Generalizations of small profinite structures

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Abstract

We generalize the model theory of small profinite structures developed by Newelski to the case of compact metric spaces considered together with compact groups of homeomorphisms and satisfying the existence of m-independent extensions (we call them compact e-structures). We analyze the relationships between smallness and different versions of the assumption of the existence of m-independent extensions and we obtain some topological consequences of these assumptions. Using them, we adopt Newelski's proofs of various results about small profinite structures to compact e-structures. In particular, we notice that a variant of the group configuration theorem holds in this context.

A general construction of compact structures is described. Using it, a class of examples of compact e-structures which are not small is constructed.

It is also noticed that in an m-stable compact e-structure every orbit is equidominant with a product of m-regular orbits.

0 Introduction

In [12, 14], Newelski introduced the notion of a profinite structure and developed a counterpart of geometric stability theory in a purely topological setting.

Recall that a profinite space X is, up to homeomorphism, the inverse limit of a system of finite discrete topological spaces, that is $X = \{\langle x_i\rangle_{i\in I} : f_{ji}(x_j) = x_i \text{ for every } j \geq i\}$, where $\{X_i, f_{ji} : i, j \in I, j \geq i\}$ is an inverse system of finite discrete spaces. The topology on X is inherited from the product of X_i's. We always assume that I is countable.

A profinite structure is a pair $(X, Aut^*(X))$ consisting of a profinite topological space X and a closed subgroup $Aut^*(X)$, called the structural group of X, of the group of all homeomorphisms of X respecting a distinguished inverse system defining X (the topology is inherited from the product topology on X^X). It is easy to see that $Aut^*(X)$ is a profinite group acting continuously on X. We say that $(X, Aut^*(X))$

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is small if for every natural number n, there are only countably many orbits on X^n under the action of $Aut^*(X)$. To develop the model theory of small profinite structures, Newelski defined m-independence, which has similar properties to those of forking independence in stable theories. He considered counterparts of such notions as Lascar U-rank, superstability or 1-basedness, and proved various results about them. The deepest result seems to be the group configuration theorem [14, Theorem 1.7 and Theorem 3.3].

Smallness and the fact that we have a basis consisting of clopen sets which are classes of finite $Aut^*(X)$-invariant equivalence relations play a prominent role in all these considerations. From the model theoretic point of view, smallness is a natural assumption, because any profinite structure interpretable in a small theory (see Definition 1.2) is small. Unfortunately, it is not easy to find explicit examples of small profinite structures, especially of small profinite groups. All known examples of small profinite groups are abelian profinite groups of finite exponent and their variants (see [3, 4] for details). So, it would be interesting to extend Newelski's approach to wider classes of profinite structures or even to "non-profinite" mathematical objects.

In this paper, we investigate pairs (X, G), where X is a compact metric space and G is a compact group acting continuously and faithfully on X (so G is just a compact subgroup of the group of all homeomorphisms of X). We call them compact structures.

Similarly to profinite structures, compact structures appear naturally as objects interpretable in some sense in first order theories. Namely, the space of classes of a bounded, type-definable equivalence relation together with the group of homeomorphisms induced by automorphisms of the monster model is a compact structure. Moreover, any compact structure is of this form (see Theorem 1.4).

In Section 1, we give definitions and prove some fundamental results about compact and profinite structures. In particular, in Proposition 1.5, we notice that profinite structures can be defined without referring to a distinguished inverse system. Namely, profinite structures are those compact structures for which the underlying space is profinite.

At the beginning of Section 2, we easily get that smallness of (X, G) would imply that (X, G) is a small profinite structure. So, in the further part of the paper, instead of smallness we assume in a sense the weakest condition necessary to develop a counterpart of geometric stability theory, namely the existence of m-independent extensions. Compact structures satisfying this condition will be called compact e-structures (see Definition 2.5). Notice that the class of compact e-structures contains profinite structures which are not necessarily small, but in which m-independent extensions exist (we call them profinite e-structures).

We also give a short, elementary proof of Kim's theorem that in small theories the finest bounded, type-definable equivalence relation equals the relation of having the same strong type. As in the proof of this theorem in [6], we use the fact that strong types are connected-components in the space of Kim-Pillay strong types. However, using this fact, we give a completely elementary proof, without applications of Haar measure and integrals.

An important part of the paper is the rest of Section 2, where we analyze the relationships between different versions of the assumption of the existence of m-independent extensions, obtain various topological consequences (in particular, that all orbits are profinite), construct a class of examples of compact e-structures which are not profinite (so not small), and a class of profinite e-structures which are not small.

In Subsection 3.1, we explain how to generalize most of the results from [14] to the case of compact structures satisfying the existence of m-independent extensions. In particular, we have a variant of the group configuration theorem.

In Subsection 3.2, working in the class of compact e-structures, we show counterparts of some results about regular types, domination and weight known for simple theories (see Sections 5.1 and 5.2 of [15]). These results are new even for small profinite structures.

1 Compact and profinite structures

In this section, we give definitions and prove some fundamental results about compact and profinite structures. We also discuss some notions of interpretability of compact and profinite structures in first order theories.

Definition 1.1 *A compact structure is a pair (X, G), where X is a compact metric space and G is a compact group acting continuously and faithfully on X.*

Equivalently, G is a compact subgroup of the group of all homeomorphisms of X with the compact-open topology. Since the group of all homeomorphisms of X with the compact-open topology is Polish, G is second countable. By the compactness of G, the pointwise convergence topology on G coincides with the compact-open topology. Of course, each profinite structure is a compact structure (recall that in the definition of a profinite structure, we always assume that the inverse system is countable).

Let (X, G) be a compact structure. Let $A \subseteq X$ be finite. By G_A we denote the pointwise stabilizer of A. We say that $V \subseteq X$ is A-invariant if $f[V] = V$ for every $f \in G_A$. If in addition V is closed, we say that V is A-definable. For $a \in X^n$ and $A \subseteq X$, we define $o(a/A) = \{f(a) : f \in G_A\}$ (the orbit of a over A) and $O_n(A) = \{o(a/A) : a \in X^n\}$. Each orbit is always a closed subset of X.

For a finite $A \subseteq X$, the algebraic [definable] closure of A, denoted by $acl(A)$ [$dcl(A)$, respectively] is the set of all elements in X with finite [1-element, respectively] orbits over A. If A is infinite, $acl(A) = \bigcup\{acl(A_0) : A_0 \subseteq \text{A is finite}\}$ and $dcl(A) = \bigcup\{dcl(A_0) : A_0 \subseteq \text{A is finite}\}$. We will introduce later an imaginary extension X^{eq} of X; acl^{eq} and dcl^{eq} are defined then as acl and dcl, but in X^{eq}.

We say that compact structures (X, G) and (Y, H) are isomorphic, if there is a homeomorphism $\phi : X \to Y$ and an (topological) automorphism $\psi : G \to H$ such that $\phi(gx) = \psi(g)\phi(x)$ for all $x \in X, g \in G$. To be precise, the definition of a profinite structure is up to isomorphism, i.e. any compact structure isomorphic to a profinite structure is also a profinite structure.

We have the following natural notion of interpretability of profinite structures in first order theories [12, 14] (for more details on this and another notion of interpretability, see [5]).

Let T be a first order, countable, complete theory T with a monster model $\mathfrak{C}$, and $A \subseteq \mathfrak{C}$ be countable. In the definition below, Y is an arbitrary A-type-definable subset of $\mathfrak{C}^{eq}$ and $E_1 \supseteq E_2 \supseteq \dots$ is an arbitrary descending sequence of finite A-definable equivalence relations on Y.

Definition 1.2 *We say that a profinite structure is interpretable in T over A if it is isomorphic to the inverse limit of spaces Y/E_i with the structural group induced by $Aut(\mathfrak{C}/A)$.*

So, $(X, Aut^*(X))$ is interpretable in T over A iff it is isomorphic to

$$\{\langle a/E_1, a/E_2, \dots \rangle : a \in Y\}$$

with the structural group induced by $Aut(\mathfrak{C}/A)$.

The main examples of profinite structures interpretable in T over A are traces of complete types over A. More precisely, for $p \in S(A)$, we consider $(Tr(p), Aut^*(Tr(p)))$, where

$$Tr(p) = \{q \in S(acl^{eq}(A)) : p \subseteq q\}$$

and $Aut^*(Tr(p))$ is induced by $Aut(\mathfrak{C}/A)$. We treat $Tr(p)$ as the inverse limit of the system of all spaces $p(\mathfrak{C})/E$, with E ranging over finite equivalence relations on $\mathfrak{C}$ definable over A. So, $Tr(p)$ is a profinite structure homogeneous under the action of $Aut^*(Tr(p))$.

It is obvious that any profinite structure interpretable in a small theory over any finite set is small. Moreover, it is easy to show that any [small] profinite structure is interpretable as the space of all strong 1-types in some [small] stable, weakly minimal theory. To see this, take any [small] profinite structure $(X, Aut^*(X))$. Then, we have the distinguished set $\{E_i : i \in I\}$ of finite, invariant equivalence relations inducing the profinite topology on X. Let $\mathcal{X}$ be the first order structure with the universe X, the relations E_i, $i \in I$, and the relations R_i, $i \in I$, which are defined as follows. Write explicitly $X/E_i = \{a^i_1/E_i, \dots, a^i_{k_i}/E_i\}$ and let $\pi : X \to X/E_i$ be the quotient map. Then, $R_i \subseteq X^{k_i}$ is defined as $(\pi \times \dots \times \pi)^{-1}[o(a^i_1/E_i, \dots, a^i_{k_i}/E_i)]$, where $o(a^i_1/E_i, \dots, a^i_{k_i}/E_i)$ is the orbit of the tuple $(a^i_1/E_i, \dots, a^i_{k_i}/E_i)$ under the action of $Aut^*(X)$. Now, we define $T = Th(\mathcal{X})$. Then, one can check that T is [small] stable, weakly minimal, and that $(X, Aut^*(X))$ is interpretable in T as the set of all strong 1-types over $\emptyset$.

For compact structures, we can also introduce a natural notion of interpretability. Let T be a first order, countable, complete theory with a monster model $\mathfrak{C}$, and $A \subseteq \mathfrak{C}$ be countable. Let Y be any A-type-definable set and E be a bounded, A-type-definable equivalence relation on Y. Then, Y/E is a compact metric space (with the so-called logic topology), and $Aut(\mathfrak{C}/A)$ induces a compact group (denoted by $Aut(\mathfrak{C}/A) \restriction Y/E$) acting continuously on Y/E (for details see [1, 6, 8]).

Definition 1.3 *We say that a compact structure is interpretable in T over A if it is isomorphic to a compact structure of the form $(Y/E, Aut(\mathfrak{C}/A) \upharpoonright Y/E)$, where E is a bounded, A-type-definable equivalence relation on an A-type-definable set Y.*

Similarly to the case of profinite structures, it turns out that any compact structure is interpretable in some first order theory. This fact is a folklore, but I have never found any published proof of it, so we give a proof below.

Theorem 1.4 *Any compact structure (X, G) is interpretable in some first order countable theory T so that X becomes $\mathfrak{C}/E$, where E is a bounded $\emptyset$-type-definable equivalence relation on a monster model $\mathfrak{C}$ of T.*

Proof. Using the Haar measure on G and a given metric on X, it is easy to produce a new metric d on X which is invariant under the action of G (see [6], the paragraph before Theorem 3.5).

We are going to consider X as a first order, relational structure. Choose a dense countable subset A of X. Let $\overline{A}$ be the set of finite tuples of elements of A. Now, we define a countable family of relational symbols and their interpretations in X:

- $U_q(x, y), q \in \mathbb{Q}^+$, and $X \models U_q(x, y)$ iff $d(x, y) < q$;
- $R_a(x), a \in \overline{A}$, and $R_a(X) = o(a)$.

We treat X as a model in the language $\mathcal{L} = \{U_q(x, y), R_a : q \in \mathbb{Q}^+, a \in \overline{A}\}$. Let $T = Th(X)$ and $\mathfrak{C}$ be a monster model of T containing X as an elementary substructure. In fact, the relations R_a will be used only in the proof of Claim 4 below.

We define a $\emptyset$-type-definable equivalence relation on $\mathfrak{C}$:

$$E(x, y) \iff \bigwedge_{q \in \mathbb{Q}^+} U_q(x, y).$$

To finish the proof, we need to show that E is a bounded, $\emptyset$-type-definable equivalence relation on $\mathfrak{C}$ and $(\mathfrak{C}/E, Aut(\mathfrak{C}) \upharpoonright \mathfrak{C}/E) \cong (X, G)$. We will prove this in successive claims.

Claim 1 $\mathfrak{C}/E = \{x/E : x \in X\}$ and for any distinct $x, y \in X$, we have $x/E \neq y/E$; hence E is bounded.

Proof. The second part is obvious. For the first part, suppose for a contradiction that there is $a \in \mathfrak{C}$ such that $[a]_E \cap X = \emptyset$. Then, for each $x \in X$, there is $q_x \in \mathbb{Q}^+$ such that $\neg U_{q_x}(x, a)$. But, since X is compact, finitely many sets $U_{q_{x_1}}(x_1, X), \dots, U_{q_{x_n}}(x_n, X)$ cover X. Hence, the sets $U_{q_{x_1}}(x_1, \mathfrak{C}), \dots, U_{q_{x_n}}(x_n, \mathfrak{C})$ cover $\mathfrak{C}$, so $U_{q_{x_i}}(x_i, a)$ for some i. This is a contradiction. □

Claim 2 X is homeomorphic to $\mathfrak{C}/E$.

Proof. Let $\pi : X \to \mathfrak{C}/E$ be defined by $\pi(x) = x/E$. By Claim 1, π is 1-1 and onto.

Since both spaces X and $\mathfrak{C}/E$ are compact and Hausdorff, it is enough to show that π is continuous. An open basis of the logic topology on $\mathfrak{C}/E$ consists of the sets $U_{b,q} = \{a/E : [a]_E \subseteq U_q(b, \mathfrak{C})\}, b \in X$. Then, $\pi^{-1}[U_{b,q}] = U_q(b, X)$ is open in X. □

From now on, we identify spaces X and $\mathfrak{C}/E$. Then, G and $Aut(\mathfrak{C}) \restriction \mathfrak{C}/E$ become compact subgroups of the group of all homeomorphisms of X.

Claim 3 G *is contained in* $Aut(\mathfrak{C}) \restriction \mathfrak{C}/E$.

Proof. It is clear, because G consists of automorphisms of the structure X. □

Claim 4 $Aut(\mathfrak{C}) \restriction \mathfrak{C}/E$ *is contained in* G.

Proof. In the following, we use the compactness of X and G, and the continuity of the action. Suppose for a contradiction that there is $f \in Aut(\mathfrak{C})$ such that $f \restriction \mathfrak{C}/E \notin G$. Then, there is $a = (a_1, \dots, a_n) \in \overline{A}$ such that $\neg R_a(b_1, \dots, b_n)$, where $\{b_i\} = [f(a_i)]_E \cap X$ for $i = 1, \dots, n$. Since $R_a(X)$ is closed, there is $q \in \mathbb{Q}^+$ such that

$$X \models (\forall x_1, \dots, x_n) \left(\bigwedge_{1 \le i \le n} U_q(b_i, x_i) \to \neg R_a(x_1, \dots, x_n) \right).$$

So, the same formula holds in $\mathfrak{C}$, but the tuple $(f(a_1), \dots, f(a_n))$ witnesses that this is impossible. ■

We see that the definition of a profinite structure depends on a distinguished inverse system. The question arises if we can define profinite structures without referring to this inverse system. The next result yields a positive answer.

Proposition 1.5 *If* (X, G) *is a compact structure such that* X *is a profinite space, then* (X, G) *is a profinite structure (hence* G *is a profinite group).*

Proof. By Theorem 1.4, there exists a countable theory T and a bounded, $\emptyset$-type-definable equivalence relation E on a monster model $\mathfrak{C}$ of T such that $(X, G) \cong (\mathfrak{C}/E, Aut(\mathfrak{C}) \restriction \mathfrak{C}/E)$. Since $\mathfrak{C}/E \approx X$ is 0-dimensional, we get that E is an intersection of countably many finite, $\emptyset$-definable equivalence relations $E_i, i \in \omega$, [6, Proposition 2.4]. We see that

$$(\mathfrak{C}/E, Aut(\mathfrak{C}) \restriction \mathfrak{C}/E) \cong (\varprojlim \mathfrak{C}/E_i, Aut(\mathfrak{C}) \restriction \varprojlim \mathfrak{C}/E_i),$$

and, of course, $Aut(\mathfrak{C})$ preserves the inverse system $\mathfrak{C}/E_i, i \in \omega$, with the natural projections. So, (X, G) is a profinite structure. ■

Hence, we can define profinite structures as those compact structures (X, G) for which X is a profinite space. One can also prove it by a purely topological argument (without referring to first order theories) by using Proposition 1.16. We leave it as an easy exercise.

One could ask here whether G being profinite implies that (X, G) is a profinite structure. Proposition 1.11 below shows that this is not the case. However, we have the following weaker conclusion.

Proposition 1.6 *If (X, G) is a compact structure such that G is a profinite group, then every orbit under the action of G is profinite.*

Proof. Choose any $x \in X^n$. It is an easy exercise to show that the function $\Phi : G \to o(x)$ given by $\Phi(g) = gx$ is continuous, closed and open.

To prove that $o(x)$ is profinite, we need to show that $o(x)$ has a basis consisting of clopen sets. Consider any U open in $o(x)$ and $y \in U$. Take any $g \in \Phi^{-1}(y)$. Since $\Phi^{-1}[U]$ is open in G and G is profinite, there is a clopen subset V of $\Phi^{-1}[U]$ containing g. Then, $y \in \Phi[V] \subseteq U$ and $\Phi[V]$ is clopen in $o(x)$. So, $o(x)$ has a basis consisting of clopen sets. ■

In model theory, we can freely use names of definable sets, because we can add imaginary sorts whose elements are classes of definable equivalence relations. We can make a similar trick for compact structures.

Remark 1.7 *Let (X, G) be a compact structure and E be a $\emptyset$-definable equivalence relation on X^n. Then X^n/E is a compact metric space, and G induces a compact group, denoted by $G \restriction X^n/E$, of homeomorphisms of X^n/E acting continuously on X^n/E. So, $(X^n/E, G \restriction X^n/E)$ is a compact structure.*

Proof. Since X^n is a compact metric space and E is closed, we easily get that X^n/E is a compact, Hausdorff, second countable space, so it is a compact metric space. The rest is an easy exercise which uses the compactness of X and G, and the continuity of the action of G on X. ■

Definition 1.8 *Let (X, G) be a compact structure. We define X^{eq} as the disjoint union of sets X^n/E with E ranging over $\emptyset$-definable equivalence relations on X^n. The sets X^n/E will be called sorts of X^{eq}.*

By the last remark, each sort of X^{eq} is a compact structure. Now, elements and sets of parameters can be taken from X^{eq}. As in model theory, $(X^{eq})^{eq} = X^{eq}$, which means that if E is a $\emptyset$-definable equivalence relation on a product of sorts $X^{n_1}/E_1 \times \cdots \times X^{n_k}/E_k$, then the set of E-classes can be identified with the sort $(X^{n_1} \times \cdots \times X^{n_k})/E'$, where

$$E'(x_1, \ldots, x_k; y_1, \ldots, y_k) \iff E(x_1/E_1, \ldots, x_k/E_k; y_1/E_1, \ldots, y_k/E_k).$$

Definition 1.9 *Let V be a definable subset of a compact structure (X, G). We say that $a \in X^{eq}$ is a name for V if any $f \in G$ fixes V as a set iff it fixes a. A name for V will be denoted by $\ulcorner V \urcorner$ (notice that it is defined up to interdefinability).*

Proposition 1.10 *Any set definable in a compact structure (X, G) has a name in X^{eq}.*

Proof. Suppose V is a-definable for some $a \in X^{eq}$. On the sort of a, we define an equivalence relation E by

$$E(a_1, a_2) \iff [a_1 = a_2 \vee (a_1, a_2) \in S(a, a))],$$

where $S = \{(f, g) \in G \times G : f[V] = g[V]\}$. It is easy to check that E is a $\emptyset$-definable equivalence relation and that a/E is a name for V. ■

A similar definition of X^{eq} was given in [12, 14] for profinite structures. By [12, Lemma 1.3], we know that all sorts of a small profinite structure are profinite structures. The next proposition shows that in general this is not the case.

Proposition 1.11 *If $(X, Aut^*(X))$ is a non-small profinite structure, then there is a $\emptyset$-definable equivalence relation E on some Cartesian power X^n such that X^n/E is not profinite; even more, each compact metric space is of the form X^n/E for some E as above.*

Proof. Replacing X by X^n, if necessary, we can assume that $O_1(\emptyset)$ is uncountable. We know that $(X, Aut^*(X))$ is interpretable as $S_1(acl^{eq}(\emptyset))$ in some first order theory T. So, we can identify X with $\mathfrak{C}/\overset{s}{\equiv}$, where $\mathfrak{C}$ is a monster model of T. Since $O_1(\emptyset)$ is uncountable, $S_1(\emptyset)$ is uncountable as well. Let Y be any compact metric space. By [6, Corollary 2.3], there is a $\emptyset$-definable equivalence relation E' on $\mathfrak{C}$ coarser than the relation of having the same type and such that $\mathfrak{C}/E' \approx Y$. Let $\pi : \mathfrak{C}/\overset{s}{\equiv} \to \mathfrak{C}/E'$ be the natural projection. Define an equivalence relation E on $X = \mathfrak{C}/\overset{s}{\equiv}$ by

$$E(a/\overset{s}{\equiv}, b/\overset{s}{\equiv}) \iff \pi(a/\overset{s}{\equiv}) = \pi(b/\overset{s}{\equiv}).$$

We see that E is $\emptyset$-definable in X and $X/E \approx Y$. ■

If $(X, Aut^*(X))$ is a profinite structure, it is natural to define X^{eq} as the disjoint union of those sorts X/E which are profinite spaces. Then, by Proposition 1.5 and Remark 1.7, $(X/E, Aut(X) \upharpoonright X/E)$ is a profinite structure. It is obvious that for such definition of X^{eq}, we still have $(X^{eq})^{eq} = X^{eq}$.

Proposition 1.12 *Let $(X, Aut^*(X))$ be a profinite space and E be a $\emptyset$-definable equivalence relation. Then X/E is profinite iff E is an intersection of finite, $\emptyset$-definable equivalence relations.*

Proof. ($\Longleftarrow$) is obvious.
($\Longrightarrow$) Since $(X/E, Aut(X) \upharpoonright X/E)$ is a profinite structure, there is a countable family $\{E_i : i \in \omega\}$ of finite $\emptyset$-definable equivalence relations on X/E whose classes form an open basis. Let $\pi : X \to X/E$ be the quotient map. Then, $(\pi \times \pi)^{-1}[E_i], i \in \omega$, are finite, $\emptyset$-definable equivalence relations on X whose intersection equals E. ■

Proposition 1.13 *Let $(X, Aut^*(X))$ be a profinite structure. If E is a $\emptyset$-definable equivalence relation on X finer than lying in the same orbit, then X/E is profinite.*

Proof. Once again we use the fact that $(X, Aut^*(X))$ is interpretable as $S_1(acl^{eq}(\emptyset))$ in some theory T, and hence X can be identified with $\mathfrak{C}/\overset{s}{\equiv}$. Let $\pi : \mathfrak{C} \to \mathfrak{C}/\overset{s}{\equiv}$ be the quotient map, and $E' = (\pi \times \pi)^{-1}[E]$. It is easy to check that E' is a $\emptyset$-type-definable equivalence relation on $\mathfrak{C}$ finer than $\equiv$ but coarser than $\overset{s}{\equiv}$. By [6, Fact 2.5], $\mathfrak{C}/E'$ is profinite. We are done since $X/E \approx \mathfrak{C}/E'$. ■

Corollary 1.14 *Any set definable in a profinite structure $(X, Aut^*(X))$ has a name in X^{eq}.*

Proof. We see that the relation E defined in the proof of Proposition 1.10 is finer than lying in the same orbit. Hence, the assertion follows from Proposition 1.13. ■

Corollary 1.15 *Let (X, G) be a compact structure. If all orbits on X^n, $n \in \omega$, are profinite, then all orbits on X^{eq} are profinite as well.*

Proof. Consider any $[a]_E \in X^{eq}$ (i.e. $a \in X^n$ and E is a $\emptyset$-definable equivalence relation). Let F be the intersection of E and the relation of being in the same orbit. The map $f : o(a)/F \to o([a]_E)$ given by $f([b]_F) = [b]_E$ is a well-defined bijection. It is easy to see that it is also a homeomorphism. Indeed, since both spaces are compact, it is enough to show that f is continuous. Take any closed $D \subseteq o([a]_E)$. Then, $D' := \{b \in X^n : [b]_E \in D\}$ is closed. So, we get that $\{b \in o(a) : [b]_F \in f^{-1}[D]\} = D' \cap o(a)$ is also closed, and hence $f^{-1}[D]$ is closed.

By assumption and Proposition 1.5, $(o(a), G/G_{o(a)})$ is a profinite structure. Thus, by Proposition 1.13, $o(a)/F$ is profinite. We finsih using the fact that f is a homeomorphism. ■

The next easy, topological observation and its corollary are very important in the following sections.

Proposition 1.16 *Let (X, G) be a compact structure and Z be an A-definable subset of X^{eq} for some finite $A \subseteq X^{eq}$. Let Y be a clopen subset of Z. Then the setwise stabilizer of Y in G_A is a clopen subgroup of G_A, and the set $\{f[Y] : f \in G_A\}$ is finite. Thus, there exists a finite, A-definable equivalence relation E on Z with open classes and such that Y is the union of some equivalence classes of E.*

Proof. Since G_A, and hence G_A/G_Z, acts continuously on Z, and G_A and Z are compact, the topology on G_A/G_Z is the compact-open topology. Therefore, as Y is compact and open in Z, the setwise stabilizer of Y in G_A, i.e. the set

$$G_{A\{Y\}} := \{f \in G_A : f[Y] = Y\} = \{f \in G_A : f[Y] \subseteq Y\} \cap \{f \in G_A : f^{-1}[Y] \subseteq Y\},$$

is an open subgroup of G_A. So, by the compactness of G_A, we get $[G_A : G_{A\{Y\}}] < \omega$. Thus, $\{f[Y] : f \in G_A\}$ is finite.

The second part of the proposition is now almost trivial. Namely, the atoms of the Boolean algebra of subsets of Z generated by $\{f[Y] : f \in G_A\}$ form a partition of Z into clopen sets which are classes of some finite, A-definable equivalence relation on Z. Of course, Y is the union of some classes of E. ■

Corollary 1.17 *Let (X, G) be a compact structure. Let $A, B, a \subseteq X^{eq}$ be finite and such that $o(a/AB)$ is open in $o(a/A)$. Then the set $\{f[o(a/AB)] : f \in G_A\}$ is finite. Thus, there exists a finite, A-definable equivalence relation E on $o(a/A)$ such that $o(a/AB)$ is the union of some equivalence classes of E.*

2 Smallness and the existence of m-independent extensions

As for profinite structures, we say that a compact structure (X, G) is small if for every natural number n, there are only countably many orbits on X^n. Equivalently, there are only countably many 1-orbits [n-orbits] over any finite set.

The next remark shows that if we want to consider a class of objects essentially wider than small profinite structures, we cannot assume smallness.

Remark 2.1 *Any small compact structure is a small profinite structure.*

Proof. Suppose (X, G) is a small compact structure which is not profinite. Then, there is a non-trivial connected component Y of X. Choose $y \in Y$. Then, Y is y-definable and it is the union of countably many orbits over y. By Baire category theorem, one of these orbits is open in Y, but it is also closed, so it must be equal to Y. Hence, $Y = \{y\}$, a contradiction. ■

The following result of Kim (see [2] or [6, Theorem 3.5]) is an immediate corollary of the last Remark and Proposition 3.1 of [6]. The advantage of the proof given here in comparison with the one from [6] is that it is a completely elementary topological argument, which does not use Haar measures and integrals.

Theorem 2.2 *In a small theory, the finest bounded, $\emptyset$-type-definable equivalence relation equals $\overset{s}{\equiv}$.*

Proof. Let $\overset{bd}{\equiv}$ denote the finest bounded, $\emptyset$-type-definable equivalence relation on a monster model $\mathfrak{C}$ of a small theory T. Then, $(\mathfrak{C}/\overset{bd}{\equiv}, Aut(\mathfrak{C}) \restriction \mathfrak{C}/\overset{bd}{\equiv})$ is a small compact structure. Hence, by Remark 2.1, $\mathfrak{C}/\overset{bd}{\equiv}$ is profinite. Since by [6, Proposition 3.1], the strong types are the connected components of $\mathfrak{C}/\overset{bd}{\equiv}$, the proof is completed. ■

In profinite structures, Newelski defined the following notion of an independence relation, which plays a similar role to forking independence in stable and simple theories. Here, we consider this notion in the more general context of compact structures.

Definition 2.3 *Let (X, G) be a compact structure, a be a finite tuple and A, B finite subsets of X^{eq}. We say that a is m-independent from B over A (written $a \overset{m}{\mathop{\smile\!\!\!\!\!\!|}}_A B$) if $o(a/AB)$ is open in $o(a/A)$. We say that a is m-dependent on B over A (written $a \overset{m}{\not\mathop{\smile\!\!\!\!\!\!|}}_A B$) if $o(a/AB)$ is nowhere dense in $o(a/A)$.*

Of course, if A, B, C are finite subsets of X^{eq}, then $A \overset{m}{\mathop{\perp\!\!\!\perp}}_C B$ means that $a \overset{m}{\mathop{\perp\!\!\!\perp}}_C B$, where a is any tuple consisting of the elements of A.

For small profinite structures, the following was proved by Newelski [13].

Fact 2.4 *In a small profinite structure $(X, Aut^*(X))$, m-independence has the following properties.*

(1) (Symmetry) For every finite $A, B, C \subseteq X^{eq}$, we have that $A \overset{m}{\mathop{\perp\!\!\!\perp}}_C B$ iff $B \overset{m}{\mathop{\perp\!\!\!\perp}}_C A$.

(2) (Transitivity) For every finite $A \subseteq B \subseteq C \subseteq X^{eq}$ and $a \subseteq X^{eq}$, we have that $a \overset{m}{\mathop{\perp\!\!\!\perp}}_A C$ iff $a \overset{m}{\mathop{\perp\!\!\!\perp}}_B C$ and $a \overset{m}{\mathop{\perp\!\!\!\perp}}_A B$.

(3) $a \in acl(A)$ implies $a \overset{m}{\mathop{\perp\!\!\!\perp}}_A B$ for every finite $B \subseteq X^{eq}$.

(4) (Extensions) For every finite $a, A, B \subseteq X^{eq}$, there is some $a' \in o(a/A)$ with $a' \overset{m}{\mathop{\perp\!\!\!\perp}}_A B$.

As in the case of forking independence, from symmetry and transitivity, we get

$$ab \overset{m}{\mathop{\perp\!\!\!\perp}}_A B \iff a \overset{m}{\mathop{\perp\!\!\!\perp}}_A B \wedge b \overset{m}{\mathop{\perp\!\!\!\perp}}_{Aa} B \qquad (*)$$

for any finite $a, b, A, B \subseteq X^{eq}$.

In fact, Properties (1), (2) and (3) are true for all compact structures (without smallness): (2) and (3) are trivial; (1) follows from the Kuratowski-Ulam theorem applied to the subset $o(ab/C)$ of the product $o(a/C) \times o(b/C)$, where a, b are any tuples of the elements of A and B. As to Property (4), it may fail without smallness, e.g. in the additive group of p-adic numbers with the standard structural group or in the unit circle S_1 with the group of all rotations.

Property (4) (at least in the home sort) seems to be a necessary assumption in order to develop a counterpart of geometric stability theory. Since it will be the main assumption in the rest of the paper, we introduce the following terminology.

Definition 2.5 *A compact e-structure is a compact structure (X, G) satisfying Property (4) in the home sort (i.e. only for tuples and subsets of X), and a compact ei-structure is a compact structure satisfying Property (4) in all imaginary sorts.*

We introduce the same terminology for profinite structures as well. It is obvious that if (X, G) is a profinite structure, then it is a profinite e-structure iff it is a compact e-structure.

Remark 2.6 *If (X, G) is a profinite structure, then it is a profinite ei-structure iff it is a compact ei-structure.*

Proof. Only ($\rightarrow$) requires a short explanation. An easy forking calculus using $(*)$ reduces the proof to showing that (X, G) satisfies the existence of m-independent extensions for orbits of finite tuples of elements from X over parameters from X^{eq} computed in (X, G) treated as a compact structure. By virtue of Proposition 1.13, in order to do that, it is enough to notice that any element b/E, where $b \in X^n$ and E is $\emptyset$-definable equivalence relation, is interdefinable with some b'/E', where E' is a $\emptyset$-definable equivalence relation finer that lying in the same orbit. We finish by taking $b' := b$ and defining E' as the intersection of E and the relation of being in the same orbit. ■

Remark 2.7 *Let (X, G) be any compact structure. If Property (4) holds in X [or, more generally, in X^{eq}] when a and B are any singletons from X, then it holds in general, even for $a, B \subseteq X^{eq}$.*

Proof. By transitivity and an easy induction, we get that (4) holds when a is a singleton and $B \subseteq X$ is finite.

Suppose now that $A, B \subseteq X$ are finite [$A \subseteq X^{eq}$, when we work in X^{eq}]. By induction on n, we will show that for any $a = (a_1, \dots, a_n) \in X^n$, there is $a' \in o(a/A)$ such that $a' \overset{m}{\perp}_A B$.

Suppose that the statement holds for $(n-1)$-tuples. So, there is a tuple $b = (a'_1, \dots, a'_{n-1}) \in o((a_1, \dots, a_{n-1})/A)$ such that $b \overset{m}{\perp}_A B$. Choose $a''_n \in X$ with $(a'_1, \dots, a'_{n-1}, a''_n) \in o(a/A)$. Once again by the inductive hypothesis, we get an element $a'_n \in o(a''_n/Ab)$ such that $a'_n \overset{m}{\perp}_{Ab} B$. So, we are done by $(*)$.

Now, the fact that Property (4) holds even for $a, B \subseteq X^{eq}$ easily follows from Properties (2), (3) and $(*)$. ■

Definition 2.8 *We say that an orbit $o(a/A)$ in a compact structure (X, G) is strongly small if for any finite $B \subseteq X$, the orbit $o(a/A)$ is a union of countably many orbits over AB. We say that it is small if the same condition holds but with $B \subseteq o(a/A)$.*

Remark 2.9 *Each 1-orbit over $\emptyset$ is strongly small iff for every natural number n, each n-orbit over any finite subset of X^{eq} is [strongly] small iff for every natural number n, each n-orbit over $\emptyset$ is small iff each orbit on any sort of X^{eq} over any finite subset of X^{eq} is [strongly] small.*

If one of the above equivalent conditions holds, we say that (X, G) has small orbits. In the next proposition, we consider a list of stronger and stronger properties between Property (4) and smallness.

Proposition 2.10 *Let us consider the following list of properties of a compact structure (X, G).*

(a) (X, G) is a compact e-structure.

(b) (X, G) is a compact ei-structure.

(c) For every finite $A \subseteq X$, for every A-definable subset D of X (equivalently, of X^{eq}) such that any two elements $a, b \in D$ lie in the same orbit over $\ulcorner D \urcorner$, there is $a \in D$ such that $o(a/A)$ is open in D.

(d) (X, G) has small orbits.

(e) *For every finite $A \subseteq X$, for every A-definable subset D of X such that any two elements $a, b \in D$ lie in the same orbit over $\emptyset$, there is $a \in D$ such that $o(a/A)$ is open in D.*

(f) *For every finite $A \subseteq X$ and for every A-definable subset D of X, there is $a \in D$ such that $o(a/A)$ is open in D.*

(g) *(X, G) is small.*

Then $(a) \Longleftarrow (b) \iff (c) \Longleftarrow (d) \iff (e) \Longleftarrow (f) \iff (g)$.

Proof. (a) $\Longleftarrow$ (b) is obvious.
(b) $\Longleftarrow$ (c). Let $a, A, B \subseteq X^{eq}$ be finite. By Remark 2.7, we can assume that $a \in X$ and $B \subseteq X$. We can identify the tuple of all elements of A with an element b/E from some sort X^n/E. Let $D = o(a/A)$. Then, D is b-definable, so it is also Bb-definable. Moreover, $o(a/A) = o(a/A\ulcorner D \urcorner) = o(a/\ulcorner D \urcorner)$. Hence, by (c), we can find an element $a' \in D$ such that $o(a'/Bb)$ is open in D. So, $o(a'/AB)$ is open in $o(a/A)$, i.e. $a' \mathop{\smile\!\!\!\!\!\!\mid}^{m}_{A} B$.
(b) $\Longrightarrow$ (c). Let D satisfy the assumptions of (c). We have $\ulcorner D \urcorner \in dcl^{eq}(A)$. Take any $a \in D$. By assumption, we have that $o(a/\ulcorner D \urcorner) = D$. Hence, from (b), it follows that there is $a' \in D$ such that $a' \mathop{\smile\!\!\!\!\!\!\mid}^{m}_{\ulcorner D \urcorner} A$, i.e. $o(a'/A)$ is open in D.

Notice that in the proof of $(b) \Longleftarrow (c)$, we used (c) for $D \subseteq X$, whereas in the proof of $(b) \Longrightarrow (c)$, we got (c) for $D \subseteq X^{eq}$. This shows that both versions of (c) are equivalent.
(c) $\Longleftarrow$ (e) is obvious.
(d) $\Longrightarrow$ (e) follows easily from Baire category theorem.
(d) $\Longleftarrow$ (e) has a similar proof to (f) $\Longrightarrow$ (g) below.
(e) $\Longleftarrow$ (f) is obvious.
(f) $\Longleftarrow$ (g) follows from Baire category theorem.
(f) $\Longrightarrow$ (g). We will show that there are only countably many 1-orbits over any finite set A. Wlog $A = \emptyset$. We construct a descending sequence $X_\alpha, \alpha \in Ord$, of $\emptyset$-definable subsets of X in the following way:

- $X_0 = X$,
- $X_{\alpha+1} = X_\alpha \setminus \bigcup\{o(a) : o(a) \text{ is open in } X_\alpha\}$,
- $X_\gamma = \bigcap_{\alpha<\gamma} X_\alpha$ for $\gamma \in Lim$.

By (f), we have that if $X_\alpha \neq \emptyset$, then $X_{\alpha+1}$ is a proper subset of X_α.

If $X_\alpha \neq \emptyset$ for all $\alpha < \omega_1$, then we get a contradiction with the fact that X is second countable. Hence, $X_{\alpha_0} = \emptyset$ for some $\alpha_0 < \omega_1$. Then, $X = \bigcup_{\alpha<\alpha_0} X_\alpha \setminus X_{\alpha+1}$. But, since X is second countable, for every α, the set $X_\alpha \setminus X_{\alpha+1}$ is a union of countably many orbits which are open in X_α. So, we get that X is the union of countably many orbits. ■

Later in this section, we give examples of compact and profinite e-structures [ei-structures] which are not small. In particular, we give examples showing that in

Proposition 2.10, (a) does not imply (b), and (e) does not imply (f). As to the implication $(c) \Longrightarrow (d)$, we have not found a counterexample yet.

Before turning to the examples, we study various topological properties of compact e-structures, which are essential in Section 3. By Example 2 below, we know that there exist compact ei-structures which are not profinite. However, we have

Proposition 2.11 *All orbits in a compact e-structure* (X, G) *are profinite.*

Proof. In fact, we claim here that all orbits on X^{eq} are profinite. But, by Corollary 1.15, it is enough to show that all orbits on X^n, $n \in \omega$, are profinite. Let o be any orbit on X^n over $\emptyset$. Put $H = G/G_o$, where G_o is the pointwise stabilizer of o. Then, (o, H) is a homogeneous compact e-structure. So, by Proposition 1.6, it is enough to show that H is profinite.

Choose a dense subset $\{a_i : i \in \omega\}$ of o. Since (o, H) is a compact e-structure, there is a sequence $\langle b_i \rangle_{i\in\omega}$ such that $o(b_0/a_0)$ is open in o and $o(b_i/a_{\leq i})$ is open in $o(b_{i-1}/a_{\leq i-1})$ for every $i \geq 1$. Put $o_i = o(b_i/a_{\leq i})$, $i \in \omega$.

Claim 1 The intersection $\bigcap_{i\in\omega} o_i$ has exactly one element; we will denote it by a.

Proof. That $\bigcap_{i\in\omega} o_i \neq \emptyset$ follows from the compactness of o and the fact that o_i's form a descending sequence of closed, non-empty sets.

Assume that $a, b \in \bigcap_{i\in\omega} o_i$. Then, there are $h_i \in H$, $i \in \omega$, such that $h_i a = b$ and $h_i \in H_{a_{\leq i}}$. Since H is a compact metric group, there is a subsequence $\langle h_{i_k} \rangle$ converging to some $h \in H$. Then, $h \in \bigcap_{i\in\omega} H_{a_{\leq i}} = \{id\}$ as $\{a_i : i \in \omega\}$ is dense. Thus, since $h_{i_k} a = b$ for all $k \in \omega$, we get $a = ha = b$. □

Now, put $S_i = \{h o_i : h \in H\}$ and $N_i = \{h \in H : hs = s \text{ for every } s \in S_i\}$. By Proposition 1.16, we see that S_i's are finite. This together with Proposition 1.16 easily gives us that N_i's are clopen, normal subgroups of H.

We will be done if we prove the following

Claim 2 The collection of all intersections of finitely many N_i's forms a basis of open neighborhoods of id.

Proof. Suppose it is not true. By the compactness of H and the closedness of N_i's, we get $\bigcap_{i\in\omega} N_i \neq \{id\}$. So, take any $f \in \bigcap_{i\in\omega} N_i \setminus \{id\}$. Then, there is $b \in o$ with $fb \neq b$. On the other hand, for every $i \in \omega$, we can choose $h_i \in H$ so that $b \in h_i o_i$.

Since $h_i o_i \in S_i$ and $f \in N_i$, we get $f h_i o_i = h_i o_i$. So, b and fb belong to $h_i o_i$, and thus, $h_i^{-1} b \in o_i$ and $h_i^{-1} fb \in o_i$.

By the compactness of H, there is a subsequence $\langle h_{i_k} \rangle$ converging to some $h \in H$. Therefore, by Claim 1, $h^{-1} b \in \bigcap_{k\in\omega} o_{i_k} = \{a\}$ and $h^{-1} fb \in \bigcap_{k\in\omega} o_{i_k} = \{a\}$. Thus, $h^{-1} b = a = h^{-1} fb$, and so $b = fb$, a contradiction. ■

Proposition 2.12 *Every compact e-structure* (X, G) *satisfies the existence of m-independent extensions over parameters from* $X \cup acl^{eq}(\emptyset)$.

Proof. Notice that using the same argument as in Remark 2.7, we easily get that it is enough to consider any orbit $o(a/AC)$, where $a \in X$, A is a finite subset of X and C a finite subset of $acl^{eq}(\emptyset)$, and to show that for any finite $B \subseteq X$, there is $b \in o(a/AC)$ such that $b \overset{m}{\downarrow}_{AC} B$.

On the one hand, $o(a/AC)$ is open in $o(a/A)$. On the other, by the existence of m-independent extensions over the real elements, there is $a' \in o(a/A)$ with $a' \overset{m}{\downarrow}_A B$, i.e. $o(a'/AB)$ is open in $o(a/A)$. Moreover, by virtue of Proposition 2.11, $o(a/A)$ is profinite, and so the compact structure $(o(a/A), G_A/G_{Ao(a/A)})$ is a profinite structure by Proposition 1.5. All these three observations imply that there exists a finite, A-invariant equivalence relation E on $o(a/A)$ with clopen classes and such that the following two conditions hold:

$$[a]_E \subseteq o(a/AC), \tag{$*$}$$

$$o(a'/AB) = [a_1]_E \cup \dots \cup [a_n]_E \text{ for some } a_1, \dots, a_n \in o(a/A). \tag{$**$}$$

By the existence of m-independent extensions over the real elements, there is $a'' \in o(a'/AB)$ such that $a'' \overset{m}{\downarrow}_{AB} a_1 \dots a_n$. By $(**)$, $a'' \in [a_i]_E$ for some $i \in \{1, \dots, n\}$, and we get that $o(a''/ABa_1 \dots a_n)$ is an open subset of $[a_i]_E$.

Consider any $g \in G_A$ mapping a_i to a. Then, $o(g(a'')/Ag[B]g(a_1) \dots g(a_n)) = g[o(a''/ABa_1 \dots a_n)]$ is an open subset of $[a]_E$. On the other hand, once again using the existence of m-independent extensions over the real elements, there is $b \in o(g(a'')/Ag[B]g(a_1) \dots g(a_n))$ such that $b \overset{m}{\downarrow}_{Ag[B]g(a_1)\dots g(a_n)} B$. Thus, we conclude that $o(b/ACBg[B]g(a_1) \dots g(a_n))$ is an open open subset of $[a]_E$. So, by $(*)$, it is an open subset of $o(a/AC)$, and in particular, $b \in o(a/AC)$. Since $o(b/ABC)$ contains $o(b/ACBg[B]g(a_1) \dots g(a_n))$, we get $b \overset{m}{\downarrow}_{AC} B$. ■

Corollary 2.13 *Let (X, G) be a compact e-structure, A and B be finite subsets of X, and $a \in X^n$. Assume that $U \neq \emptyset$ is an open subset of $o(a/A)$. Then there is $b \in U$ with $b \overset{m}{\downarrow}_A B$.*

Proof. By Propositions 2.11 and 1.5, $(o(a/A), G_A/G_{Ao(a/A)})$ is a profinite structure, and so there is a finite, A-definable equivalence relation E on $o(a/A)$ with $[a]_E \subseteq U$. On the other hand, it is obvious that (X, G_A) is a compact e-structure, and thus by Proposition 2.12, there is $b \in o(a/A[a]_E) = [a]_E$ with $b \overset{m}{\downarrow}_{A[a]_E} B$. Then, $b \in U$ and $b \overset{m}{\downarrow}_A B$. ■

Following Newelski, we say that a sequence $\langle a_i : i \in \omega \rangle$ is a flat Morley sequence in an orbit $o = o(a/A)$ in a compact structure (X, G) if it is m-independent (i.e. $o(a_n/Aa_{<n})$ is open in o for every n) and dense in o.

The following follows immediately from Corollary 2.13.

Corollary 2.14 *A flat Morley sequence exists in every orbit of a real tuple over a real subset in any compact e-structure.*

One more corollary will be useful later.

Corollary 2.15 *In every compact e-structure (X, G), if $A \subseteq B$ are finite subsets of X^{eq} and $o(a/B)$ is open in $o(a/A)$, then there is $b \in acl^{eq}(\emptyset)$ such that $o(a/B) = o(a/Ab) = o(a/Bb)$.*

Proof. By Proposition 2.11 and 1.5, there is a finite, $\emptyset$-definable equivalence relation E on $o(a)$ and $a_1, \dots, a_n \in o(a/B)$ such that $o(a/B) = o(a/A) \cap ([a_1]_E \cup \dots \cup [a_n]_E)$ and for every i, $o(a/A) \cap [a_i]_E \neq \emptyset$. Now, define b as a name for $[a_1]_E \cup \dots \cup [a_n]_E$. It is clear that b satisfies our requirements. ■

Now, we turn to examples. Since in the rest of this section we will often consider products and projections, let us fix some notation. For a product $X \times Y$, let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ denote the projections on the first and on the second coordinate, respectively. If $a \in X \times Y$ and $A \subseteq X \times Y$, then $a_1 := \pi_1(a)$, $a_2 := \pi_2(a)$, $A_1 := \pi_1[A]$ and $A_2 := \pi_2[A]$. Moreover, if X is the inverse limit of a system indexed by a set I, then for any $x = \langle x_i \rangle_{i \in I} \in X$, x_i is of course the i-th coordinate of x.

A trivial example, showing that (e) does not imply (f) in Proposition 2.10, is any uncountable compact [profinite] structure with the trivial structural group. A little bit more complicated (but still rather trivial) examples are the following.

Example 1 Let $(X, Aut^*(X))$ be any small profinite structure and let Y be an uncountable profinite space. Consider the profinite structure $(X \times Y, Aut^*(X))$ with the trivial action of $Aut^*(X)$ on Y and the given action of $Aut^*(X)$ on X. Then $(X \times Y, Aut^*(X))$ is not small, but it has small orbits. So, (e) does not imply (f).

Example 2 Let $(X, Aut^*(X))$ be a small profinite structure and Y be a compact metric space which is not profinite. Consider the compact structure $(X \times Y, Aut^*(X))$ with the trivial action of $Aut^*(X)$ on Y. Then $(X \times Y, Aut^*(X))$ is a compact, non-profinite (hence non-small) structure which has small orbits.

More generally, it is trivial that if (X, G) and (Y, H) are small [or with small orbits] compact structures, then so is $(X \times Y, G \times H)$ with the natural action of $G \times H$ on $X \times Y$. Similarly, if (X, G) and (Y, G) are compact e-structures, so is $(X \times Y, G \times H)$. The next remark tells us that the same is also true in imaginary sorts, but the proof of this fact is less straightforward.

Remark 2.16 *If (X, G) and (Y, G) are compact ei-structures, so is $(X \times Y, G \times H)$.*

Proof. By Proposition 2.10, we need to show that for every finite $A \subseteq X \times Y$ and every A-definable subset D of $X \times Y$ such that any two elements from D lie in the same orbit over $\ulcorner D \urcorner$, there is $d \in D$ such that $o(d/A)$ is open in D.

By the choice of D, D_1 is an orbit over $\ulcorner D \urcorner$, so also over $\ulcorner D_1 \urcorner$. Since (X, G) is a compact ei-structure and D_1 is A_1-definable, there is $d_1 \in D_1$ such that $o_1 := o(d_1/A_1)$ is open in D_1.

Put $D' = D \cap (o_1 \times Y)$. Then, D' is a clopen subset of D. So, by Proposition 1.16, $\ulcorner D' \urcorner \in acl^{eq}(\ulcorner D \urcorner)$. Hence, if we choose any $d_2' \in D_2'$ such that $(d_1, d_2') \in D'$ (the last condition holds anyway, because using the A-definability of D', we get $D' = D'_1 \times D'_2$), then $D'' := o(d_1, d'_2/\ulcorner D \urcorner \ulcorner D' \urcorner)$ is open in $o(d_1, d'_2/\ulcorner D \urcorner) = D$. Moreover, we see that D' is A-definable, and so is D''.

We conclude that D''_2 is A_2-definable and it is an orbit over $\ulcorner D''_2 \urcorner$. Since (Y, H) is a compact *ei*-structure, there is $d_2 \in D''_2$ such that $o_2 := o(d_2/A_2)$ is open in D''_2.

Define $d = (d_1, d_2)$. Since $D'' \subseteq D'$ is A-definable, we get $D''_1 = o_1$. This easily implies that $D'' = D''_1 \times D''_2$, and so $d \in D''$. On the other hand, $o(d/A) = o(d_1/A_1) \times o(d_2/A_2) = o_1 \times o_2$. Since $o_1 = D''_1$ and o_2 is open in D''_2, we conclude that $o(d/A)$ is an open subset of D''. Using the fact that D'' is open in D, we obtain the desired conclusion that $o(d/A)$ is open in D. ■

Below we describe a general construction which will yield non-trivial examples of non-small compact e-structures. We will get even more – examples showing that (a) does not imply (b) in Proposition 2.10.

Construction $(*)$ We start from two compact structures (X, H) and (Y, G). Equip the set $Cont(X, G) \subseteq G^X$ of all continuous functions from X to G with the compact-open topology. Let G^* be a subgroup (with the group structure inherited from the product group structure on G^X) of $Cont(X, G)$ which is a compact subset. Assume that $G^* \circ H = G^*$, i.e. for any $g \in G^*$ and $h \in H$, $g \circ h \in G^*$.

Notice that by compactness of G^* and the fact that G^* is equipped with the compact-open topology, we get that this is also the pointwise convergence topology. It is a classical fact that if G and X are compact metric spaces, then $Cont(X, G)$ with the compact-open topology is metrizable. Thus, G^* is metrizable.

In several claims, the proofs of which are left as easy exercises, we will explain how the above assumptions produce a new compact structure $(X \times Y, H \ltimes G^*)$.

Claim 1 G^* is a topological group.

We have the natural actions of H and G^* on $X \times Y$ as groups of permutations. Namely, H acts by $h(x, y) = (hx, y)$ and G^* by $g^*(x, y) = (x, g^*(x)y)$.

Claim 2 Inside $Sym(X \times Y)$, G^* is normalized by H. More precisely, for any $g^* \in G^*$ and $h \in H$, we have $h^{-1}g^*h = g^* \circ h \in G^*$.

Of course, H acts on G^* on the right by $g * h := g \circ h$. Claim 2 tells us that this action coincides with the action by conjugation if we treat H and G^* as subgroups of $Sym(X \times Y)$. Since inside $Sym(X \times Y)$ the intersection of H and G^* is trivial, $H \ltimes G^*$ can be considered as a subgroup of $Sym(X \times Y)$, and so it acts on $X \times Y$ by $(h, g^*)(x, y) = (hx, g^*(x)y)$. Using Claim 1, one can show

Claim 3 $H \ltimes G^*$ equipped with the product topology is a compact topological group and it acts continuously on $X \times Y$.

Therefore, we have obtained a new compact structure $(X \times Y, H \ltimes G^*)$, and Construction $(*)$ is completed. ■

In fact, the following special case of the above construction will be used to produce our examples.

Construction $(*')$ Let $(X, Aut^*(X))$ be a profinite structure with a distinguished inverse system $\mathcal{X} = \{X_i, f_{ji} : i \leq j; i, j \in I\}$, i.e. X is the inverse limit of $\mathcal{X}$ and $Aut^*(X)$ respects $\mathcal{X}$. Let (Y, G) be a compact structure, and let $\mathcal{Y} = \{Y_i, g_{ji} : i \leq j; i, j \in I\}$ be an inverse system of compact metric spaces such that Y is the inverse limit of $\mathcal{Y}$ and G respects $\mathcal{Y}$.

It is clear that $X \times Y$ can be considered as the inverse limit of the system $\mathcal{X} \times \mathcal{Y} := \{X_i \times Y_i, (f_{ji}, g_{ji}) : i \leq j; i, j \in I\}$.

G^X acts on $X \times Y$ as a group of permutations by $\overline{g}(x, y) = (x, \overline{g}(x)y)$. Define G^* as the subgroup of G^X consisting of the elements of G^X respecting the inverse system $\mathcal{X} \times \mathcal{Y}$. Equip G^X with the product topology and G^* with the inherited topology.

By routine arguments (using the finiteness of X_i's in Claim 2), we get

Claim 1 G^* is a compact topological group.

Claim 2 The function $\Phi : G^* \times X \to G$ defined by $\Phi(g^*, x) = g^*(x)$ is continuous.

By Claims 1 and 2, the topology on G^* coincides with the compact-open topology inherited from $Cont(X, G)$. Of course, we also have $G^* \circ Aut^*(X) = G^*$. Therefore, Construction $(*)$ yields the compact structure $(X \times Y, Aut^*(X) \ltimes G^*)$.

We can also get $Aut^*(X) \ltimes G^*$ in a little bit different way. Namely, inside $Sym(X \times Y)$, G^X is normalized by $Aut^*(X)$, so we can build $Aut^*(X) \ltimes G^X$. Define $G(X, Y)$ as the subgroup of $Aut^*(X) \ltimes G^X$ consisting of the permutations from $Aut^*(X) \ltimes G^X$ respecting the inverse system $\mathcal{X} \times \mathcal{Y}$, and equip it with the topology inherited from $Aut^*(X) \ltimes G^X$ with the product topology. One can easily check that

$$G(X, Y) = Aut^*(X) \ltimes G^*.$$

Let us make a few additional observations. Let $G_i(X_i, Y_i)$ be the group of permutations of $X_i \times Y_i$ induced by $G(X, Y)$. If we equip $G_i(X_i, Y_i)$ with the pointwise convergence topology coming from the action on $X_i \times Y_i$, then the natural map from $G(X, Y)$ onto $G_i(X_i, Y_i)$ is continuous, so $G_i(X_i, Y_i)$ is compact. It is easy to see that $G_i(X_i, Y_i)$ is a compact topological group acting continuously on $X_i \times Y_i$. Finally, $G(X, Y)$ is the inverse limit of all the groups $G_i(X_i, Y_i)$. ■

Before we use Construction $(*')$ to produce our examples, it is convenient to make some general observations. In all applications of Construction $(*')$, we assume that $I = \omega$, i.e. the inverse systems are indexed by natural numbers.

Lemma 2.17 *Let $(X, Aut^*(X))$ and (Y, G) be as in Construction $(*')$. Suppose that for some $x^1, \dots, x^n \in X$ we have elements $g_1, \dots, g_n \in G$ such that, whenever $x^k_i = x^j_i$, g_k and g_j induce the same permutation of Y_i. Then we can find $g^* \in G^*$ such that $g^*(x^j) = g_j$ for $j = 1, \dots, n$.*

Proof. Extend the set $\{x^j : j \leq n\}$ to a dense subset $D := \{x^j : j \in \omega\}$ of X. By an easy recursion, we extend the given sequence $\langle g_j \rangle_{j \leq n}$ to a sequence $\langle g_j \rangle_{j \in \omega}$ of elements of G so that, whenever $x_i^k = x_i^j$, then g_k and g_j induce the same permutation of Y_i.

Now, we are going to define g^*. Consider any $x \in X$. For every $i \in \omega$, choose any j_i such that $x_i = x_i^{j_i}$ (such a j_i exists because D is dense). Let h_i be the permutation of Y_i induced by g_{j_i}. By the choice of g_j's, h_i does not depend on the choice of j_i, and for $i_1 > i_2$, the permutation induced by h_{i_1} on Y_{i_2} is exactly h_{i_2}. So, the h_i's determine some $h_x \in G$ in the sense that for every i, the permutation induced by h_x on Y_i equals h_i.

As a result, we obtain a function $g^* \in G^X$ defined by $g^*(x) = h_x$. It is obvious that $g^* \in G^*$ and $g^*(x^j) = g_j$ for $j = 1, \ldots, n$. ■

Lemma 2.18 *Suppose $(X, Aut^*(X))$ and (Y, G) are as in Construction $(*')$. In (iii) and (iv) below, assume additionally that G induces a finite group of permutations of Y_i for every $i \in \omega$ (e.g. it is the case if all Y_i's are finite). Consider any $(x, y) \in X \times Y$. Take any finite sets A and B such that $A \subseteq \{x\} \times Y$ and $B \subseteq (X \setminus \{x\}) \times Y$. Then we have the following description of orbits in $(X \times Y, G(X, Y))$.*

(i) $o(x, y) = o(x) \times o(y)$.

(ii) If $A \neq \emptyset$*, then* $o(x, y/A) = \{x\} \times o(y/A_2)$.

(iii) $o(x, y/B) = o(x/B_1) \times U$*, where* U *is open in* $o(y)$.

(iv) If $A \neq \emptyset$*, then* $o(x, y/AB) = \{x\} \times U$*, where* U *is open in* $o(y/A_2)$.

(v) If C *is any finite subset of* $X \times Y$*, then* $o(x, y/C) \supseteq o(x/C_1) \times o(y/C_2)$.

Proof. (i) The inclusion $(\subseteq)$ is obvious. For the opposite inclusion, take any $x^1 \in o(x)$ and $y^1 \in o(y)$. Then, there are $f \in Aut^*(X)$ and $g \in G$ such that $f(x) = x^1$ and $g(y) = y^1$. Let $g^* \in G^X$ be the function constantly equal to g. We see that $(f, g^*) \in G(X, Y)$ and it maps (x, y) to (x^1, y^1).

(ii) The inclusion $(\subseteq)$ is obvious. For the opposite inclusion, we argue as in (i).

(iii) Let

$$S = \{g^* \in G^* : (\forall b \in B_1)(g^*(b) = id)\}.$$

It is enough to show that $\{g^*(x)y : g^* \in S\}$ is open in $o(y)$. So, we will be done if we prove that

$$P := \{g^*(x) : g^* \in S\}$$

is open in G.

Let i be the maximal index for which there is $b \in B_1$ such that $b_i = x_i$. We shall show that P is the set of all elements of G inducing the trivial permutation of Y_i. Since G induces a finite group of permutations of Y_i, this will complete the proof.

Of course, every element of P induces the trivial permutation of Y_i. For the other direction, take any $g \in G$ inducing the trivial permutation of Y_i. By Lemma 2.17, there is $g^* \in S$ such that $g^*(x) = g$, and so $g \in P$.

(iv) We argue as in (iii).

(v) We argue as in (i). ■

Lemma 2.19 *Suppose we have (X, H), (Y, G) and G^* as in Construction $(*)$. Assume additionally that:*

(i) for any $(x, y), (x_1, y_1), \dots, (x_n, y_n) \in X \times Y$ with x different from all x_i's, we have that $\{g^*(x)y : g^*(x_1)y_1 = y_1, \dots, g^*(x_n)y_n = y_n\}$ is open in $o(y)$,

(ii) (X, H) is a compact e-structure,

(iii) for every finite $A \subseteq X$, $acl(A) = A$,

(iv) for every $y \in Y$, $G_y := \{g \in G : gy = y\} = \{id\}$.

Then $(X \times Y, H \ltimes G^*)$ obtained in Construction $(*)$ is a compact e-structure, but it is not a compact ei-structure whenever G is infinite.

Proof. First, we prove that it is a compact e-structure. So, take any finite subsets $A, B \subseteq X \times Y$ and $a \in X \times Y$. We need to find $a' \in o(a/A)$ such that $o(a'/AB)$ is open in $o(a/A)$.

Case 1 $a_1 \in A_1$.
Then, by (iv), $o(a/A)$ is a singleton, and so $a' := a$ works.

Case 2 $a_1 \notin A_1$.
By (ii), there is $a_1' \in o(a_1/A_1)$ such that $o(a_1'/A_1B_1)$ is open in $o(a_1/A_1)$. By (iii), we have that $a_1 \notin acl(A_1)$, and so $a_1' \notin A_1 \cup B_1$. Take any $a_2' \in \pi_2[o(a/A)]$, and put $a' := (a_1', a_2')$. Thus, by virtue of (i), we get that $o(a'/AB)$ is open in $o(a_1'/A_1) \times o(a_2)$, which implies that $o(a'/AB)$ is open in $o(a/A)$.

It remains to show that $(X \times Y, H \ltimes G)$ is not a compact ei-structure whenever G is infinite. In order to do that, choose any $a \in X \times Y$ and define $D = \{a_1\} \times G^*(a_1)a_2$. Since G is infinite and stabilizers of all singletons in Y are trivial, $o(a_2)$ is infinite, and so, using (i), D is also infinite. We also see that D is definable over a and all elements in D lie in the same orbit over $\ulcorner D \urcorner$, whereas all orbits in D over a are singletons. So, by Proposition 2.10, $(X \times Y, H \ltimes G^*)$ is not a compact ei-structure. ■

We leave it as an easy exercise to check that if we replace Assumption (iii) by $A \subsetneq acl(A)$ for some finite $A \subseteq X$, then $(X \times Y, H \ltimes G^*)$ is not a compact e-structure whenever G is infinite.

Now, we use Construction $(*')$ to produce examples. Recall that in all applications of Construction $(*')$, we assume that $I = \omega$.

Example 3 Let $(X, Aut^*(X))$ be any small profinite structure [or, more generally, profinite e-structure] such that $acl(A) = A$ for every finite $A \subseteq X$ (e.g. it is the case when $X = \{0, 1\}^\omega$ and $Aut^*(\{0, 1\}^\omega)$ is the group of all homeomorphisms of X

preserving the inverse system consisting of all initial subproducts). Let $Y = S_1^\omega$ be treated as the inverse limit of S_1^n, $n \geq 1$, and $G = \mathbb{Z}_2^\omega$ act on Y by $\langle g_i \rangle \langle y_j \rangle = \langle z_k \rangle$, where

$$z_k = \begin{cases} y_k & \text{if} \quad g_k = 0 \\ -y_k & \text{if} \quad g_k = 1. \end{cases}$$

Then $(X \times Y, G(X,Y))$ is a compact e-structure which is not a compact ei-structure, and, of course, it is not a profinite structure. This shows that Condition (a) in Proposition 2.10 does not imply (b). Moreover, $G(X,Y)$ induces finite groups of permutations on every $X_i \times Y_i$.

Proof. It is an easy application of Lemma 2.19 once one uses Lemma 2.18(iii) in order to prove that Assumption (i) of Lemma 2.19 is satisfied. ■

Notice that although the orbits in the structures obtained in the above example are profinite (which, by Proposition 2.11, is always the case in compact e-structures), no such structure is isomorphic to a structure obtained by the trivial product construction described in Example 2. Even more, any structure obtained in Example 3 is not isomorphic to a compact structure of the form $(Z_1 \times Z_2, K)$, where Z_1 is profinite, Z_2 is compact, and for every $k \in K$, $k(z_1, z_2) \in Z_1 \times \{z_2\}$. To see this, suppose for a contradiction that it is isomorphic to such a structure. The connected components of $X \times Y$ are the sets $\{x\} \times Y$, where $x \in X$, and the connected components of $Z_1 \times Z_2$ are the sets $\{z_1\} \times C$, where $z_1 \in Z_1$ and C is any connected component of Z_2. Since the structures are isomorphic, the connected components of $X \times Y$ are mapped to the connected components of $Z_1 \times Z_2$, and the induced actions of $G(X,Y)$ and K on the corresponding components coincide. However, on each $\{x\} \times Y$, every orbit under the action of the setwise stabilizer of $\{x\} \times Y$ in $G(X,Y)$ is uncountable, whereas on each $\{z_1\} \times C$, the setwise stabilizer of $\{z_1\} \times C$ in K acts trivially. This is a contradiction.

Example 4 Let $(X, Aut^*(X))$ be any small profinite structure [or, more generally, a profinite e-structure] such that $acl(A) = A$ for every finite $A \subseteq X$. Let $(Y, Aut^*(Y))$ be a profinite structure with an infinite structural group and trivial stabilizers of all singletons (it is always the case when $(Y, Aut^*(Y))$ is an infinite, strongly 1-transitive profinite structure; e.g. Y is any infinite profinite group, and $Aut^*(Y) = Y$ acts on Y by left translations). Then $(X \times Y, G(X,Y))$ is a profinite e-structure, but it is not a profinite ei-structure. This shows that Condition (a) in Proposition 2.10 does not imply (b), even in the class of profinite structures.

Proof. The same argument as in Example 3 works here. ■

Notice that any structure obtained in Example 4 is not isomorphic to a structure obtained in Example 1. This is because the structures obtained in Example 4 do not have small orbits.

Now, we will show that Construction $(*')$ preserves the properties that we are interested in (smallness, having small orbits, being e-structure or ei-structure). In particular, starting from Examples 3 or 4, and iterating Construction $(*')$, we produce a class of examples of compact [profinite] e-structures, which are not small.

Proposition 2.20 *Let $(X, Aut^*(X))$ be a profinite and (Y, G) a compact structure.*

(i) If $(X, Aut^*(X))$ and (Y, G) are small, then so is $(X \times Y, G(X, Y))$.

(ii) If $(X, Aut^*(X))$ and (Y, G) have small orbits, then so has $(X \times Y, G(X, Y))$.

(iii) If $(X, Aut^*(X))$ and (Y, G) are compact e-structures and G induces a finite group of permutations of every Y_i, then $(X \times Y, G(X, Y))$ is also a compact e-structure, and $G(X, Y)$ induces a finite group of permutations of every $X_i \times Y_i$.

(iv) If $(X, Aut^*(X))$ and (Y, G) are compact ei-structures and G induces a finite group of permutations of every Y_i, then $(X \times Y, G(X, Y))$ is also a compact ei-structure, and $G(X, Y)$ induces a finite group of permutations of every $X_i \times Y_i$.

Before we start to prove the proposition, notice that if (Y, G) is a profinite structure, then the extra assumption about G in (iii) and (iv) is automatically satisfied.

Proof. (i) Consider any $a \in X \times Y$ and any finite $A \subseteq X \times Y$. By Lemma 2.18, $o(a/A) \supseteq o(a/A_1) \times o(a/A_2)$. Since in $(X, Aut^*(X))$ there are countably many orbits over A_1, and in (Y, G) there are countably many orbits over A_2, we get that in $(X \times Y, G(X, Y))$ there are countably many orbits over A.

(ii) Take any 1-orbit $o(a)$ in $X \times Y$ and any finite subset A of $X \times Y$. Then, $o(a) = o(a_1) \times o(a_2)$ and for any $a' \in o(a)$, $o(a'/A) \supseteq o(a'_1/A_1) \times o(a'_2/A_2)$. To finish the proof, notice that by assumption, $o(a_1)$ and $o(a_2)$ are unions of countably many orbits over A_1 and A_2, respectively.

(iii) Let $A \subseteq B$ be finite subsets of $X \times Y$ and $a \in X \times Y$. We need to find $a' \in o(a/A)$ such that $o(a'/B)$ is open in $o(a/A)$.

Case 1 $a_1 \in A_1$.

Let $C = A \cap (\{a_1\} \times Y))$. By Lemma 2.18, $o(a/A) = \{a_1\} \times U$, where U is open in $o(a_2/C_2)$. Since (Y, G) is a compact e-structure, by Corollary 2.13, we can find $a'_2 \in U$ such that $o(a'_2/B_2)$ is open in $o(a_2/C_2)$. Put $a' := (a_1, a'_2)$. As $o(a'/B) \supseteq \{a_1\} \times o(a'_2/B_2)$, we get that $o(a'/B)$ is open in $o(a/A)$.

Case 2 $a_1 \notin A_1$.

By Lemma 2.18, $o(a/A) = o(a_1/A_1) \times U$, where U is an open subset of $o(a_2)$. Since $(X, Aut^*(X))$ is a compact e-structure, there is $a'_1 \in o(a_1/A_1)$ such that $o(a'_1/B_1)$ is open in $o(a_1/A_1)$. As (Y, G) is a compact e-structure, by Corollary 2.13, there is $a'_2 \in U$ such that $o(a'_2/B_2)$ is open in $o(a_2)$. Put $a' = (a'_1, a'_2)$. We see that $a' \in o(a/A)$. We also have $o(a'/B) \supseteq o(a'_1/B_1) \times o(a'_2/B_2)$, and the last product is open in $o(a_1/A_1) \times o(a_2)$. Hence, $o(a'/B)$ is open in $o(a/A)$.

(iv) Let $D \subseteq X \times Y$ be A-definable for some finite subset A of $X \times Y$, and assume that D is an orbit over $\ulcorner D \urcorner$. By Proposition 2.10, it is enough to find $a \in D$ with $o(a/A)$ open in D.

We see that D_1 is an A_1-definable subset of X whose elements lie in the same orbit over $\ulcorner D_1 \urcorner$. Since $(X, Aut^*(X))$ is a compact *ei*-structure, we can find $a_1 \in D_1$ such that $o(a_1/A_1)$ is open in D_1.

Case 1 $a_1 \in A_1$.
Then, D_1 is finite. Put $E = \pi_2[(\{a_1\} \times Y) \cap D]$. Then, E is A_2-definable and it is an orbit over its name. Since (Y, G) is a compact *ei*-structure, there is $a_2 \in E$ with $o(a_2/A_2)$ open in E. Hence, $a := (a_1, a_2)$ does the job.

Case 2 $a_1 \notin A_1$.
Take any $a_2 \in Y$ with $(a_1, a_2) \in D$. Put $a = (a_1, a_2)$. By Lemma 2.18, $o(a/A) = o(a_1/A_1) \times U$, where U is an open subset of $o(a_2)$. On the other hand, since all elements of D lie in the same orbit over $\emptyset$, we have $D \subseteq D_1 \times o(a_2)$. As $o(a_1/A_1)$ is open in D_1 and U is open in $o(a_2)$, we see that $o(a/A)$ is open in D. ■

We have constructed a class of examples of compact [profinite] *e*-structures which are not small. Another problem, which we leave for future considerations, is to find examples of non-small, compact [profinite] *e*-groups, which are defined as follows.

Definition 2.21 *A compact [profinite] e-group is a compact [profinite] e-structure (H, G), where H is a compact group and G acts on H by automorphisms. Compact [profinite] ei-groups are defined analogously.*

Let us only make here a few comments and observations about groups. There are some results [12, 16, 7] which describe the structure of small profinite groups and rings, and which significantly restrict the class of possible examples of small profinite groups (recall that all known examples of small profinite groups are some variants of abelian profinite groups of finite exponent [3, 4]). Most of these results are based on various chain conditions proved in Section 2 of [12], and on the fact that small profinite groups are locally finite [12, Proposition 2.4]. For compact [profinite] *ei*-groups, these results are false, because, for example, any infinite compact metric group considered with the trivial structural group is always a compact *ei*-group. More generally, Examples 1 and 2 as well as the comments immediately after them (including Remark 2.16) work for compact [profinite] groups. In particular, we see that a compact *ei*-group is not necessarily locally finite, even if the structural group is infinite. But, of course, it would be interesting to find more complicated examples.

It is pretty clear that any topologically finitely generated, compact metric group with an infinite structural group is not a compact *e*-group. Indeed, if $a_1, \dots, a_n$ are topological generators, then $o(a_i)$ must be infinite for at least one $i = 1, \dots, n$, whereas all orbits over $\{a_1, \dots, a_n\}$ are trivial.

However, we do not know whether the free profinite group on ω-many generators (symbolically $\widehat{F}_\omega$) with some natural, infinite (standard) structural group is a profinite *e*-group. Of course, we know that it is not small since it is not locally finite. We can show even more.

Remark 2.22 *Let $(\widehat{F}_\omega, G)$ be a profinite group for which G contains all inner automorphisms of $\widehat{F}_\omega$ (it is always the case when G is the standard structural group* *with respect to any distinguished inverse system). Then* $(\widehat{F}_\omega, G)$ *is not a profinite ei-group.*

Proof. Let e_i, $i \in \omega$, be the set of free generators of $\widehat{F}_\omega$. Let $F < \widehat{F}_\omega$ be the closure of $\langle e_0, e_1\rangle$. Then, $F \cong \widehat{F}_2$. For $a \in \widehat{F}_\omega$, let $f_a \in G$ be defined by $f_a(x) = a^{-1}xa$. Since F_2 is residually finite, we get $f_{e_0^n}(e_1) \neq e$ for every $n \geq 1$. We also have $f_{e_0^n}[F] = \overline{\langle e_0, e_0^{-n}e_1e_0^n\rangle} = F$. So, we conclude that $o(e_1/\ulcorner F\urcorner)$ is infinite. On the other hand, for any $a \in F$, $o(a/e_0, e_1) = \{a\}$. Thus, $o(e_1/\ulcorner F\urcorner)$ does not have an m-independent extension over $\{e_0, e_1\}$. ■

The above proof gives us the following general remark about compact ei-groups. However, we do not know if this remark holds for compact e-groups.

Remark 2.23 *If* (H, G) *is a compact ei-group, then for every topologically finitely generated, closed* $F < H$, $G_{\ulcorner F\urcorner}$ *induces a finite group of automorphisms of* F. *Similarly, if* (X, G) *is a compact ei-structure and* $A \subseteq X$ *is finite, then* $G_{\ulcorner dcl(A)\urcorner}$ *induces a finite group of homeomorphisms of* $dcl(A)$.

Notice that any compact structure with finite orbits is a compact ei-structure. Below we give an example of a profinite ei-structure with this property and with infinite structural group. It would be also interesting to find a compact ei-group with infinite structural group and with finite orbits.

Example 5 Let $X = \mathbb{Z}_2^\omega$ and $Aut^*(X) = \mathbb{Z}_2^\omega$, where the action of $Aut^*(X)$ on X is defined by

$$f(\eta) = \begin{cases} \eta + (0, f(0), f(0), f(0), f(0), \dots) & \text{if } \eta(0) = 0, \\ \eta + (0, 0, f(0), f(1), f(0), f(1), \dots) & \text{if } \eta(0) = 1, \eta(1) = 0, \\ \eta + (0, 0, 0, f(0), f(1), f(2), f(0), f(1), f(2), \dots) & \text{if } \eta(0) = 1, \eta(1) = 1, \eta(2) = 0, \\ \vdots & \vdots \\ \eta & \text{if } \eta \equiv 1. \end{cases}$$

It is easy to check that $(X, Aut^*(X))$ is a profinite structure with finite orbits, and, of course, $Aut^*(X)$ is infinite.

3 Model theory of compact e-structures

3.1 Newelski's results in a wider context

In this section, we explain why most of the results (including the group configuration theorem) proved by Newelski for small profinite structures hold in the more general context of compact [profinite] e-structures (in some cases, we need to work in a compact ei-structure). In many cases, instead of repeating the proofs, we will briefly explain why they work, referring the reader to Newelski's papers for details.

Essentially, in the model theory of small profinite structures, one uses the following consequences of smallness of a profinite structure $(X, Aut^*(X))$:

(a) the existence of m-independent extensions in the home sort,

(b) the existence of m-independent extensions over parameters from $X \cup acl^{eq}(\emptyset)$,

(c) the existence of m-independent extensions in imaginary sorts,

(d) the existence of open orbits in definable sets.

Of course, any result about small profinite structures whose proof uses only Consequences (a) or (b) [or (c)] holds for any profinite e-structure [ei-structure]. If a proof uses (d), we cannot automatically generalize it to the case of profinite ei-structures. Moreover, Propositions 1.16, 2.11, 2.12 and Corollary 1.17 enable us to pass, in many cases, from a profinite to a compact situation.

The following theorem is a generalization of [10, Theorem 1.5].

Theorem 3.1 *Let (X, G) be any compact structure. Let I be a ternary relation defined on finite tuples from X^{eq}, invariant under G and satisfying Properties (1), (2), (3) and (4) listed in Fact 2.4. Then $\overset{m}{\perp}$ implies I. Similarly, if all orbits are profinite (e.g. (X, G) is a compact e-structure) and I is a ternary relation defined on finite tuples from $X \cup acl^{eq}(\emptyset)$, invariant under G and satisfying Properties (1), (2), (3) and (4) listed in Fact 2.4, then $\overset{m}{\perp}$ implies I.*

Proof. Suppose for a contradiction that $a \overset{m}{\perp}_C b$, but $\neg a I_C b$ for some finite $a, b, C \subseteq X^{eq}$. Then, $o(a/Cb)$ is open in $o(a/C)$. By Corollary 1.17, we get that $d := \ulcorner o(a/Cb) \urcorner \in acl^{eq}(C)$.

Let us prove now that $\neg a I_{Cd} b$. Otherwise, by (1), we get $b I_{Cd} a$. By (3), we have $d I_C a$. So, by (1) and (2), we get $bd I_C a$, hence $a I_C bd$, and finally $a I_C b$, a contradiction.

Since d is a name for $o(a/Cb)$, we have that $o(a/Cd) = o(a/Cdb)$. Take any $a' \in o(a/Cd)$. Then, $a' \in o(a/Cdb)$. Since I is invariant under G, we get $\neg a' I_{Cd} b$; this contradicts Property (4) for I.

The second part of the theorem can be proved similarly. Namely, suppose for a contradiction that $a \overset{m}{\perp}_C b$, but $\neg a I_C b$ for some finite $a, b, C \subseteq X \cup acl^{eq}(\emptyset)$. The proof of Corollary 2.15 yields $d \in acl^{eq}(\emptyset)$ such that $o(a/Cb) = o(a/Cbd) = o(a/Cd)$. Then, we proceed as in the first part to get a contradiction. ■

From now on, assume that **(X, G) is a compact e-structure**. Notice that by Proposition 2.12, we can freely work in $X \cup acl^{eq}(\emptyset)$. Sometimes, e.g. working essentially in X^{eq}, we assume that (X, G) is a compact ei-structure.

We can measure the size of orbits in X (or X^{eq}) by means of $\mathcal{M}$-rank defined as follows.

Definition 3.2 *The rank $\mathcal{M}$ is the unique function from the collection of orbits over finite sets to the ordinals together with ∞, satisfying*

$$\mathcal{M}(a/A) \geq \alpha + 1 \quad \textit{iff} \quad \text{there is a finite set } B \supseteq A \text{ such that } o(a/B) \text{ is nowhere dense in } o(a/A) \text{ and } \mathcal{M}(a/B) \geq \alpha.$$

The following results follow from a standard forking calculation (e.g. see [15, Lemma 5.1.4 and Theorem 5.1.6].

Proposition 3.3 *Let a, b, A be finite tuples (subsets) of $X \cup acl^{eq}(\emptyset)$ [or of X^{eq}, if (X, G) is a compact ei-structure].*

1. *If $a \overset{m}{\perp}_A b$, then $\mathcal{M}(a/Ab) = \mathcal{M}(a/A)$. The converse holds whenever $\mathcal{M}(a/A) < \infty$.*
2. *$\mathcal{M}(a/bA) + \mathcal{M}(b/A) \leq \mathcal{M}(ab/A) \leq \mathcal{M}(a/bA) \oplus \mathcal{M}(b/A)$.*
3. *Suppose $\mathcal{M}(a/Ab) < \infty$ and $\mathcal{M}(a/A) \geq \mathcal{M}(a/Ab) \oplus \alpha$. Then $\mathcal{M}(b/A) \geq \mathcal{M}(b/Aa) + \alpha$.*
4. *Suppose $\mathcal{M}(a/Ab) < \infty$ and $\mathcal{M}(a/A) \geq \mathcal{M}(a/Ab) + \omega^\alpha n$. Then $\mathcal{M}(b/A) \geq \mathcal{M}(b/Aa) + \omega^\alpha n$.*
5. *If $a \overset{m}{\perp}_A b$, then $\mathcal{M}(ab/A) = \mathcal{M}(a/bA) \oplus \mathcal{M}(b/A)$.*

As in stable or simple theories, the (in)equalities in (2)-(5) are called Lascar (in)equalities.

Remark 3.4 *Let $a, A \subseteq X$ be finite. Then the $\mathcal{M}$-rank of $o(a/A)$ computed in X is the same as the $\mathcal{M}$-rank of $o(a/A)$ computed in $X \cup acl^{eq}(\emptyset)$. Moreover, if (X, G) is a compact ei-structure, then the $\mathcal{M}$-rank of $o(a/A)$ computed in X is the same as the $\mathcal{M}$-rank of $o(a/A)$ computed in X^{eq}.*

Definition 3.5 *(X, G) is m-stable if every 1-orbit has an ordinal $\mathcal{M}$-rank.*

Equivalently, there is no infinite sequence $A_1 \subseteq A_2 \subseteq \dots$ of finite subsets of X and $a \in X$ such that $o(a/A_{i+1})$ is nowhere dense in $o(a/A_i)$ for every i.

By Lascar inequalities we easily get

Remark 3.6 *(X, G) is m-stable iff each n-orbit has an ordinal $\mathcal{M}$-rank (iff each orbit in X^{eq} has an ordinal $\mathcal{M}$-rank, assuming that (X, G) is a compact ei-structure).*

Following the lines of the proof of [13, Lemma 2.6] and using Proposition 1.16 in the appropriate places, one gets

Proposition 3.7 *Assume that (X, G) is m-stable and $a, A \subseteq X \cup acl^{eq}(\emptyset)$ [or $a, A \subseteq X^{eq}$, if (X, G) is a compact ei-structure]. Then $o(a/A)$ is invariant over finitely many parameters from $o(a/A)$ which can be chosen m-independent from a given set B over A.*

Let us recall that the proof of [13, Lemma 2.6] is by induction on $\ulcorner o(a/A) \urcorner$. In our context, even without the existence of m-independent extensions over imaginary sorts, we can also make such an induction, because m-stability implies that for every $d \in X^{eq}$, $\mathcal{M}(d) < \infty$, where $\mathcal{M}$ is computed while working only over parameters from X.

Let us add that, elaborating the proof of the above proposition and applying Corollary 2.13, the parameters from $o(a/A)$ can be chosen to be m-independent over A.

The next definition (see [14]) is in the same spirit as the above result; in fact, it is a counterpart of the notion of 1-basedness in geometric stability theory. However, in compact structures we have to formulate it in a slightly different way than in [14] (both definitions coincide in profinite structures).

Definition 3.8 (X, G) *is m-normal if for every finite $a, A \subseteq X$, there is $U \ni a$ clopen in $o(a/A)$ and with finitely many conjugates under G_a.*

Notice that by Proposition 1.16, in the above definition we can replace U by any clopen subset of U containing a. Hence, by Propositions 2.11 and 1.5, we can take $U = o(a/A) \cap [a]_E$, where E is a finite, $\emptyset$-definable equivalence relation on $o(a)$ with clopen classes.

The next two results are more or less Theorem 2.3 of [13]. However, we will prove them in our context to illustrate how Proposition 1.16 works.

Remark 3.9 (X, G) *is m-normal iff for every finite $a, A \subseteq X \cup acl^{eq}(\emptyset)$, there is $U \ni a$ clopen in $o(a/A)$ and with finitely many conjugates under G_a. Moreover, if (X, G) is a compact ei-structure, then (X, G) is m-normal iff for every finite $a, A \subseteq X^{eq}$, there is $U \ni a$ clopen in $o(a/A)$ and with finitely many conjugates under G_a.*

Proof. $(\Longleftarrow)$ is obvious.
$(\Longrightarrow)$. Let us prove the 'moreover' part (the proof applies also in the first part). Take any $a, A \in X^{eq}$. Then, $A = A_0/F$, where A_0 is a finite tuple from X and F is a $\emptyset$-definable equivalence relation. Wlog $A_0 \overset{m}{\perp}_A a$, hence $a \overset{m}{\perp}_A A_0$. This means that $o(a/A_0)$ is clopen in $o(a/A)$. Hence, wlog $A = A_0$.

Now, $a = a_0/E$ for some finite tuple a_0 from X and a $\emptyset$-definable equivalence relation E. Wlog

$$(*) \quad a_0 \overset{m}{\perp}_a A.$$

Since (X, G) is m-normal, we can find a set $V \ni a_0$ clopen in $o(a_0/A)$ and with finitely many conjugates under G_{a_0}.

Let $b = \ulcorner V \urcorner$. Then, $b \in acl^{eq}(a_0)$. On the other hand, by Proposition 1.16, $b \in acl^{eq}(A)$. This together with $(*)$ gives us

$$(**) \quad b \in acl^{eq}(a).$$

As we noticed right below Definition 3.8, by Proposition 1.16, we can assume that V is the equivalence class $[a_0]_{E_0}$ of some finite A-definable equivalence relation E_0 on $o(a_0/A)$.

Let $T : X \to X/E$ be the quotient map.

Claim $T[V]$ is a clopen subset of $o(a/A)$.

Proof. G_{Ab} acts transitively on V, so it acts transitively on $T[V]$. On the other hand, since $o(a_0/A)$ is covered by finitely many translates of V by elements of G_A, $o(a/A)$ is also covered by the translates of $T[V]$ by these same elements of G_A. Hence, $T[V]$ has a non-empty interior in $o(a/A)$, so it is open by the first line of the proof. The fact that $T[V]$ is closed follows from continuity of T and compactness of X. □

The Claim tells us that $T[V]$ is a clopen neighborhood of a inside $o(a/A)$. By (∗∗), we get that $T[V]$ has finitely many conjugates under G_a. ■

The next result shows that m-normality corresponds to the notion of 1-basedness.

Proposition 3.10 *(X, G) is m-normal iff for all finite $a, b \subseteq X$ [or $a, b \subseteq X^{eq}$, if (X, G) is a compact ei-structure], there exists $c \in acl^{eq}(a) \cap acl^{eq}(b)$ with $a \overset{m}{\perp}_c b$.*

Proof. ($\Longrightarrow$) Take any finite a, b. Then, we can find a set $U \ni a$ clopen in $o(a/b)$ and with finitely many conjugates under G_a. Let $a^+ = \ulcorner U \urcorner$. By Proposition 1.16, we get $a^+ \in acl^{eq}(a) \cap acl^{eq}(b)$. But, we can assume that $U = [a]_E$ for some finite, b-definable equivalence relation E on $o(a/b)$. Hence, we see that $U = o(a/a^+b) = o(a/a^+)$, so $a \overset{m}{\perp}_{a^+} b$.
($\Longleftarrow$) Take any finite a and A. By assumption, there is $c \in acl^{eq}(a) \cap acl^{eq}(A)$ such that $a \overset{m}{\perp}_c A$. So, $o(a/Ac)$ is clopen in both $o(a/c)$ and $o(a/A)$. Put $U = o(a/Ac)$. Then, U is a clopen neighborhood of a in $o(a/A)$. By Proposition 1.16, it has finitely many conjugates under G_c. Since $c \in acl^{eq}(a)$, U has finitely many conjugates over a. ■

The next remark (see [14, Remark 1.4]) follows from Proposition 2.11.

Remark 3.11 *For any finite $A \subseteq X^{eq}$, $acl^{eq}(A) = dcl^{eq}(A \cup acl^{eq}(\emptyset))$.*

Proof. ($\supseteq$) is obvious.
($\subseteq$). Take $[a]_E \in acl^{eq}(A)$. By Proposition 2.11, $Y := o([a]_E)$ is profinite. On the other hand, we know that $o([a]_E/A) = \{[a_1]_E, \dots, [a_n]_E\}$ is finite. So, by Remark 1.7 and Proposition 1.5, there is a finite, $\emptyset$-definable equivalence relation R on Y such that the $[a_i]_E$'s lie in different classes modulo R; denote these classes by $c_1, \dots, c_n$. Then, $[a]_E \in dcl^{eq}(Ac_1 \dots c_n)$ and $c_1, \dots, c_n \in acl^{eq}(\emptyset)$. ■

So far, we have discussed some basic results of Newelski which can be generalized to compact e-structures. Now, we turn to some deeper results.

Lemma 3.1 of [14] is true for compact ei-structures. Namely, we have

Proposition 3.12 *Let (X, G) be a compact ei-structure and o be an orbit over $\emptyset$. Then $acl^{eq}(a) \cap o$ is finite for every $a \in o$.*

The proof of Lemma 3.1 of [14] works here, except for one step that will be explained now. As in [14], we define Y_a as the topological closure of $acl^{eq}(a) \cap o$, and, using Proposition 2.11, we prove that $Y_a = Y_b$ for $b \in Y_a$. This means that Y_a is a-definable and any two elements of Y_a lie in the same orbit over $\ulcorner Y_a \urcorner$. Hence, by Proposition 2.10, there is an orbit over a which is open in Y_a. This orbit meets $acl^{eq}(a) \cap o$, so it is finite. Hence, there is an element $b \in Y_a$ which is isolated in Y_a. Since Y_a is an orbit over $\ulcorner Y_a \urcorner$, all points in Y_a are isolated, and hence Y_a is finite.

Examples 3 and 4 show that the above result is not true for compact [profinite] e-structures. Indeed, by Lemma 2.18(i), in both these examples $o(x, y) = o(x) \times o(y)$ for any $(x, y) \in X \times Y$. We also have that $o(y)$ is infinite. Hence, $o(x, y) \cap (\{x\} \times Y)$ is infinite. On the other hand, $\{x\} \times Y \subseteq dcl(x, y)$. So, we see that $acl(x, y) \cap o(x, y)$ is infinite.

Recall that $A = \{a, b, c\}$ is a dcl-triangle in (X, G) if for every $x \in A$, we have $x \in dcl(A \setminus \{x\}) \setminus acl(\emptyset)$, and every two elements of A are m-independent. Replacing dcl by acl, we get the definition of an acl-triangle.

Having the last two results and using Proposition 1.16 and 2.11 in the appropriate places, we can repeat the proof of Theorem 3.3 of [14]. So, we have the following group configuration theorem for compact ei-structures.

Theorem 3.13 *Assume $\{a, b, c\}$ is an acl-triangle in an m-normal compact ei-structure (X, G). Then there is a group H, which is open in $o(a)$, a is the neutral element of H, and H is definable over a and finitely many parameters from $acl^{eq}(\emptyset)$.*

If we want to have such a theorem in the wider class of compact [profinite] e-structures, we have to formulate it in a slightly weaker form. The reason is that in this situation we do not have Proposition 3.12.

Definition 3.14 *A compact structure (Y, H) is interpretable in (X, G) over a finite subset A of X if there is a closed subgroup H^* of H and A-definable subset Z of X^{eq} such that (Y, H^*) is isomorphic to $(Z, G_A/G_Z)$.*

Theorem 3.15 *Assume $\{a, b, c\}$ is an acl-triangle in an m-normal compact e-structure (X, G). Then there is a compact group (H, K) interpretable in (X, G) over finitely many parameters from $acl^{eq}(\emptyset)$.*

To prove this theorem, it is enough to apply Newelski's proof of [14, Theorem 3.3], using in the appropriate places Propositions 1.16, 2.11, 2.12 and the results discussed before Theorem 3.13 in this section. The group H obtained in Newelski's proof together with K defined as the group of automorphisms of H induced by the pointwise stabilizer in G of finitely many parameters from $acl^{eq}(\emptyset)$ over which H is defined do the job.

One of the main open problems about small profinite structures is the so-called $\mathcal{M}$-gap conjecture saying that for every orbit o, $\mathcal{M}(o) \in \omega \cup \{\infty\}$. This conjecture has been proved by Newelski for small, m-normal profinite structures [9, Theorem 1.4], and by Wagner for small, m-stable profinite groups [16, Theorem 18]. Below we prove it in the context of m-normal compact ei-structures. We give a proof because the proof of [9, Theorem 1.4] uses Consequence (d) of smallness formulated at the beginning of Section 3. In fact, we simplify the proof of [9, Theorem 1.4] a bit, eliminating the application of m-normality from the final part of this proof. Notice, however, that the final part of our proof does not work for compact e-structures.

Theorem 3.16 *Assume (X, G) is an m-normal compact ei-structure. Then there is no orbit o on X^{eq} such that $\omega \leq \mathcal{M}(o) < \infty$.*

Proof. Suppose for a contradiction that such an orbit exists. Then, $\mathcal{M}(a/A) = \omega$ for some $a \in X^{eq}$ and finite $A \subseteq X^{eq}$. Wlog $A = \emptyset$.

Consider a relation $\sim$ on $o(a)$ defined by

$$a_1 \sim a_2 \iff a_1 \overset{m}{\not\perp} a_2.$$

Claim 1 $\sim$ is an $\emptyset$-invariant equivalence relation.

Proof. We just repeat the proof of (a) from the proof of [9, Theorem 1.4]. Only transitivity requires an explanation. Assume $a_1 \sim a_2 \sim a_3$, i.e. $a_1 \overset{m}{\not\perp} a_2$ and $a_2 \overset{m}{\not\perp} a_3$. By Lascar inequalities, we get

$$\mathcal{M}(a_3/a_1) \leq \mathcal{M}(a_2a_3/a_1) \leq \mathcal{M}(a_3/a_2a_1) \oplus \mathcal{M}(a_2/a_1) \leq \mathcal{M}(a_3/a_2) \oplus \mathcal{M}(a_2/a_1) < \omega.$$

But $\mathcal{M}(a_3) = \omega$. Therefore, $a_3 \overset{m}{\not\perp} a_1$, i.e. $a_1 \sim a_3$. □

Claim 2 $\sim$ is $\emptyset$-definable, and thus $[a]_\sim$ is closed.

Proof. Consider any $a_1, a_2 \in o(a)$ with $a_1 \nsim a_2$. Then, $a_1 \overset{m}{\perp} a_2$. So, by the Kuratowski-Ulam theorem, $V := o(a_1a_2)$ is open in $o(a) \times o(a)$. Also, for any $(b_1, b_2) \in V$, $b_1 \nsim b_2$. Thus, $\sim$ is a closed subset of $o(a) \times o(a)$.

The fact that $[a]_\sim$ is closed can be also seen without the Kuratowski-Ulam theorem. Namely, suppose $b \in \overline{[a]_\sim} \setminus [a]_\sim$. Then, $b \overset{m}{\perp} a$, so $o(b/a)$ is open in $o(b) = o(a)$. Hence, $o(b/a) \cap [a]_\sim \neq \emptyset$. But for $c \in o(b/a) \cap [a]_\sim$, we get $c \overset{m}{\perp} a$ and $c \overset{m}{\not\perp} a$, a contradiction. □

Claim 3 For any $n \in \omega$, there is $b \in [a]_\sim$ such that $\mathcal{M}(b/a) > n$.

Proof. This time, we repeat the proof of (d) from the proof of [9, Theorem 1.4]. Choose a finite set B such that $a \overset{m}{\not\perp} B$ and $\mathcal{M}(a/B) > n$. Lemma 1.3 of [9] goes through for compact e-structures (this is the only place where m-normality is used), so there is $c \in o(a)$ such that $a \overset{m}{\perp}_c B$ and $a \overset{m}{\perp}_B c$. It follows that $\mathcal{M}(a/c) = \mathcal{M}(a/B) < \omega$, so $a \sim c$. Applying an authomorphim mapping c to a, we do not change $[a]_\sim$, and so a is mapped to some $b \in [a]_\sim$ such that $\mathcal{M}(b/a) > n$. □

Now, we will show that there is a bound on $\mathcal{M}(b/a)$ for $b \in [a]_\sim$, which gives a contradiction with Claim 3. Take any $b \in [a]_\sim$. Then, $\mathcal{M}(b/a) \leq \mathcal{M}(b/[a]_\sim)$. Since $[a]_\sim$ is a definable set which is an orbit over its name, $\mathcal{M}(b/[a]_\sim) = \mathcal{M}(a/[a]_\sim) < \omega$ (if the last inequality was false, then taking $c \in [a]_\sim$ with $c \overset{m}{\perp}_{[a]_\sim} a$, we would get that $\mathcal{M}(c/a) = \omega$, a contradiction). So, $\mathcal{M}(b/a)$ is bounded by $\mathcal{M}(a/[a]_\sim) < \omega$. ■

In [10, 11, 13, 14], Newelski considered *acl*-pregeometry on an orbit of $\mathcal{M}$-rank 1. To have some good properties (e.g. homogeneity) of this pregeometry, one needs to localize it at a flat Morley sequence. Since by Corollary 2.14 flat Morley sequences exist in every compact e-structure, we can also localize *acl* at flat Morley sequences in this context and easily check the basic properties of such pregeometries. Newelski introduced the notion of a full [weak] coordinatization and he proved [13, Theorem 3.3] that a small profinite structure of finite $\mathcal{M}$-rank is m-normal iff it has full [weak] coordinatization and each orbit of $\mathcal{M}$-rank 1 is locally modular. Analyzing Newelski's proof and modifying it appropriately, we can conclude that this equivalence is also true for compact ei-structures (in the proof, we use Theorem 3.16). It is not clear if this also holds for compact e-structures, because in some places in [13] forking calculus over non-algebraic imaginary elements was used. Anyway, as in [14], we get the following corollary of Theorem 3.13, which is another form of the group configuration theorem.

Corollary 3.17 *If (X, G) is a compact ei-structure with a non-trivial locally modular orbit o of $\mathcal{M}$-rank 1, then some open subset o' of o is a definable group.*

3.2 Regular orbits, domination and weight

In this subsection, we study counterparts of some model theoretic notions (which have not been considered yet, even in the context of small profinite structures) in our general context of compact e-structures. The main result of this subsection is Theorem 3.24 saying that each orbit in an m-stable compact e-structure is equidominant with a product of finitely many m-regular orbits.

From now on, **(X, G) is an m-stable compact e-structure**. Assuming that (X, G) is a compact ei-structure, everywhere below one can work in X^{eq}.

The general scheme is the same as in Sections 5.1 and 5.2 of [15]. However, caution has to be taken, because we do not have stationary (Lascar strong) types, independence theorem and canonical bases. It turns out that Proposition 3.7 will allow us to omit all such obstacles.

Recall that two orbits $o(a/A)$ and $o(b/B)$ are said to be m-orthogonal over a set C containing $A \cup B$ if for any $a' \in o(a/A)$ and $b' \in o(b/B)$ with $a' \overset{m}{\perp}_A C$ and $b' \overset{m}{\perp}_B C$ we have $a' \overset{m}{\perp}_C b'$. These orbits are said to be m-orthogonal if they are m-orthogonal over every finite set containing $A \cup B$. As in stable theories, we say that $o(a/A)$ is m-regular if it is m-orthogonal to all its m-dependent extensions.

It turns out that, as in stable theories, m-regular orbits are exactly the orbits on which m-dependence induces a pregeometry. More precisely, let $o(a/A)$ be any orbit.

For a finite $B \subseteq o(a/A)$, define $cl(B) := \{b \in o(a/A) : b \overset{m}{\not\perp}_A B\}$; if B is infinite, put $cl(B) = \bigcup\{cl(B_0) : B_0$ is a finite subset of $B\}$.

Proposition 3.18 *Assume that $a \notin acl^{eq}(A)$. Then the orbit $o(a/A)$ is m-regular iff $(o(a/A), cl)$ is a pregeometry.*

Proof. ($\Longrightarrow$) Only $cl(cl(B)) = cl(B)$ requires a proof; this can be shown by an easy forking calculus (exactly as in stable theories), without using m-stability.
($\Longleftarrow$) This is the place where Proposition 3.7 (and hence m-stability) plays an important role. Suppose for a contradiction that $o(a/A)$ is not m-regular, i.e. there are $a, b \in o(a/A)$ and $C \supseteq A$ with $a \overset{m}{\perp}_A C$, $b \overset{m}{\not\perp}_A C$ and $a \overset{m}{\not\perp}_C b$.

By Proposition 3.7, we can find a finite sequence $I \subseteq o(ab/C)$ over which $o(ab/C)$ is invariant and with $I \overset{m}{\perp}_C ab$. Hence, $o(ab/IC)$ is open in $o(ab/C)$ and $o(ab/I) \subseteq o(ab/C)$. So, $ab \overset{m}{\perp}_I C$.

A simple forking calculus yields: $a \overset{m}{\perp}_A I$, $b \overset{m}{\not\perp}_A I$ and $a \overset{m}{\not\perp}_{IA} b$. This means that $a \notin cl(I)$, $b \in cl(I)$ and $a \in cl(Ib)$, a contradiction. ■

The above proposition is also true for m-normal compact e-structures. The proof is similar to the above one, except that instead of Proposition 3.7 we use the fact that for each orbit $o(a/A)$ in any m-normal compact e-structure, we can find $b \in acl^{eq}(\emptyset)$ such that $o(a/Ab)$ is invariant over ab. To prove the last fact, apply the proof of [9, Remark 0.2] using Proposition 2.11.

The proof of [15, Proposition 5.1.11] works in our context, and so we get

Proposition 3.19 *Each non-algebraic orbit is not m-orthogonal to an m-regular orbit (of an element from X).*

Definition 3.20 *The weight, denoted by w, is the unique function from the collection of all orbits over finite sets to $\omega \cup \{\infty\}$ such that for every $n \in \omega$, we have $w(o(a/A)) \geq n$ iff there is a finite set $A' \supseteq A$, $a' \in o(a/A)$ with $a' \overset{m}{\perp}_A A'$, and a sequence $\langle a_0, \dots, a_{n-1}\rangle$ m-independent over A' and satisfying $a' \overset{m}{\not\perp}_{A'} a_i$ for every $i < n$.*

An easy forking calculation shows that every orbit in any m-stable compact e-structure has finite weight (a precise finite bound for the weight of an orbit is provided by [15, Theorem 5.2.5] whose proof works in our context). The following properties of weight can be proved as in stable or simple theories [15].

Proposition 3.21 *1. If $a \overset{m}{\perp}_A B$, then $w(a/A) = w(a/B)$.*

2. $w(ab/A) \leq w(a/A) + w(b/Aa)$.

3. If $a \overset{m}{\perp}_A b$, then $w(ab/A) = w(a/A) + w(b/A)$.

The next proposition is a counterpart of a well-known result for stable theories, but to prove it, we use Proposition 3.7 instead of canonical bases and Morley sequences.

Proposition 3.22 *Each m-regular orbit $o(a/A)$ has weight 1.*

Proof. Suppose for a contradiction that there are $A' \supseteq A$, $a' \in o(a/A)$ with $a' \overset{m}{\smile\!\!\!\!\!|}_A A'$, and elements b, c with $b \overset{m}{\smile\!\!\!\!\!|}_{A'} c$, $a' \overset{m}{\not\smile\!\!\!\!\!|}_{A'} b$ and $a' \overset{m}{\not\smile\!\!\!\!\!|}_{A'} c$. By Proposition 3.7, we can find a finite sequence $\langle b_i : i \leq n \rangle \subseteq o(a'/A'b)$ which is m-independent from c over $A'b$, and over which $o(a'/A'b)$ is invariant. So, there is $k \leq n$ such that $a' \overset{m}{\smile\!\!\!\!\!|}_{A'} b_{<k}$ and $a' \overset{m}{\not\smile\!\!\!\!\!|}_{A'b_{<k}} b_k$. The rest of the proof is the same as in the proof of Lemma 5.2.11(1) in [15]. By m-regularity of $o(a'/A') = o(b_k/A')$, we have $b_k \overset{m}{\smile\!\!\!\!\!|}_{A'} b_{<k}$. Since $c \overset{m}{\smile\!\!\!\!\!|}_{A'} b_{\leq k}$, we get $b_k \overset{m}{\smile\!\!\!\!\!|}_{A'} c b_{<k}$. By m-regularity of $o(b_k/A') = o(a'/A')$, we obtain $b_k \overset{m}{\smile\!\!\!\!\!|}_{A'cb_{<k}} a'$; this yields $b_k \overset{m}{\smile\!\!\!\!\!|}_{A'} a' c b_{<k}$, a contradiction. ■

We define the notions of domination exactly as in [15, Definition 5.2.6]. a and b below are elements (or finite tuples) and C is a finite set containing sets A and B.

Definition 3.23 *1. We say that a dominates b over A, written $a \gtrdot_A b$, if $b \overset{m}{\smile\!\!\!\!\!|}_A c$ for all $c \overset{m}{\smile\!\!\!\!\!|}_A a$. a and b are equidominant over A, denoted $a \doteq_A b$, if $a \gtrdot_A b$ and $b \gtrdot_A a$.*
2. We say that $o(a/A)$ is more dominant than $o(b/B)$ over C, in symbols $o(a/A) \gtrdot_C o(b/B)$, if there are $a' \in o(a/A)$ and $b' \in o(b/B)$ such that $a' \overset{m}{\smile\!\!\!\!\!|}_A C$, $b' \overset{m}{\smile\!\!\!\!\!|}_B C$ and $a' \gtrdot_C b'$. $o(a/A)$ is more dominant than $o(b/B)$, written $o(a/A) \gtrdot_0 o(b/B)$, means that $o(a/A) \gtrdot_C o(b/B)$ for some C.
3. We say that $o(a/A)$ is equidominant with $o(b/B)$ over C, in symbols $o(a/A) \doteq_C o(b/B)$, if there are $a' \in o(a/A)$ and $b' \in o(b/B)$ such that $a' \overset{m}{\smile\!\!\!\!\!|}_A C$, $b' \overset{m}{\smile\!\!\!\!\!|}_B C$ and $a' \doteq_C b'$. Finally, $o(a/A)$ and $o(b/B)$ are equidominant, written $o(a/A) \doteq_0 o(b/B)$, if $o(a/A) \doteq_C o(b/B)$ for some C.

Equidominanation over a fixed set is an equivalence relation on elements, whereas equidomination may not be an equivalence relation on orbits (it is easy to find an example of an infinite orbit with two m-independent extensions which are m-orthogonal).

We have all the basic properties of domination, e.g.: if $w(o_1) = 1$ and o_1 is not m-orthogonal to o_2 over a set A (containing the domains of o_1 and o_2), then $o_2 \gtrdot_A o_1$. If additionally $w(o_2) = 1$, we get $o_2 \doteq_A o_1$. In particular, by Proposition 3.22, if an m-regular orbit is not m-orthogonal over A to an orbit of weight 1, then these orbits are equidominant over A (which together with Proposition 3.19 is used in the proof of the next theorem).

Having all these results, we can repeat the proof of Theorem 5.2.18 of [15] to get the main theorem of this subsection.

Theorem 3.24 *Each orbit in an m-stable compact e-structure is equidominant with a product of finitely many m-regular (hence of weight 1) orbits.*

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