates under $Aut^*(X/a)$.*

In the above definition we can choose as U a canonical open set definable over a, where by a canonical open set in

$$X = \lim_{\longleftarrow} X_i \subseteq \prod X_i$$

we mean the set of elements of X with the i-th coordinate fixed (i is arbitrary). A canonical open set in X^n is a product of canonical open sets in X.

It is worth noticing that in [N1] and [N2], m-normality and m-stability have been investigated only under the assumption of smallness. This is because of good model theoretic properties of these notions under the assumption of smallness.

At the end I would to thank the anonymous referee for some interesting comments and suggestions.

1 Description of orbits and the main result

This section is constructed as follows: first we recall the notion of abelian structure; then, using it, we obtain a description of orbits in abelian profinite groups under the action of the standard structural group; as a conclusion we obtain the main result – Theorem 1.9.

Definition 1.1 *An abelian structure $\mathcal{A} = (A, +, P_i)_{i\in I}$ is an abelian group $(A, +)$ expanded by predicates P_i, $i \in I$, being subgroups of A^{n_i} for $n_i \in \omega$.*

If we add to the language function symbols defining homomorphisms from some cartesian powers of A into A, then the arising structure can also be regarded as an abelian structure. Namely, each homomorphism $f : A^n \to A$ can be identified with its graph $G_f = \{(a, f(a)) : a \in A^n\}$, a subgroup of A^{n+1}.

Now we list several examples of abelian structures.

1. A pure abelian group $\mathcal{A} = (A, +)$.
2. A pure R-module $\mathcal{A} = (A, +, r\cdot)_{r\in R}$ for an arbitrary ring R.
3. An R-module $\mathcal{A} = (A, +, r\cdot, P_i)_{r\in R, i\in I}$, where each P_i is a unary predicate defining a subgroup of $(A, +)$.

   (i) Let $X = \prod_{i\in\omega} X_i$ be a product of countably many finite abelian groups, and predicates P_n, $n \in \omega$, be defined in the following way

   $$P_n(\eta) \iff \eta \restriction n = (0, \ldots, 0),$$

   where for $\eta \in X$, by $\eta \restriction n$ we denote the first n coordinates of η. Then $\mathcal{X} := (X, +, r\cdot, P_n)_{r\in\mathbb{Z}, n\in\omega}$ is an abelian structure such that $f \in Aut^*(X) \iff f \in Aut(\mathcal{X})$, where $Aut^*(X)$ is the standard structural group of X. The same is also true when X is a product of finitely many groups.

   (ii) Let X be the inverse limit of an inverse system of finite abelian groups X_i, $i \in \omega$. Let $f_i : X \to X_i$ be the natural projection and $P_i = f_i^{-1}(0) < X$. Then $\mathcal{X} := (X, +, r\cdot, P_i)_{r\in\mathbb{Z}, i\in\omega}$ is an abelian structure such that $f \in Aut^*(X) \iff f \in Aut(\mathcal{X})$, where $Aut^*(X)$ is the standard structural group of X. The same is true if X is the inverse limit of a finite system.

Let $\mathcal{A} = (A, +, P_i, f_j)_{i\in I, j\in J}$ be an abelian structure (i.e. P_i's are predicates being subgroups of cartesian powers of A and f_j's are homomorphisms from cartesian powers of A into A).

Definition 1.2 *A formula $\phi(\overline{x})$ (in the language of $\mathcal{A}$) is a p.p. formula if it is equivalent modulo $Th(\mathcal{A})$ to a formula of the form*

$$\exists \overline{y} \bigwedge_{i<k} P_{n_i}(t_1^i(\overline{y}, \overline{x}), \ldots, t_{k_i}^i(\overline{y}, \overline{x})),$$

where $\overline{y}, \overline{x}$ are finite tuples of variables, $k \in \omega$, $n_i \in I \cup \{=\}$ ($\models P_=(a) \iff a = 0$) and $t^i_1(\overline{y}, \overline{x}), \ldots, t^i_{k_i}(\overline{y}, \overline{x})$ are terms in the language $\{+, f_j\}_{j \in J}$.

A particular case is when $\mathcal{A} = (A, +, r\cdot)_{r \in R}$ is a pure module over R. Then $\phi(x_1, \ldots, x_m)$ is a p.p. formula if it is equivalent modulo $Th(\mathcal{A})$ to a formula of the form

$$\exists y_1, \ldots, y_k \bigwedge_{i<k} \sum_{j=1}^{k} a_{ij} \cdot y_j + \sum_{j=1}^{m} b_{ij} \cdot x_j = 0,$$

where $a_{ij}\cdot$ and $b_{ij}\cdot$ are the scalar multiplication by elements of R. The following fact is known as the Baur-Garavaglia-Monk's theorem.

Fact 1.3 *In any pure module, each formula $\phi(\overline{x})$ is equivalent to a Boolean combination of p.p. formulas.*

The proof of the Baur-Garavaglia-Monk's theorem ([Ba], [Z]) is made by induction on structure of formula $\phi(\overline{x})$. It uses B.H. Neumann's lemma on covering of a group by a set of cosets and some basic combinatorial tricks. One can check that to prove this theorem it is enough to have it for atomic formulas and to use the following properties of p.p. formulas (it is not important what p.p. formulas look like, we use only their properties). Let $\mathcal{A} = (A, +, \ldots)$ be a model in which we formulate the theorem ($(A, +)$ is an abelian group).

(i) P.p. formulas are closed under existential quantification and conjunction.

(ii) Each p.p. formula $\varphi(x_1, \ldots, x_m)$ defines a subgroup of A^m.

(iii) If $\varphi(x, y)$ is a p.p. formula, then $\varphi(0, y)$ is also a p.p. formula.

(iv) If $\varphi(x, y)$ is a p.p. formula and a is a tuple from $\mathcal{A}$ of the same length as y, then $\varphi(\mathcal{A}, a) = \emptyset$ or $\varphi(\mathcal{A}, a)$ is a coset of $\varphi(\mathcal{A}, 0)$.

We easily see that in any abelian structure p.p. formulas have all these properties, so as a conclusion we obtain the Baur-Garavaglia-Monk's theorem for abelian structures.

While considering abelian structures we should mention that they were introduced and originally studied by E. Fischer ([F]). The p.p. elimination for abelian structures is mentioned in [We]. In modern model theory abelian stuctures are basic examples of stable structures, more precisely they are 1-based groups ([Bu]).

We start our consideration on abelian profinite groups from the following easy remark.

Remark 1.4 *If an abelian profinite group is small, then it has a finite exponent.*

Proof. Let G be a small abelian profinite group. By Proposition 2.4 from [N1] we know that G is locally finite. Hence it is a torsion abelian profinite group. Lemma 4.3.7 from [RZ] tells us that in such situation G has a finite exponent. ■

From now on we deal with abelian profinite groups, which are inverse limits of inverse systems

$$X_0 \longleftarrow X_1 \longleftarrow \dots$$

(indexed by ω) of finite abelian groups. We will consider such a group X with the standard structural group $Aut^*(X)$.

For an element $\eta \in X$, $\eta(n) \in X_n$ denotes the n-th coordinate of η and for $\eta = (\eta_1, \dots, \eta_k) \in X^k$, by $\eta(n)$ we denote the tuple $(\eta_1(n), \dots, \eta_k(n)) \in X_n^k$. Let $\alpha = (\alpha_1, \dots, \alpha_m) \in X^m$ and A be a finite submodule (over $\mathbb{Z}$) of X.

Lemma 1.5 (Description of orbits) *$o(\alpha/A) = U$, where U consists of elements $\beta \in X^m$ such that for all $k \geq 1$, natural numbers $n_1 \leq \dots \leq n_k$, $(a_{ij})_{1\leq i,j \leq k} \in M_{k\times k}(\mathbb{Z})$, $(b_{ij})_{i\leq k, j\leq m} \in M_{k\times m}(\mathbb{Z})$ and $(\eta_1, \dots, \eta_k) \in A^k$ we have*

$$\left(\exists \overline{y} \in X^k\right) \bigwedge_{i=1}^{k} \left(\sum_{j=1}^{k} a_{ij} y_j(n_i) + \sum_{j=1}^{m} b_{ij} \alpha_j(n_i) + \eta_i(n_i) = 0 \right)$$

$$\Updownarrow$$

$$\left(\exists \overline{y} \in X^k\right) \bigwedge_{i=1}^{k} \left(\sum_{j=1}^{k} a_{ij} y_j(n_i) + \sum_{j=1}^{m} b_{ij} \beta_j(n_i) + \eta_i(n_i) = 0 \right).$$

Proof. ($\subseteq$) is obvious.
($\supseteq$) First, in Step 1, we will show this inclusion for the variant of the lemma with X being the inverse limit of a finite system $X_0 \longleftarrow \dots \longleftarrow X_n$ of finite abelian groups considered with the standard structural group $Aut^*(X)$ (that is, after an obvious identification, $X = X_n$ and $Aut^*(X) = Aut^*(X_n)$ is the standard structural group of X_n, which may be smaller than $Aut(X_n)$). Then, in Step 2, we will conclude that the desired inclusion from the lemma holds in general.

Step 1 Assume that X is the inverse limit of a finite system $X_0 \longleftarrow \dots \longleftarrow X_n$ of finite abelian groups considered with the standard structural group $Aut^*(X)$.

By example 3 (ii) from our list of examples of abelian structures we can replace X by a suitable finite abelian structure $\mathcal{X}$. Since $\mathcal{X}$ is finite, we get

$$tp(\alpha/A) = tp(\beta/A) \text{ iff } (\exists f \in Aut(\mathcal{X}/A))(f(\alpha) = \beta).$$

So to finish the proof we have to show that the type $tp(\alpha/A)$ in variables $x_1, \dots, x_m$ is determined by the set of these formulas of the form

$$\left(\exists \overline{y} \in X^k\right) \bigwedge_{i=1}^{k} P_{n_i}\left(\sum_{j=1}^{k} a_{ij} y_j + \sum_{j=1}^{m} b_{ij} x_j + \eta_i\right),$$

where $n_1 \leq \dots \leq n_k \leq n$, $(a_{ij})_{1\leq i,j\leq k} \in M_{k\times k}(\mathbb{Z})$, $(b_{ij})_{i\leq k, j\leq m} \in M_{k\times m}(\mathbb{Z})$ and $(\eta_1, \dots, \eta_k) \in A^k$, which are satisfied by α. This follows from the following statement

Every formula $\varphi(x_1, \ldots, x_m)$ is equivalent modulo $Th(\mathcal{X})$ to a Boolean combination of formulas of the form

$$(\exists \overline{y} \in X^k) \bigwedge_{i=1}^{k} P_{n_i}(\sum_{j=1}^{k} a_{ij} y_j + \sum_{j=1}^{m} b_{ij} x_j),$$

which is exactly the Baur-Garavaglia-Monk's theorem applied to $\mathcal{X}$.

Step 2 Now consider the general situation from the lemma.

Assume $\beta \in U$. We want to find an $f \in Aut^*(X/A)$ mapping β to α. For each n, β satisfies the set of conditions guaranteeing (in virtue of Step 1) existence of a $g_n \in Aut^*(X_n)$ mapping $\beta(n)$ to $\alpha(n)$ and fixing the set $A(n) = \{\eta(n) : \eta \in A\}$ pointwise.

Since $Aut^*(X_n)$ is the standard structural group of X_n regarded as the inverse limit of the system $X_0 \longleftarrow \ldots \longleftarrow X_n$, each g_n determines a sequence of automorphisms $g_{n,k} \in Aut^*(X_k)$, $k \leq n$.

Now we define recursively sequences $f_n \in Aut^*(X_n)$ and $I_n \subseteq \omega \setminus n$, $n \in \omega$, such that

1. $I_0 \supseteq I_1 \supseteq \ldots$ are all infinite,
2. for $k \in I_n$, we have $f_n = g_{k,n}$.

The construction is straightforward, using the pigeon-hole rule.

We see that the sequence f_n, $n \in \omega$, is an automorphism of the system converging to an $f \in Aut^*(X/A)$ such that $f(\beta) = \alpha$. ■

The above lemma is too weak to show Theorem 1.9. We need here a deeper insight in the structure of orbits (as in Lemma 3.1.1 in [K1]).

Assume that X has a finite exponent e. For $n \in \omega$, by ${}^{n}X$ we denote the inverse limit of the system

$$X_n \longleftarrow X_{n+1} \longleftarrow \ldots$$

and we consider it with the standard structural group $Aut^*_n(X)$ (of course ${}^{n}X$ is naturally homeomorphic with X, so these spaces can be identified). $o^n(\cdot)$ denotes an orbit in $({}^{n}X, Aut^*_n(X))$.

Let $\alpha = (\alpha_1, \ldots, \alpha_m) \in X^m$ and A be a finite submodule (over $\mathbb{Z}_e := \mathbb{Z}/e\mathbb{Z}$) of X. For $k, l_1, \ldots, l_m \in \mathbb{Z}_e$ and $a \in A$ we define

$$n_{k,l,a} = \begin{cases} max\{n \in \omega : k \mid (\sum_{i=1}^{m} l_i \alpha_i(n) - a(n))\} & \text{, when such a maximal } n \text{ exists} \\ -1 & \text{, otherwise.} \end{cases}$$

Let $N_{\alpha,A} = max\{n_{k,l,a} : (k,l,a) \in \mathbb{Z}_e \times \mathbb{Z}_e^m \times A\} + 1$. Take $n \geq N_{\alpha,A}$.

Lemma 1.6 *$o^n(\alpha/A\alpha(n)) = U$, where U consists of elements $\beta \in X^m$ such that $\beta(n) = \alpha(n)$ and for all $a \in A, k \in \mathbb{Z}_e, l_1, \ldots, l_m \in \mathbb{Z}_e$ we have*

$$k \mid \left(\sum_{i=1}^{m} l_i \alpha_i - a \right) \Longrightarrow k \mid \left(\sum_{i=1}^{m} l_i \beta_i - a \right).$$

Proof. ($\subseteq$) is obvious.
($\supseteq$) Take a $\beta \in U$. Let $k \geq 1$, $n \leq n_1 \leq \ldots \leq n_k$, $(a_{ij})_{1\leq i,j\leq k} \in M_{k\times k}(\mathbb{Z}_e)$, $(b_{ij})_{i\leq k, j\leq m} \in M_{k\times m}(\mathbb{Z}_e)$ and $\eta = (\eta_1, \ldots, \eta_k) \in A^k$.

We define the $\mathbb{Z}_e$-linear combinations $t_i(\overline{y}, \overline{x}, \overline{z})$, $i = 1, \ldots k$, of variables $\overline{x} = (x_1, \ldots, x_m)$, $\overline{y} = (y_1, \ldots, y_k)$ and $\overline{z} = (z_1, \ldots, z_k)$ in the following way.

$$t_i(\overline{y}, \overline{x}, \overline{z}) = \sum_{j=1}^{k} a_{ij} y_j + \sum_{j=1}^{m} b_{ij} x_j + z_i.$$

We need the following fact, which can be found for example in [A]. Its proof uses the Euclidean property of $\mathbb{Z}$.

Fact 1 *For every matrix $H \in M_{n\times n}(\mathbb{Z})$ there exist invertible matrices $L, R \in M_{n\times n}(\mathbb{Z})$ such that the matrix LHR is diagonal.*

Using Fact 1 we get that there are $\mathbb{Z}_e$-linear combinations $t'_i(\overline{x}, \overline{z})$, $i = 1, \ldots, k$, and elements $k_1, \ldots, k_k \in \mathbb{Z}_e$ such that in each $\mathbb{Z}_e$-module M, for each tuples $s = (s_1, \ldots, s_m) \in M^m$ and $r = (r_1, \ldots, r_k) \in M^k$, we have

$$M \models \exists \overline{y} \bigwedge_{i=1}^{k} t_i(\overline{y}, s, r) = 0 \iff M \models \bigwedge_{i=1}^{k} k_i \mid t'_i(s, r).$$

By Lemma 1.5 we see that to finish our proof we need to show that

$$\exists \overline{y} \in X^k \left(\bigwedge_{i=1}^{k} t_i(\overline{y}(n_i), \alpha(n_i), \eta(n_i)) = 0 \right) \iff \exists \overline{y} \in X^k \left(\bigwedge_{i=1}^{k} t_i(\overline{y}(n_i), \beta(n_i), \eta(n_i)) = 0 \right).$$

($\Rightarrow$) Assume that for some $\overline{y} \in X^k$ we have

$$\bigwedge_{i=1}^{k} t_i(\overline{y}(n_i), \alpha(n_i), \eta(n_i)) = 0.$$

As $n \leq n_1 \leq \ldots \leq n_k$, we see that

$$\bigwedge_{i=1}^{k} t_i(\overline{y}(n), \alpha(n), \eta(n)) = 0.$$

By the definition of $t'_i(\overline{x}, \overline{z})$, $i = 1, \ldots, k$, we get

$$\bigwedge_{i=1}^{k} k_i \mid t'_i(\alpha(n), \eta(n)).$$

By the choice of n we conclude that

$$\bigwedge_{i=1}^{k} k_i \mid t'_i(\alpha, \eta).$$

From the assumption that $\beta \in U$ we see that

$$\bigwedge_{i=1}^{k} k_i \mid t_i'(\beta, \eta).$$

Once again by the definition of $t_i'(\overline{x}, \overline{z})$, $i = 1, \ldots, k$, we get that there is a $\overline{y}' \in X^k$ such that

$$\bigwedge_{i=1}^{k} t_i(\overline{y}', \beta, \eta) = 0,$$

let alone

$$\bigwedge_{i=1}^{k} t_i(\overline{y}'(n_i), \beta(n_i), \eta(n_i)) = 0.$$

($\Leftarrow$) Assume that for some $\overline{y} \in X^k$ we have

$$\bigwedge_{i=1}^{k} t_i(\overline{y}(n_i), \beta(n_i), \eta(n_i)) = 0.$$

Hence

$$\bigwedge_{i=1}^{k} t_i(\overline{y}(n), \beta(n), \eta(n)) = 0,$$

but $\alpha(n) = \beta(n)$, so

$$\bigwedge_{i=1}^{k} t_i(\overline{y}(n), \alpha(n), \eta(n)) = 0.$$

As in the proof of ($\Rightarrow$) we get

$$\bigwedge_{i=1}^{k} k_i \mid t_i'(\alpha, \eta).$$

By definition of $t_i'(\overline{x}, \overline{z})$, $i = 1, \ldots, k$, we get that there is a $\overline{y}' \in X^k$ such that

$$\bigwedge_{i=1}^{k} t_i(\overline{y}', \alpha, \eta) = 0,$$

let alone

$$\bigwedge_{i=1}^{k} t_i(\overline{y}'(n_i), \alpha(n_i), \eta(n_i)) = 0. \ \blacksquare$$

Now we will show some general remark.

Remark 1.7 *Let X be an arbitrary profinite structure, α be a finite tuple from X, $A \subseteq X$ (or even $A \subseteq X^{eq}$) and $a \in acl^{eq}(A)$. Then*
(i) $o(\alpha/A)$ is a union of finitely many orbits over Aa.
(ii) $o(\alpha/Aa)$ is an open subset of $o(\alpha/A)$.

Proof. (i) Since $o(a/A)$ is finite, it splits over $A\alpha$ into finitely many orbits. That is, there are finitely many $a_0, \ldots, a_n \in o(a/A)$ such that

$$o(a/A) = o(a_1/A\alpha) \cup \ldots \cup o(a_n/A\alpha).$$

For each $i \leq n$ choose $\alpha_i \in o(\alpha/A)$ with $o(a\alpha_i/A) = o(a_i\alpha/A)$. Then for every $\beta \in o(\alpha/A)$ there is an $i \leq n$ such that $o(a\beta/A) = o(a_i\alpha/A) = o(a\alpha_i/A)$, hence $o(\beta/Aa) = o(\alpha_i/Aa)$. We see that

$$o(\alpha/A) = o(\alpha_0/Aa) \cup \ldots \cup o(\alpha_n/Aa).$$

(ii) We have

$$o(\alpha/A) = \bigcup_{f \in Aut^*(X/A)} f[o(\alpha/Aa)] = \bigcup_{f \in Aut^*(X/A)} o(f(\alpha)/Af(a)).$$

So by (i), the last union has only finitely many distinct summands and one of them must be open in $o(\alpha/A)$. Hence $o(\alpha/Aa)$ is open in $o(\alpha/A)$. ■

Now we come back to our profinite group X. Let $n \in \omega$, $\alpha \in X$ and A be a subset of X (or even of X^{eq}).

Corollary 1.8 *(i) $o^n(\alpha/A\alpha(n))$ is a union of finitely many orbits in X over $A\alpha(n)$.*
(ii) $o(\alpha/A\alpha(n))$ is open in $o^n(\alpha/A\alpha(n))$.
(iii) $o(\alpha/A)$ is open in $o^n(\alpha/A)$.

Proof. Of course (i) and (iii) follows from (ii), so it is enough to prove (ii). First we see that

$$o^n(\alpha/AX_n) = o(\alpha/AX_n) \subseteq o(\alpha/A\alpha(n)) \subseteq o^n(\alpha/A\alpha(n))$$

and $o(\alpha/A\alpha(n))$ is either open or nowhere dense in $o^n(\alpha/A\alpha(n))$. Hence it is enough to show that $o^n(\alpha/AX_n)$ is open in $o^n(\alpha/A\alpha(n))$.

Since X_n can be identified with an element from $acl^{eq}(\emptyset)$, we use Remark 1.7 and we are done. ■

Theorem 1.9 *Assume that X is the inverse limit of a system indexed by ω of finite abelian groups and X has a finite exponent e. Let $Aut^*(X)$ be the standard structural group of X. Then $(X, Aut^*(X))$ is small, m-normal and m-stable.*

Proof. (1) smallness
Suppose for a contradiction that $|O_1(A)| > \omega$ for some finite subset A of X, so $|O_1(A)| = 2^\omega$. W.l.o.g. we can assume that A is a finite $\mathbb{Z}_e$-submodule of X.

For an $\alpha \in X$, let n_α be equal to the number $N_{\alpha,A}$ defined before Lemma 1.6. We have that there is a natural number n and 2^ω-many elements $\alpha \in X$ from distinct orbits over A for which $n_\alpha = n$. Hence there is an $\alpha \in X$ and $\alpha_i \in X$, $i \in 2^\omega$, such that

1. $\alpha_i(n) = \alpha(n)$ and $n_{\alpha_i} = n_\alpha = n$, for all $i \in 2^\omega$,

2. $\alpha_i \notin o(\alpha_j/A)$, for all $i \neq j$.

Let
$$S = \{\beta \in X : \beta(n) = \alpha(n) \ \wedge \ n_\beta = n\}.$$
We see that S is invariant under $Aut^*(X/A\alpha(n))$ and $Aut^*_n(X/A\alpha(n))$.

By Lemma 1.6 and Corollary 1.8 (i) we obtain respectively:

(i) there are only finitely many orbits on S under the action of $Aut^*_n(X/A\alpha(n))$,

(ii) for each $\beta \in S$, the orbit $o^n(\beta/A\alpha(n))$ is a union of finitely many orbits in X over $A\alpha(n)$.

By (i) and (ii) we get that there are only finitely many orbits on S under the action of $Aut^*(X/A\alpha(n))$. This is a contradiction with 1 and 2. □

(2) m-normality
Suppose for a contradiction that X is not m-normal and a lack of m-normality is witnessed by an $\alpha \in X^m$ and a finite submodule A of X, i.e. for each open neighbourhood U of α, the set $U \cap o(\alpha/A)$ has infinitely many conjugates under $Aut^*(X/\alpha)$. Let $n \in \omega$.

Claim $({}^n\!X, Aut^*_n(X))$ *is not m-normal and it is also witnessed by α and A.*

Proof. It is enough to show that for an arbitrarily small canonical open neighbourhood U of α, the set $U \cap o^n(\alpha/A)$ has infinitely many conjugates under $Aut^*_n(X/\alpha)$.

By Corollary 1.8 (iii) we have that $o(\alpha/A)$ is open in $o^n(\alpha/A)$. Let U be an arbitrary canonical open neighbourhood of α such that
$$U \cap o^n(\alpha/A) = U \cap o(\alpha/A).$$
Since $Aut^*(X/\alpha) \subseteq Aut^*_n(X/\alpha)$, the proof is completed. □

Now we choose an n for α and A as before Lemma 1.6. By the claim we see that there is an $f \in Aut^*_n(X/\alpha)$ and $\beta \in o^n(\alpha/A\alpha(n))$ such that $f(\beta) \notin o^n(\alpha/A\alpha(n))$.

Showing that $f(\beta) \in o^n(\alpha/A\alpha(n))$ we will get a contradiction. By Lemma 1.6 it is enough to prove that for all $a \in A, k \in \mathbb{Z}_e, l_1, \ldots, l_m \in \mathbb{Z}_e$ we have
$$k \mid \left(\sum_{i=1}^{m} l_i\alpha_i - a\right) \Longrightarrow k \mid \left(\sum_{i=1}^{m} l_i f(\beta_i) - a\right).$$
Let $k \mid (\sum_{i=1}^m l_i\alpha_i - a)$. Then $k \mid (\sum_{i=1}^m l_i\alpha_i - f(a))$, so
$$k \mid (a - f(a)).$$
Now $\beta \in o^n(\alpha/A\alpha(n))$, so by Lemma 1.6 we get that $k \mid (\sum_{i=1}^m l_i\beta_i - a)$. Hence
$$k \mid \left(\sum_{i=1}^{m} l_i f(\beta_i) - f(a)\right)$$

and finally $k \mid (\sum_{i=1}^m l_i f(\beta_i) - a)$. $\square$

(3) m-stability
Suppose for a contradiction that there is an infinite sequence $A_1 \subseteq A_2 \subseteq \ldots$ of finite subsets of X and $\alpha \in X$ such that $o(\alpha/A_{i+1})$ is nowhere dense in $o(\alpha/A_i)$ for every i. W.l.o.g. each A_i is a submodule of X.

Let $C_i = \{(k,l) \in \mathbb{Z}_e^2 : (\exists a \in A_i)(k \mid (l\alpha - a))\}$. Let n_i be equal to the number N_{α,A_i} defined before Lemma 1.6. So $n_1 \leq n_2 \leq \ldots$.

We have that $o(\alpha/A_{i+1}\alpha(n_{i+1}))$ is nowhere dense in $o(\alpha/A_i\alpha(n_{i+1}))$. By Corollary 1.8 (ii) we conclude that $o^{n_{i+1}}(\alpha/A_{i+1}\alpha(n_{i+1}))$ is nowhere dense in $o^{n_{i+1}}(\alpha/A_i\alpha(n_{i+1}))$.

Using Lemma 1.6 one can conclude that $C_i \subsetneq C_{i+1}$ for every i. So we get an infinite increasing sequence $C_1 \subsetneq C_2 \subsetneq \ldots \subseteq \mathbb{Z}_e^2$, a contradiction. $\blacksquare$

At the end we would like to say that the description of orbits in products of finite abelian groups, which is given in [K1] (Lemma 3.1.1), can also be obtained by means of model theory of abelian structures. Namely, we can apply Lemma 1.5 (in fact, we use here the Baur-Garavaglia-Monk's theorem for abelian structures from example 3 (i)) and Fact 1 formulated in the proof of Lemma 1.6 (for the details look at the proof of Lemma 2.1.6 in [K3]). In [K1] I proved this description using a group theoretic result ([K1], Lemma 2.1).

For the reader's convenience, we recall here both the group theoretic result mentioned above and the description of orbits in products of finite abelian groups.

Fact 1.10 *Let $X = X_0 \times X_1 \times \ldots \times X_w$ be a finite product of finite abelian groups, e be the exponent of X and $n \leq w$. If we have finite submodules A and B of X, an isomorphism f between them and $\alpha \in X$, $\beta_n \in X \restriction n$ such that*

$$(\forall a \in A)(\forall k \in \mathbb{Z}_e)(\forall m \leq w+1)(k \mid a \restriction m \iff k \mid f(a) \restriction m)$$

and

$$(\forall a \in A)(\forall k, l \in \mathbb{Z}_e)(\forall m \leq n)(k \mid (l\alpha \restriction m - a \restriction m) \iff k \mid (l\beta_n \restriction m - f(a) \restriction m)),$$

then there exists a $\beta_{n+1} \in X \restriction n+1$ extending β_n, such that

$$(\forall a \in A)(\forall k, l \in \mathbb{Z}_e)(k \mid (l\alpha \restriction n+1 - a \restriction n+1) \iff k \mid (l\beta_{n+1} - f(a) \restriction n+1)).$$

For $n = 0$ there is no β_0 and then Fact 1.10 says that there exists a $\beta_1 \in X_0$ satisfying the last condition.

Let $X = \prod_{i \in \omega} X_i$, where all X_i's are finite abelian groups and let the exponent e of X be finite. So X is a module over $\mathbb{Z}_e$. Let $\alpha = (\alpha_1 \ldots \alpha_m)$ be a tuple from X and A be a finite submodule of X.

Fact 1.11 (Description of orbits) *$o(\alpha/A) = U$, where U consists of elements $\beta \in X^m$ such that for all $a \in A, k \in \mathbb{Z}_e, l_1, \ldots, l_m \in \mathbb{Z}_e$ and $n \geq 1$ we have*

$$k \mid \left(\sum_{i=1}^m l_i \alpha_i \restriction n - a \restriction n \right) \iff k \mid \left(\sum_{i=1}^m l_i \beta_i \restriction n - a \restriction n \right).$$

2 Modifications of the structural groups

In this section we will show several results, similar to Theorem 1.9, for abelian profinite groups with some non-standard structural groups. To do this we need a description of orbits more general than in Lemma 1.6. We will obtain such a description by means of model theory of many-sorted abelian structures. Such structures were introduced by E. Fischer ([F]). The following definition can be found for example in [PRZ].

Definition 2.1 *A many-sorted abelian structure $\mathcal{A} = ((A_s, +)_{s\in S}, P_i)_{i\in I}$ is a disjoint union of abelian groups $(A_s, +)$ (these are sorts of $\mathcal{A}$) expanded by predicates P_i, $i \in I$, being subgroups of cartesian products of finitely many sorts.*

Similarly as for one-sorted abelian structures, if we add to the language function symbols defining homomorphisms from cartesian products of finitely many sorts into sorts, then the arising many-sorted structure can also be regarded as a many-sorted abelian structure.

Now we continue our list of examples of abelian structures, this time of many-sorted abelian structures.

4. A one-sorted abelian structure.

5. A structure $\mathcal{A} = ((A_i, +, r\cdot)_{r\in R, i\in\omega}, f_{ij})_{i,j\in\omega}$, whose sorts are pure R-modules $(A_i, +, r\cdot)_{r\in R}$, $i \in \omega$, and each $f_{ij} : A_i \to A_j$ is a homomorphism of R-modules.

   (i) Let $\{X_i\}_{i\in\omega}$ be a family of finite abelian groups. Let $X = \prod_{i\in\omega} X_i$ and $A_n = X \restriction n$ for $n \geq 1$. For $n \geq m$ we define $f_{nm} : A_n \to A_m$ as the restriction. Then $\mathcal{A} := ((A_n, +, r\cdot)_{r\in\mathbb{Z}, n\in\omega}, f_{nm})_{n\geq m}$ is a many-sorted abelian structure. Let $Aut^*(X)$ be the standard structural group of X. We have that each $f \in Aut(\mathcal{A})$ induces an automorphism $\hat{f} \in Aut^*(X)$. On the other hand, all the restrictions to A_n's of an arbitrary $f \in Aut^*(X)$ form an automorphism of $\mathcal{A}$. The same is also true if there are only finitely many groups X_i.

   (ii) Let X be the inverse limit of an inverse system $(X_i, f_{ij})_{j\leq i\in\omega}$ of finite abelian groups. Then $\mathcal{A} := ((X_i, +, r\cdot)_{r\in\mathbb{Z}, i\in\omega}, f_{ij})_{i\geq j}$ is a many-sorted abelian structure. As always we assume that all f_{ij}'s are surjections. As above we have the natural correspondence between $Aut(\mathcal{A})$ and the standard structural group $Aut^*(X)$ of X.

Let $\mathcal{A} = ((A_s, +)_{s\in S}, P_i, f_j)_{i\in I, j\in J}$ be a many-sorted abelian structure. In the many-sorted language of $\mathcal{A}$ we consider formulas and we define p.p. formulas in the same way as in Definition 2.1, remembering that each variable ranges over a specified sort.

As for one-sorted abelian structures, it is easy to check that in each many-sorted abelian structure atomic formulas are p.p. formulas and that p.p. formulas have properties (i), (iii), (iv) and (ii)' Each p.p. formula $\varphi(x_1,\ldots,x_m)$ defines a subgroup of a product of appropriate sorts $A_{s_1}\times\ldots\times A_{s_m}$.

So the Baur-Garavaglia-Monk's theorem holds for many-sorted abelian structures.

We recall that we deal with abelian profinite groups, which are inverse limits of inverse systems

$$X_0 \longleftarrow X_1 \longleftarrow \ldots$$

(indexed by ω) of finite abelian groups and we consider such a group X with the standard structural group $Aut^*(X)$.

For natural numbers $m\le n$ and an element $\eta\in X_n$, $\eta(m)\in X_m$ denotes the image of η by the connecting map $f_{nm}:X_n\to X_m$ (for $m>n$, $\eta(m)$ means nothing and is skipped in any expression below). Let $\alpha=(\alpha_1,\ldots,\alpha_m)\in X_{m_1}\times\ldots\times X_{m_m}$ for some $m_1,\ldots,m_m\in\omega\cup\{\omega\}$, where $X_\omega=X$. Let $A\subseteq\bigcup_{n\in\omega\cup\{\omega\}}X_n$. We will consider orbits over A so w.l.o.g. we can assume that A is closed on connecting maps, on projections from X onto each X_n, $n\in\omega$, and for each $n\in\omega\cup\{\omega\}$, $A\cap X_n$ is a submodule of X_n.

Lemma 2.2 (Generalized description of orbits) *$o(\alpha/A)=U$, where U consists of elements $\beta\in X_{m_1}\times\ldots\times X_{m_m}$ such that for all $k\ge 1$, natural numbers $n_1\le\ldots\le n_k$, $(a_{ij})_{1\le i,j\le k}\in M_{k\times k}(\mathbb{Z})$, $(b_{ij})_{i\le k,j\le m}\in M_{k\times m}(\mathbb{Z})$ and $(\eta_1,\ldots,\eta_k)\in A^k$ we have*

$$(\exists\overline{y}\in X^k)\bigwedge_{i=1}^{k}\left(\sum_{j=1}^{k}a_{ij}y_j(n_i)+\sum_{j=1}^{m}b_{ij}\alpha_j(n_i)+\eta_i(n_i)=0\right)$$

$$\Updownarrow$$

$$(\exists\overline{y}\in X^k)\bigwedge_{i=1}^{k}\left(\sum_{j=1}^{k}a_{ij}y_j(n_i)+\sum_{j=1}^{m}b_{ij}\beta_j(n_i)+\eta_i(n_i)=0\right).$$

Proof. The proof is similar to the proof of Lemma 1.5. This time we use the Baur-Garavaglia-Monk's theorem for many sorted abelian structures from example 5 (ii) from our list of examples of abelian structures. ■

Assume now that X has a finite exponent e. Let $\alpha=(\alpha_1,\ldots,\alpha_m)\in X^m$ and A be a finite submodule (over $\mathbb{Z}_e$) of X. Moreover, let $A_0\subseteq\bigcup_{n\in\omega\cup\{\omega\}}X_n$ be closed on connecting maps, on projections from X onto each X_n, $n\in\omega$, and assume that for each $n\in\omega\cup\{\omega\}$, $A_0\cap X_n$ is a submodule of X_n. For $n\in\omega$, we define $A_0^{\ge n}=A_0\cap(X\cup\bigcup_{i\ge n}X_i)$.

For $k,l_1,\ldots,l_m\in\mathbb{Z}_e$ and $a\in A$ we define

$$n_{k,l,a}=max\{n\in\omega:(\exists\eta\in A_0\cap X_n)(k\mid(\sum_{i=1}^{m}l_i\alpha_i(n)-a(n)-\eta)\}$$

when such a maximal n exists, or -1 otherwise.
Let $N_{\alpha,A,A_0}=max\{n_{k,l,a}:(k,l,a)\in\mathbb{Z}_e\times\mathbb{Z}_e^m\times A\}+1$. Take $n\ge N_{\alpha,A,A_0}$.

Lemma 2.3 *$o^n(\alpha/AA_0^{\geq n}\alpha(n)) = U$, where U consists of elements $\beta \in X^m$ such that $\beta(n) = \alpha(n)$ and for all $n' \geq n$, $a \in A$, $\eta \in A_0 \cap X_{n'}$, $k \in \mathbb{Z}_e, l_1, \ldots, l_m \in \mathbb{Z}_e$ we have*

$$k \mid \left(\sum_{i=1}^{m} l_i \alpha_i(n') - a(n') - \eta \right) \Longrightarrow k \mid \left(\sum_{i=1}^{m} l_i \beta_i(n') - a(n') - \eta \right).$$

Proof. ($\subseteq$) is obvious.
($\supseteq$) The proof is an elaboration of the proof of Lemma 1.6.

Take a $\beta \in U$. Let $k \geq 1$, $n \leq n_1 \leq \ldots \leq n_k$, $(a_{ij})_{1 \leq i,j \leq k} \in M_{k\times k}(\mathbb{Z}_e)$, $(b_{ij})_{i \leq k, j \leq m} \in M_{k \times m}(\mathbb{Z}_e)$, $a = (a_1, \ldots, a_k) \in A^k$ and $\eta = (\eta_1, \ldots, \eta_k) \in (A_0 \cap X_{n_1}) \times \ldots \times (A_0 \cap X_{n_k})$.

We define the $\mathbb{Z}_e$-linear combinations $t_i(\overline{y}, \overline{x}, \overline{t}, \overline{z})$, $i = 1, \ldots k$, of variables $\overline{x} = (x_1, \ldots, x_m)$, $\overline{y} = (y_1, \ldots, y_k)$, $t = (t_1, \ldots, t_k)$ and $\overline{z} = (z_1, \ldots, z_k)$ in the following way.

$$t_i(\overline{y}, \overline{x}, \overline{t}, \overline{z}) = \sum_{j=1}^{k} a_{ij} y_j + \sum_{j=1}^{m} b_{ij} x_j + t_i + z_i.$$

Now we choose $t'_i(\overline{x}, \overline{t}, \overline{z})$, $i = 1, \ldots, k$, similarly as in the proof of Lemma 1.6.

By Lemma 2.2, to finish our proof we need to show that

$$\exists \overline{y} \in X^k \left(\bigwedge_{i=1}^{k} t_i(\overline{y}(n_i), \alpha(n_i), a(n_i), \eta) = 0 \right) \Leftrightarrow \exists \overline{y} \in X^k \left(\bigwedge_{i=1}^{k} t_i(\overline{y}(n_i), \beta(n_i), a(n_i), \eta) = 0 \right).$$

($\Rightarrow$) Assume that for some $\overline{y} \in X^k$ we have

$$(A) \qquad \bigwedge_{i=1}^{k} t_i(\overline{y}(n_i), \alpha(n_i), a(n_i), \eta) = 0.$$

As in the proof of Lemma 1.6 we get

$$\bigwedge_{i=1}^{k} k_i \mid t'_i(\alpha(n), a(n), \eta(n)).$$

By the choice of n one can conclude that there is an $\eta' \in (A_0 \cap X_{n_k})^k$ such that

$$\bigwedge_{i=1}^{k} k_i \mid t'_i(\alpha(n_k), a(n_k), \eta'(n_k)).$$

From the assumption that $\beta \in U$ we see that

$$\bigwedge_{i=1}^{k} k_i \mid t'_i(\beta(n_k), a(n_k), \eta'(n_k)).$$

The last two statements imply that we can choose $\overline{y}', \overline{y}'' \in X^k$ so that

$$\begin{array}{ll} (*) & \bigwedge_{i=1}^{k} t_i(\overline{y}'(n_i), \alpha(n_i), a(n_i), \eta'(n_i)) = 0, \\ (**) & \bigwedge_{i=1}^{k} t_i(\overline{y}''(n_i), \beta(n_i), a(n_i), \eta'(n_i)) = 0. \end{array}$$

By $(*)$ and (A) we get that for $\overline{y} - \overline{y}' = (y_1 - y_1', \dots, y_k - y_k')$ and $\theta = (\eta_1 - \eta'(n_1), \dots, \eta_k - \eta'(n_k))$ we have

$$\bigwedge_{i=1}^{k} t_i((\overline{y} - \overline{y}')(n_i), 0, 0, \theta) = 0$$

Combining this with $(**)$ we obtain that for $\overline{y}''' = \overline{y}'' + \overline{y} - \overline{y}'$ we have

$$\bigwedge_{i=1}^{k} t_i(\overline{y}'''(n_i), \beta(n_i), a(n_i), \eta) = 0.$$

The proof of $(\Leftarrow)$ is similar. ■

In the next corollary n is at least the maximum of the two numbers $N_{\alpha,A}$ and N_{α,A,A_0} defined before Lemmas 1.6 and 2.3, for α, A and A_0 as above.

Corollary 2.4 *If we additionally assume that for each $i \geq n$, the set $A_0 \cap X_i$ is contained in the kernel of the connecting map f_{ii-1}, then $o^n(\alpha/AA_0^{\geq n}\alpha(n)) = o^n(\alpha/A\alpha(n))$.*

Proof. $(\subseteq)$ is obvious.
$(\supseteq)$ Let us take a $\beta \in o^n(\alpha/A\alpha(n))$. Suppose for a contradiction that $\beta \notin o^n(\alpha/A_0^{\geq n}A\alpha(n))$. From Lemma 2.3 we get that there is an $a_0 \in A_0$ (more precisely $a_0 \in X_{n'}$ for some $n' > n$), $a \in A$ and $k, l_1, \dots, l_m \in \mathbb{Z}_e$ such that

$$k \mid (\sum_{i=1}^{m} l_i\alpha_i(n') - a(n') - a_0) \text{ and } k \nmid (\sum_{i=1}^{m} l_i\beta_i(n') - a(n') - a_0).$$

Since $a_0(n) = 0$, we get that $k \mid (\sum_{i=1}^{m} l_i\alpha_i(n) - a(n))$. Hence, by the choice of n we get $k \mid (\sum_{i=1}^{m} l_i\alpha_i - a)$. As a consequence $k \mid (\sum_{i=1}^{m} l_i\alpha_i(n') - a(n'))$, so $k \mid a_0$ and $k \mid (\sum_{i=1}^{m} l_i\beta_i(n') - a(n'))$. Finally, $k \mid (\sum_{i=1}^{m} l_i\beta_i(n') - a(n') - a_0)$. ■

Proposition 2.5 *Assume that X is the inverse limit of a system indexed by ω of finite abelian groups and X has a finite exponent e. Let $Aut^*(X)$ be the standard structural group of X and A_0 be an arbitrary subset of $X \cup \bigcup_{n\in\omega} X_n$. Then $(X, Aut^*(X/A_0))$ is m-normal and m-stable.*

Proof. We use Lemma 2.3 and we proceed similarly as in the proofs of Theorem 1.9 and Propositon 3.2.8 in [K1]. ■

Proposition 2.6 *Take all the assumptions of Proposition 2.5. Assume additionally that $A_0 \subseteq \bigcup_{i\in\omega} X_n$ and for each $i \geq 1$, the set $A_0 \cap X_i$ is contained in the kernel of the connecting map f_{ii-1}. Then $(X, Aut^*(X/A_0))$ is small, m-normal and m-stable.*

Proof. By Proposition 2.5 only smallness requires an explanation. For two orbits o_1 and o_2, $o_1 \subseteq_o o_2$ will mean that o_1 is open in o_2.

It is enough to show that for each $\alpha \in X$ and finite set $A \subseteq X$ we have $o(\alpha/AA_0) \subseteq_o o(\alpha/A)$. W.l.o.g. we can assume that for each $i \in \omega$, $A_0 \cap X_i$ is a submodule of X_i.

Take an n as in Corollary 2.4. Then

$$o^n(\alpha/A\alpha(n)) = o^n(\alpha/AA_0^{\geq n}\alpha(n)).$$

Since $o(\alpha/A\alpha(n)) \subseteq o^n(\alpha/A\alpha(n))$, we get

$$o(\alpha/A\alpha(n)) \subseteq o^n(\alpha/AA_0^{\geq n}\alpha(n)).$$

On the other hand, by Remark 1.7 and Corollary 1.8 we see that

$$o(\alpha/AA_0\alpha(n)) \subseteq_o o(\alpha/AA_0^{\geq n}\alpha(n)) \subseteq_o o^n(\alpha/AA_0^{\geq n}\alpha(n)).$$

Hence

$$o(\alpha/AA_0\alpha(n)) \subseteq_o o(\alpha/A\alpha(n)).$$

Since we also have

$$o(\alpha/AA_0\alpha(n)) \subseteq o(\alpha/AA_0) \text{ and } o(\alpha/A\alpha(n)) \subseteq_o o(\alpha/A),$$

we get $o(\alpha/AA_0) \subseteq_o o(\alpha/A)$. ■

Now we will give some final comments. First of all, Theorem 1.9 and Proposition 2.6 yield new classes of examples of small profinite groups. The next remark is connected with the following question, which appears in [N1] and [N2].

(*) Does there exist a small profinite group (structure) which is not m-normal?

Theorem 1.9 and Proposition 2.5 together with Remark 1.4 yield the negative answer to the above question for an arbitrary abelian profinite group X, being the inverse limit of a system $(X_i)_{i\in\omega}$ indexed by ω, with the standard structural group or with the structural group being the pointwise stabilizer of an arbitrary subset of $X \cup \bigcup_{n\in\omega} X_n$.

The next natural step in our investigations is to consider the profinite groups which are inverse limits of arbitrary countable systems (not necessarily indexed by ω) of finite abelian groups of bounded exponent with the standard structural group. We could try to describe when such groups are small and to show that if such a group is small, then it is also m-normal (like in [K1], Theorem 4.2 and Proposition 4.3). I think that the techniques from this paper together with some combinatorics should work here.

The situation is much more complicated when we consider abelian profinite groups (or even products of finite abelian groups) with arbitrary structural groups. Propositions 2.5 and 2.6 are some partial results describing this situation. It seems that it is very difficult to classify small abelian profinite groups with arbitrary structural groups. There is still a chance that there exists an abelian profinite group (or even a product of finite abelian groups) with some (rather complicated) structural group which is small, but not m-normal.

Similarly as at the end of Section 1, we would like to say, that using many-sorted abelian structures we can give a new proof of the generalized description of orbits in products of finite abelian groups formulated in [K1] (Lemma 3.1.2). Namely, we can apply Lemma 2.2 (in fact, we use here the Baur-Garavaglia Monk's theorem for many-sorted abelian structures from example 5 (i)) and some linear algebra (for the proof see [K3], Lemma 2.1.6).

At the very end of the paper I would like to explain, that from some point of view, considering only abelian profinite groups is not a huge restriction. We have to remember that our aim is to find examples of small profinite groups and that for such examples m-normality and m-stability are also considered.

In the case of a product $X = \prod_{i\in\omega} X_i$ of finite groups, Remark 4.3 from [N1] shows that while searching for small products of finite groups, w.l.o.g. we can assume that almost all X_i's are abelian. Then arguing as in the proof of Corollary 3.2.2 in [K1] we can assume that all X_i's are abelian.

In the case of arbitrary profinite groups we have the following fact proved by Wagner [Wa].

Fact 2.7 *Each small m-stable profinite group contains an open abelian subgroup.*

So searching for small and m-stable profinite groups, we can consider only profinite groups with an open abelian subgroup. When we have such a group X, then it is easy to see that X is interpretable in its abelian canonical open subgroups and its smallness is equivalent to smallness of its arbitrary abelian canonical open subgroup (with some structural group). Assuming smallness and arguing similarly as in the proof of Corollary 3.2.2 in [K1], we get that m-normality and m-stability of X are also equivalent to m-normality and m-stability of its arbitrary abelian canonical open subgroup (with the structural group as in the above brackets). So from the point of view of small and m-stable profinite groups (with arbitrary structural groups) we can consider only abelian profinite groups.

Without m-stability we do not have Wagner's theorem. Instead we have the open question if each small profinite group contains an open abelian subgroup. It seems that considering non-abelian profinite groups (even with the standard structural group) is much more complicated and the techniques used in the description of orbits in abelian profinite groups do not work.

On the other hand, Newelski proved (see [N1] and [N3]) that any small m-normal profinite group has an open abelian subgroup. Hence, if we were able to find an example of a small profinite group without an open abelian subgroup (so also not m-stable), then we would get the positive answer to the question $(*)$.

References

[A] M. Artin, *Algebra*, Prentice Hall, Englewood Cliffs, New Jersey, 1991.

[Ba] J. T. Baldwin, *Fundamentals of stability theory*, Springer-Verlag, Heidelberg, 1988.

[Bu] S. Buechler, *Essential stability theory*, Springer-Verlag, Berlin, 1996.

[F] E. Fischer, *Abelian structures I*, Lecture Notes in Mathematics (616), 270-322, 1976.

[K1] K. Krupiński, *Products of finite abelian groups as profinite groups*, Journal of Algebra, to appear.

[K2] K. Krupiński, *Profinite structures interpretable in fields*, Ann. Pure and Appl. Logic, submitted.

[K3] K. Krupiński, *Profinite structures with various model theoretic properties*, Ph.D. Thesis, Wrocław, 2004.

[N1] L. Newelski, *Small profinite groups*, J. Symb. Logic (66), 859-872, 2001.

[N2] L. Newelski, *Small profinite structures*, Trans. Amer. Math. Soc. (354), 925-943, 2002.

[N3] L. Newelski, *$\mathcal{M}$-gap conjecture and m-normal theories*, Israel Journal of Math. (106), 285-311, 1998.

[PRZ] M. Prest, P. Rothmaler, M. Ziegler, *Extensions of elementary duality*, J. Pure and Appl. Algebra (93), 33-56, 1994.

[RZ] L. Ribes, P. Zaleski, *Profinite groups*, vol. 40, Springer, 2000.

[Wa] F. Wagner, *Small profinite m-stable groups*, Fund. Math. (176), 181-191, 2003.

[We] V. Weispfenning, *Quantifier elimination for modules*, Arch. Math. Logik Grundlagenforsch (25), 1-11, 1985.

[Z] M. Ziegler, *Model theory of modules*, Ann. Pure and Appl. Logic (26), 149-213, 1984.

Mathematical Institute, University of Wrocław
pl. Grunwaldzki 2/4, 50-384 Wrocław, Poland
e-mail: kkrup@math.uni.wroc.pl
and
Mathematical Institute of the Polish Academy of Sciences