RAMSEY THEORY AND TOPOLOGICAL DYNAMICS FOR FIRST ORDER THEORIES

KRZYSZTOF KRUPIŃSKI, JUNGUK LEE, AND SLAVKO MOCONJA

Abstract. We investigate interactions between Ramsey theory, topological dynamics, and model theory. We introduce various Ramsey-like properties for first order theories and characterize them in terms of the appropriate dynamical properties of the theories in question (such as [extreme] amenability of a theory or some properties of the associated Ellis semigroups). Then we relate them to profiniteness and triviality of the Ellis groups of first order theories. In particular, we find various criteria for [pro]finiteness and for triviality of the Ellis group of a given theory from which we obtain wide classes of examples of theories with [pro]finite or trivial Ellis groups. We also find several concrete examples illustrating the lack of implications between some fundamental properties. In the appendix, we give a full computation of the Ellis group of the theory of the random hypergraph with one binary and one 4-ary relation. This example shows that the assumption of NIP in the version of Newelski's conjecture for amenable theories (proved in [16]) cannot be dropped.

1. Introduction

In their seminal paper [15], Kechris, Pestov and Todorčević discovered surprising interactions between dynamical properties of the group of automorphisms of a Fraïssé structure and Ramsey-theoretic properties of its age. For example, they proved that this group is extremely amenable iff the age has the structural Ramsey property and consists of rigid structures (equivalently, the age has the embedding Ramsey property in the terminology used by Zucker in [35]). This started a wide area of research of similar phenomena. Recently, Pillay and the first author [18] gave a model-theoretic account for the fundamental results of Kechris-Pestov-Todorčević (shortly KPT) theory, generalizing the context to arbitrary, possibly uncountable, structures. However, KPT theory (including such generalizations) is not really about model-theoretic properties of the underlying theory, because: on the dynamical side, it talks about the topological dynamics of the topological group of automorphisms of a given structure, which can be expressed in terms of the action of this group on the universal ambit rather than on type spaces of the underlying theory, and, on the Ramsey-theoretic side, it considers arbitrary colorings (without any definability properties) of the finite subtuples of a given model. Definitions of Ramsey properties for a

2010 *Mathematics Subject Classification.* 03C45, 05D10, 54H20, 54H11, 20E18.

Key words and phrases. Ramsey property, Ramsey degree, Ellis group, [extremely] amenable theory, profinite group.

All authors are supported by National Science Center, Poland, grant 2016/22/E/ST1/00450. The first author is also supported by National Science Center, Poland, grant 2018/31/B/ST1/00357. The third author is also supported by the Ministry of Education, Science and Technological Development of Serbia grant no. ON174018.

given structure stated in [18] suggest the corresponding definitions for first order theories just by applying them to a monster model. In this paper, we go much further and define various "definable" versions of Ramsey properties for first order theories by restricting the class of colorings to "definable" ones. And then we find the appropriate dynamical characterizations of our "definable" Ramsey properties in terms of the dynamics of the underlying theory (in place of the dynamics of the group of automorphisms of a given model) some of which are surprising and different comparing to classical KPT theory.

The classes of amenable and extremely amenable theories introduced and studied in [13] are defined in a different way than typical Shelah-style, combinatorially defined classes of theories (such as NIP, simple, NTP_2). In this paper, we give Ramsey-theoretic characterizations of [extremely] amenable theories; these characterizations are clearly combinatorial, but still of different flavor than Shelah's definitions. Also, the new classes of theories introduced in this paper via some Ramsey-theoretic properties or via their dynamical characterizations do not follow the usual Shelah-style way of defining new classes of theories. This makes the whole topic rather novel in model theory.

We find the interaction between "definable" Ramsey properties and the dynamics of first order theories natural and interesting in its own right. However, our original motivation to introduce the "definable" Ramsey properties has some specific origins in model theory and topological dynamics in model theory, which we explain in the next paragraph.

Some methods of topological dynamics were introduced to model theory by Newelski in [25]. Since then a wide research on this topic has been done by Chernikov, Hrushovski, Newelski, Pillay, Rzepecki, Simon, the first author, and others. For any given theory T, a particularly important place in this research is reserved for the investigation of the flow $(\mathrm{Aut}(\mathfrak{C}), S_{\bar{c}}(\mathfrak{C}))$, where $\bar{c}$ is an enumeration of a monster model $\mathfrak{C} \models T$ and $S_{\bar{c}}(\mathfrak{C})$ is the space of global types extending $\mathrm{tp}(\bar{c}/\emptyset)$, as it turns out that topological properties of this flow carry important information about the underlying theory. In particular, in [19] it was proved that there exists a topological quotient epimorphism from the Ellis group of the flow $(\mathrm{Aut}(\mathfrak{C}), S_{\bar{c}}(\mathfrak{C}))$ (also called the Ellis group of T, as it does not depend on the choice of the monster model by [16]) to $\mathrm{Gal}_{\mathrm{KP}}(T)$ (the Kim-Pillay Galois group of T), and even to the larger group $\mathrm{Gal}_{\mathrm{L}}(T)$ (the Lascar Galois group of T); in particular, the Ellis group of T captures more information about T than the Galois groups of T. This was the starting point for the research in this paper. Namely, from the aforementioned result from [19] one easily deduces that profiniteness of the Ellis group implies profiniteness of $\mathrm{Gal}_{\mathrm{KP}}(T)$, which in turn is known to be equivalent to the equality of the Shelah and Kim-Pillay strong types. The question for which theories the Shelah and Kim-Pillay strong types coincide is fundamental in model theory. This is known to be true in e.g. stable or supersimple theories, but remains a well-known open question in simple theories in general. This led us to the question for which theories the Ellis group is profinite, which is also interesting in its own rights (keeping in mind that the Ellis group of T captures more information than any of the Galois groups of T). And among the main outcomes of this paper are results saying that various Ramsey-like properties of T imply profiniteness of the Ellis group.

Let us briefly discuss the Ramsey properties which we investigate in this paper. They are given with respect to a monster model $\mathfrak{C}$ of a first-order theory T, but we will show that they do not depend on the choice of $\mathfrak{C}$, so they are really properties of T. We say that T has *separately finite elementary embedding Ramsey degree (sep. fin. EERdeg)* if for every finite $\bar{a} \subseteq \mathfrak{C}$ there exists $l < \omega$ such that for every finite $\bar{b} \supseteq \bar{a}$, $r < \omega$, and coloring $c : \binom{\mathfrak{C}}{\bar{a}} \to r$ there exists $\bar{b}' \in \binom{\mathfrak{C}}{\bar{b}}$ such that $\#c[\binom{\bar{b}'}{\bar{a}}] \leqslant l$. Here, for a tuple $\bar{a}$ and a set B (or a tuple which is treated as the set of coordinates), $\binom{B}{\bar{a}}$ denotes the set of all $\bar{a}' \subseteq B$ such that $\bar{a}' \equiv \bar{a}$. If l above can be taken to be 1 for every finite $\bar{a}$, we say that T has the *elementary embedding Ramsey property (EERP)*. If l can be taken to be 1 and we restrict ourselves to considering only [externally] definable colorings (see Section 4 for definitions), we say that T has the *[externally] definable elementary embedding Ramsey property ([E]DEERP)*. If for every finite set of formulae Δ and every finite $\bar{a}$ the above holds (for some l) for the Δ-externally definable colorings, then we say that T has *separately finite externally definable elementary embedding Ramsey degree (sep. fin. EDEERdeg)*.

Theories with *EERP* and sep. fin. *EERdeg* are generalizations of the classical notions of embedding Ramsey property and finite embedding Ramsey degree in the following sense: If K is an $\aleph_0$-saturated locally finite Fraïssé structure, then its age has the embedding Ramsey property [sep. fin. embedding Ramsey degree] iff $\mathrm{Th}(K)$ has *EERP* [sep. fin. *EERdeg*].

We also consider the following convex Ramsey-like properties. We say that that T has the *elementary embedding convex Ramsey property (EECRP)* if for every $\epsilon \geqslant 0$ and finite $\bar{a} \subseteq \bar{b} \subseteq \mathfrak{C}$, $n < \omega$, and coloring $c : \binom{\mathfrak{C}}{\bar{a}} \to 2^n$ there exist $k < \omega$, $\lambda_0, \dots, \lambda_{k-1} \in [0, 1]$ with $\lambda_0 + \dots + \lambda_{k-1} = 1$, and $\sigma_0, \dots, \sigma_{k-1} \in \mathrm{Aut}(\mathfrak{C})$ such that for any two tuples $\bar{a}', \bar{a}'' \in \binom{\bar{b}}{\bar{a}}$ the convex combinations $\sum_{j<k} \lambda_j c(\sigma_j(\bar{a}'))(i)$ and $\sum_{j<k} \lambda_j c(\sigma_j(\bar{a}''))(i)$ differ by at most ϵ for every $i < n$. If we restrict ourselves to definable colorings, we say that T has the *definable elementary embedding convex Ramsey property (DEECRP)*.

To state our main results, we need to use a natural refinement of the usual space of Δ-types, denoted by $S_{\bar{c},\Delta}(\bar{p})$ for a finite set of formulae $\Delta = \{\varphi_0(\bar{x}, \bar{y}), \dots, \varphi_{k-1}(\bar{x}, \bar{y})\}$ and a finite set (or sequence) of types $\bar{p} = \{p_0(\bar{y}), \dots, p_{m-1}(\bar{y})\} \subseteq S_{\bar{y}}(\emptyset)$. (It is defined after Lemma 2.18.) For a flow (G, X), by $EL(X)$ we denote the Ellis semigroup of this flow. By $\mathrm{Inv}_{\bar{c}}(\mathfrak{C})$, we denote the space of global invariant types extending $\mathrm{tp}(\bar{c}/\emptyset)$. (All these notations and definitions can be found in Section 2.) Our main result yields dynamical characterizations of the introduced Ramsey properties.

Theorem 1. *Let T be a complete first-order theory and $\mathfrak{C}$ its monster model. Then:*

(i) T has DEERP iff T is extremely amenable (in the sense of [13]).
(ii) T has EDEERP iff there exists $\eta \in \mathrm{EL}(S_{\bar{c}}(\mathfrak{C}))$ such that $\mathrm{Im}(\eta) \subseteq \mathrm{Inv}_{\bar{c}}(\mathfrak{C})$.
(iii) T has sep. fin. EDEERdeg iff for every finite set of formulae Δ and finite sequence of types $\bar{p}$ there exists $\eta \in \mathrm{EL}(S_{\bar{c},\Delta}(\bar{p}))$ such that $\mathrm{Im}(\eta)$ is finite.
(iv) T has DEECRP iff T is amenable (in the sense of [13]).

How is it related to the Ellis group of the theory? The answer is given by the next corollary.

Corollary 2. *(i) Each theory with EDEERP has trivial Ellis group (see Corollary 4.16).*
(ii) Each theory with sep. fin. EDEERdeg has profinite Ellis group (see Corollary 5.1).

Item (i) is an easy consequence of Theorem 1(ii). Item (ii) follows from Theorem 1(iii) and the implication (D) $\implies$ (A) in Theorem 3 below.

In the next theorem, $\mathcal{M}$ denotes a minimal left ideal in $\mathrm{EL}(S_{\bar{c}}(\mathfrak{C}))$ and u an idempotent in this ideal, so $u\mathcal{M}$ is the Ellis group of T; $u\mathcal{M}/H(u\mathcal{M})$ is the canonical Hausdorff quotient of $u\mathcal{M}$ (see Section 2). Analogously, $u_{\Delta,\bar{p}}\mathcal{M}_{\Delta,\bar{p}}$ is the Ellis group of the flow $(\mathrm{Aut}(\mathfrak{C}), S_{\bar{c},\Delta}(\bar{p}))$. The main idea behind the next result is that a natural way to obtain that the Ellis group of T is profinite is to present the flow $S_{\bar{c}}(\mathfrak{C})$ as the inverse limit of some flows each of which has finite Ellis group, and if it works, it should also work for the standard presentation of $S_{\bar{c}}(\mathfrak{C})$ as the inverse limit of the flows $S_{\bar{c},\Delta}(\bar{p})$ (where Δ and $\bar{p}$ vary).

Theorem 3. *Consider the following conditions:*

(A") $\mathrm{Gal}_{\mathrm{KP}}(T)$ *is profinite;*

(A') $u\mathcal{M}/H(u\mathcal{M})$ *is profinite;*

(A) $u\mathcal{M}$ *is profinite;*

(B) The $\mathrm{Aut}(\mathfrak{C})$*-flow* $S_{\bar{c}}(\mathfrak{C})$ *is isomorphic to the inverse limit* $\varprojlim_{i\in I} X_i$ *of some* $\mathrm{Aut}(\mathfrak{C})$*-flows* X_i *each of which has finite Ellis group;*

(C) for every finite sets of formulae Δ *and types* $\bar{p} \subseteq S(\emptyset)$*,* $u_{\Delta,\bar{p}}\mathcal{M}_{\Delta,\bar{p}}$ *is finite;*

(D) for every finite sets of formulae Δ *and types* $\bar{p} \subseteq S(\emptyset)$*, there exists* $\eta \in \mathrm{EL}(S_{\bar{c},\Delta}(\bar{p}))$ *with* $\mathrm{Im}(\eta)$ *finite.*

Then (D) $\implies$ *(C)* $\iff$ *(B)* $\implies$ *(A)* $\implies$ *(A')* $\implies$ *(A").*

We also find several other criteria for [pro]finiteness of the Ellis group. Applying Corollary 2 or our other criteria together with some well-known theorems from structural Ramsey theory (saying that various Fraïssé classes have the appropriate Ramsey properties), we get wide classes of examples of theories with [pro]finite or sometimes even trivial Ellis groups. But we also find some specific examples illustrating interesting phenomena, e.g. we give examples showing that in Theorem 3: (A") does not imply (A'), and (A') does not imply (B). The example showing that (A") does not imply (A') is supersimple of SU-rank 1, so it shows that even for supersimple theories the Ellis group of the theory need not be profinite. We have not found examples showing that (C) does not imply (D), and (A) does not imply (B), which we leave as open problems.

In the appendix, we give a precise computation of the Ellis group of the theory of the random hypergraph with one binary and one 4-ary relation. This group turns out to be the cyclic two-element group. This example is interesting for various reasons. Firstly, by classical KPT theory, we know that it has sep. finite *EERdeg*, so the Ellis group is profinite by the above results (in fact, it satisfies the assumptions of some other criteria that we found, which implies that the Ellis group is finite), and the example shows that it may be non-trivial. A variation of this example (see Example 6.9) yields an infinite Ellis group, which shows that in some of our criteria for profiniteness, we cannot expect to get finiteness of the Ellis group. Finally, this example is easily seen to be extremely amenable in the sense of [13], so its KP-Galois group is trivial. But the Ellis group is non-trivial. Hence, the epimorphism (found in [19]) from the Ellis group to the KP-Galois group is not an isomorphism. On the other hand, by [16, Theorem 0.7], we know that under NIP, even amenability of the theory is sufficient for this epimorphism to be an isomorphism. So our example shows that one cannot drop the NIP assumption in [16, Theorem 0.7], which was not known so far.

Using our observations that both properties $EERP$ and $EECRP$ do not depend on the choice of the monster model, or even an $\aleph_0$-saturated and strongly $\aleph_0$-homogeneous model $M \models T$, and the results from [18] saying that $EERP$ (defined in terms of M) is equivalent to extreme amenability of the topological group $\mathrm{Aut}(M)$, and $EECRP$ (defined in terms of M) is equivalent to amenability of $\mathrm{Aut}(M)$, we get the following corollary.

Corollary 4. *Let T be a complete first-order theory. The group $\mathrm{Aut}(M)$ is [extremely] amenable as a topological group for some $\aleph_0$-saturated and strongly $\aleph_0$-homogeneous model $M \models T$ iff it is [extremely] amenable as a topological group for all $\aleph_0$-saturated and strongly $\aleph_0$-homogeneous models $M \models T$.*

This means that [extreme] amenability of the group of automorphisms of an $\aleph_0$-saturated and strongly $\aleph_0$-homogeneous structure is actually a property of its theory, which seems to be a new observation.

The paper is organized as follows. In Section 2, we recall or introduce all the notions from model theory, topological dynamics and classical structural Ramsey theory that we need throughout this paper. Furthermore, we point out and prove several fundamental and useful observations. In Section 3, we recall a result from [19] which guarantees profiniteness of $\mathrm{Gal}_{\mathrm{KP}}(T)$ provided that the Ellis group (or just its canonical Hausdorff quotient) is profinite, and we further investigate conditions for profiniteness of the Ellis groups in a general setting. Section 4 is the central part of the paper. We introduce and characterize all the aforementioned Ramsey properties for first order theories. We prove Theorem 1 and Corollary 4. In Section 5, we prove Theorem 3 and find some other conditions which imply [pro]finiteness of the Ellis group of the theory. In Section 6, we give a long list of examples to which our results apply, and find several examples with some specific properties, e.g. the aforementioned examples showing the lack of two of the implications between the items of Theorem 3. In the appendix, we give a complete computation of the Ellis group of the theory of the random hypergraph with one binary and one 4-ary symbol.

Some "definable" versions of Ramsey properties were also introduced and considered in a recent paper by Nguyen Van Thé [28]; also, Ehud Hrushovski has very recently written an interesting paper [12], where he introduces some version of Ramsey properties in a first-order setting. But all these notions seem to be different and they are introduced for different reasons. It would be interesting to see in the future if there are any relationships.

2. Preliminaries and fundamental observations

Most of this section consists of definitions, notations and facts needed in this paper. But there are also some new ingredients, especially in Subsection 2.4, where we obtain some new reductions and introduce the type spaces $S_{\Delta,\bar{c}}(\bar{p})$ playing a key role in this paper.

2.1. **Model theory.**

We use standard model-theoretic concepts and terminology. By a theory we always mean a complete first-order theory T in a first-order language L. For simplicity, we will be assuming that L is one-sorted, but the whole theory developed in this paper works almost the same for many-sorted languages. We usually work in a monster model $\mathfrak{C}$ of T, i.e. a κ-saturated and strongly κ-homogeneous model of T for a large enough cardinal κ (called the degree of saturation of $\mathfrak{C}$). Elements of $\mathfrak{C}$ are denoted by $a, b, \dots$ and tuples (finite or infinite) of $\mathfrak{C}$ are denoted by $\bar{a}, \bar{b}, \dots$. By a small set [model] we mean a subset [elementary submodel] of $\mathfrak{C}$ of cardinality less than κ; small subsets of $\mathfrak{C}$ are denoted by $A, B, \dots$, and small submodels by $M, N, \dots$.

By an L-formula we mean any formula in the language L; by an $L(A)$-formula (where A is not necessarily small) we mean a formula with parameters from A. For a formula $\varphi(\bar{x})$, by $\varphi(\mathfrak{C})$ we denote the set of its solutions (or realizations) in $\mathfrak{C}$. A set is definable over A if it is the set of solutions of some $L(A)$-formula. A type over a (not necessarily small) set A is any finitely satisfiable set $\pi(\bar{x})$ of $L(A)$-formulae with free variables $\bar{x}$. A complete type over A is a maximal finitely satisfiable set $p(\bar{x})$ of $L(A)$-formulae with free variables $\bar{x}$. Global types are complete types over $\mathfrak{C}$. For a tuple $\bar{a}$ we write $\mathrm{tp}(\bar{a}/A)$ for the complete type over A realized by $\bar{a}$; $\mathrm{tp}(\bar{a})$ denotes $\mathrm{tp}(\bar{a}/\emptyset)$. By $S_{\bar{x}}(A)$ we denote the space of all complete types over A in variables $\bar{x}$; if $A = \emptyset$, we also write $S_{\bar{x}}(T)$ for $S_{\bar{x}}(\emptyset)$. For a type $\pi(\bar{x})$ over some $B \subseteq A$, $S_\pi(A)$ denotes the subspace of $S_{\bar{x}}(A)$ consisting of all types extending $\pi(\bar{x})$. For a tuple $\bar{a}$, $S_{\bar{a}}(A)$ denotes the space of all complete types over A extending $\mathrm{tp}(\bar{a})$; in other words, $S_{\bar{a}}(A) = S_{\mathrm{tp}(\bar{a})}(A)$. These spaces are naturally compact, Hausdorff, 0-dimensional topological spaces. For tuples $\bar{a}, \bar{b}$, $\bar{a} \equiv \bar{b}$ means that $\bar{a}$ and $\bar{b}$ have the same type over $\emptyset$. A set is type-definable over A if it is the set of realizations of some (not necessarily complete) type over A.

$\mathrm{Aut}(\mathfrak{C})$ and $\mathrm{Aut}(\mathfrak{C}/A)$ denote respectively the group of all automorphism of $\mathfrak{C}$ and the group of all automorphisms of $\mathfrak{C}$ fixing A pointwise. A subset of a power of $\mathfrak{C}$ is invariant [A-invariant] if it is invariant under $\mathrm{Aut}(\mathfrak{C})$ [$\mathrm{Aut}(\mathfrak{C}/A)$]. Having the same type over $\emptyset$ [small A] is the equivalence relation of lying in the same orbit of $\mathrm{Aut}(\mathfrak{C})$ [$\mathrm{Aut}(\mathfrak{C}/A)$] on the appropriate power of $\mathfrak{C}$. By $\equiv_{\mathrm{Sh}}$ on a fixed power of $\mathfrak{C}$ we denote the intersection of all $\emptyset$-definable finite equivalence relations (i.e. with finitely many classes) on this power; the classes of $\equiv_{\mathrm{Sh}}$ are called Shelah strong types. By $\mathrm{Autf}_{\mathrm{Sh}}(\mathfrak{C})$ we denote the group of all Shelah strong automorphisms of $\mathfrak{C}$, i.e. the group of all automorphisms of $\mathfrak{C}$ fixing all Shelah strong types. An equivalence relation is bounded if it has less than κ-classes. $\equiv_{\mathrm{KP}}$ and $\equiv_{\mathrm{L}}$ are respectively the finest bounded $\emptyset$-type-definable equivalence relation and the finest bounded $\emptyset$-invariant equivalence relation (on a fixed power of $\mathfrak{C}$); the classes of $\equiv_{\mathrm{KP}}$ and $\equiv_{\mathrm{L}}$ are called Kim-Pillay strong types and Lascar strong types, respectively. By $\mathrm{Autf}_{\mathrm{KP}}(\mathfrak{C})$ and $\mathrm{Autf}_{\mathrm{L}}(\mathfrak{C})$ we denote respectively the group of all Kim-Pillay strong automorphism and the group of all Lascar strong automorphisms, i.e. the groups of automorphisms of $\mathfrak{C}$ fixing all $\equiv_{\mathrm{KP}}$-classes and all $\equiv_{\mathrm{L}}$-classes, respectively. It turns out that $\equiv_{\mathrm{Sh}}$, $\equiv_{\mathrm{KP}}$, and $\equiv_{\mathrm{L}}$ are the orbit equivalence relations of $\mathrm{Autf}_{\mathrm{Sh}}(\mathfrak{C})$, $\mathrm{Autf}_{\mathrm{KP}}(\mathfrak{C})$, and $\mathrm{Autf}_{\mathrm{L}}(\mathfrak{C})$, respectively.

$\mathrm{Autf}_{\mathrm{Sh}}(\mathfrak{C})$, $\mathrm{Autf}_{\mathrm{KP}}(\mathfrak{C})$, $\mathrm{Autf}_{\mathrm{L}}(\mathfrak{C})$ are normal subgroups of $\mathrm{Aut}(\mathfrak{C})$, and the corresponding quotients do not depend on the choice of the monster $\mathfrak{C}$ and are called respectively the Shelah Galois group, the Kim-Pillay Galois group, and the Lascar Galois group of T; we denote them by $\mathrm{Gal}_{\mathrm{Sh}}(T)$, $\mathrm{Gal}_{\mathrm{KP}}(T)$, and $\mathrm{Gal}_{\mathrm{L}}(T)$, respectively. Since $\mathrm{Autf}_{\mathrm{L}}(\mathfrak{C}) \leqslant \mathrm{Autf}_{\mathrm{KP}}(\mathfrak{C}) \leqslant \mathrm{Autf}_{\mathrm{Sh}}(\mathfrak{C})$, we have natural epimorphisms $\mathrm{Gal}_{\mathrm{L}}(T) \to \mathrm{Gal}_{\mathrm{KP}}(T) \to \mathrm{Gal}_{\mathrm{Sh}}(T)$.

All the above Galois groups of a theory are topological groups. The topology on $\mathrm{Gal}_{\mathrm{L}}(T)$ is defined as follows. Let M be a small model and let $\bar{m}$ be an enumeration of M. The natural projection $\mathrm{Aut}(\mathfrak{C}) \to \mathrm{Gal}_{\mathrm{L}}(T)$ factors through $S_{\bar{m}}(M)$: $\mathrm{Aut}(\mathfrak{C}) \to S_{\bar{m}}(M) \to \mathrm{Gal}_{\mathrm{L}}(T)$, and we equip $\mathrm{Gal}_{\mathrm{L}}(T)$ with the quotient topology induced by $S_{\bar{m}}(M) \to \mathrm{Gal}_{\mathrm{L}}(T)$. This does not depend on the choice of the model M. It turns out that $\mathrm{Gal}_{\mathrm{L}}(T)$ is a compact (but not necessarily Hausdorff) topological group. The topologies on $\mathrm{Gal}_{\mathrm{Sh}}(T)$ and $\mathrm{Gal}_{\mathrm{KP}}(T)$ are defined in similar fashion. $\mathrm{Gal}_{\mathrm{Sh}}(T)$ is a compact, Hausdorff and 0-dimensional topological group (i.e. a profinite group), while $\mathrm{Gal}_{\mathrm{KP}}(T)$ is a compact, Hausdorff group. The epimorphisms $\mathrm{Gal}_{\mathrm{L}}(T) \to \mathrm{Gal}_{\mathrm{KP}}(T) \to \mathrm{Gal}_{\mathrm{Sh}}(T)$ are topological quotient maps.

Furthermore, $\mathrm{Gal}_{\mathrm{Sh}}(T)$ is the largest profinite quotient of $\mathrm{Gal}_{\mathrm{KP}}(T)$. Thus, $\mathrm{Gal}_{\mathrm{KP}}(T)$ is profinite iff it equals $\mathrm{Gal}_{\mathrm{Sh}}(T)$, and the last condition is clearly equivalent to saying that $\equiv_{\mathrm{KP}}$ and $\equiv_{\mathrm{Sh}}$ are equal on all powers of $\mathfrak{C}$, i.e. the Kim-Pillay and Shelah strong types coincide. The general question when $\mathrm{Gal}_{\mathrm{KP}}(T)$ is profinite was an initial motivation behind this paper.

For more details concerning strong types and Galois groups the reader is referred to [4], [34], or [29, Chapter 2.5].

2.2. **Topological dynamics.** We quickly introduce and state some facts from topological dynamics; we also provide some proofs. As a general reference we can recommend [1] and [8].

By a *G-flow* we mean a pair (G, X) where G is a topological group acting continuously on a compact Hausdorff space X. The *Ellis semigroup* of a G-flow (G, X) is the closure of the set $\{\pi_g \mid g \in G\}$ in X^X (equipped with the topology of pointwise convergence), where π_g is the function given by $x \mapsto gx$, with composition as the semigroup operation; this semigroup operation is continuous in the left coordinate. We denote the Ellis semigroup of (G, X) by $\mathrm{EL}(X)$. The Ellis semigroup of (G, X) itself is a G-flow, where the action is defined by $g\eta = \pi_g \circ \eta$ for $g \in G$ and $\eta \in \mathrm{EL}(X)$. By abusing notation, we denote π_g simply by g, treat G as a subset of $\mathrm{EL}(X)$ (although this "inclusion" is not necessarily 1-1), and then $\mathrm{EL}(X) = \mathrm{cl}(G)$. The minimal G-subflows of $\mathrm{EL}(X)$ coincide with the minimal left ideals of $\mathrm{EL}(X)$. If $\mathcal{M}$ is any minimal left ideal of $\mathrm{EL}(X)$, then the set of $\mathcal{J}(\mathcal{M})$ of all idempotents in $\mathcal{M}$ is non-empty. Furthermore, $\mathcal{M}$ is a disjoint union of subsets $u\mathcal{M}$ for $u \in \mathcal{J}(\mathcal{M})$. For each $u \in \mathcal{J}(\mathcal{M})$, $u\mathcal{M}$ is a group with respect to the composition of functions (a subgroup of $\mathrm{EL}(X)$) with neutral u. Moreover, the isomorphism type of this group does not depend on the choice of $\mathcal{M}$ and $u \in \mathcal{J}(\mathcal{M})$, and it is called the *Ellis group* of the flow (G, X); abusing terminology, any $u\mathcal{M}$ is also called an (or the) Ellis group of (G, X).

The existence of an element in the Ellis semigroup with finite image will be one of the key properties in this paper. The fundamental observation in this situation is given by the next fact, which follows from Lemmas 4.2 and 4.3 in [16], but we give a proof.

Fact 2.1. *If there exists $\eta \in \mathrm{EL}(X)$ with $\mathrm{Im}(\eta)$ finite, then the Ellis group is also finite.*

Proof. Let $\eta \in \mathrm{EL}(X)$ be an element with $\mathrm{Im}(\eta)$ finite. Let $\mathcal{M}$ be a minimal left ideal in $\mathrm{EL}(X)$ and $u \in \mathcal{J}(\mathcal{M})$. By considering $u\eta u \in u\mathcal{M}$, we may assume that $\eta \in u\mathcal{M}$: indeed, $\mathrm{Im}(u\eta u) \subseteq u[\mathrm{Im}(\eta)]$, so $\mathrm{Im}(u\eta u)$ is finite, as $\mathrm{Im}(\eta)$ is finite.

Further, note that for every $\tau \in u\mathcal{M}$ we have $\mathrm{Im}(\tau) = \mathrm{Im}(u)$. This follows since in $u\mathcal{M}$ we have $\tau = u\tau$ and $u = \tau\tau^{-1}$. So we get that $\mathrm{Im}(u) = \mathrm{Im}(\eta)$ is finite.

Consider the mapping $u\mathcal{M} \to \mathrm{Sym}(\mathrm{Im}(u))$ given by the restriction: $\tau \mapsto \tau_{\restriction \mathrm{Im}(u)}$. To see that $\tau_{\restriction \mathrm{Im}(u)}$ indeed belongs to $\mathrm{Sym}(\mathrm{Im}(u))$, note that for any $\tau \in u\mathcal{M}$, $\mathrm{id}_{\mathrm{Im}(u)} = u_{\restriction \mathrm{Im}(u)} = \tau_{\restriction \mathrm{Im}(u)} \circ \tau^{-1}{}_{\restriction \mathrm{Im}(u)} = \tau^{-1}{}_{\restriction \mathrm{Im}(u)} \circ \tau_{\restriction \mathrm{Im}(u)}$. Moreover, our mapping is injective as $\tau_{1\restriction \mathrm{Im}(u)} = \tau_{2\restriction \mathrm{Im}(u)}$ iff $\tau_1 u = \tau_2 u$ iff $\tau_1 = \tau_2$. Therefore, $u\mathcal{M}$ is finite. □

On an Ellis group $u\mathcal{M}$ we have a topology inherited from $\mathrm{EL}(X)$. Besides this topology, a coarser, so-called *τ-topology* is defined. First, for $a \in \mathrm{EL}(X)$ and $B \subseteq \mathrm{EL}(X)$ we define $a \circ B$ to be the set of all limits of the nets $(g_i b_i)_i$ such that $g_i \in G$, $b_i \in B$ and $\lim_i g_i = a$. For $B \subseteq u\mathcal{M}$ we define $\mathrm{cl}_\tau(B) = u\mathcal{M} \cap (u \circ B)$. cl_τ is a closure operator on $u\mathcal{M}$; the *τ-topology* is a topology on $u\mathcal{M}$ induced by cl_τ. $u\mathcal{M}$ with the τ-topology is a compact, T_1 semitopological group (i.e. group operation is separately continuous). The isomorphism types of the Ellis groups (for all $\mathcal{M}$ and $u \in \mathcal{J}(M)$) as semitopological groups do not depend on the choice of u and $\mathcal{M}$. Put $H(u\mathcal{M}) = \bigcap_U \mathrm{cl}_\tau(U)$, where the intersection is taken over all τ-open neighbourhoods of u in $u\mathcal{M}$. This is a τ-closed normal subgroup of $u\mathcal{M}$, and the quotient $u\mathcal{M}/H(u\mathcal{M})$ is a compact, Hausdorff topological group. Moreover, $H(u\mathcal{M})$ is the smallest τ-closed normal subgroup of $u\mathcal{M}$ such that $u\mathcal{M}/H(u\mathcal{M})$ is Hausdorff. $u\mathcal{M}/H(u\mathcal{M})$ will be called the *canonical Hausdorff quotient* of $u\mathcal{M}$.

A mapping $\Phi : X \to Y$ between two G-flows (G, X) and (G, Y) is a *G-flow homomorphism* if it is continuous and for every $g \in G$ and $x \in X$ we have $\Phi(gx) = g\Phi(x)$. A surjective [bijective] G-flow homomorphism is a *G-flow epimorphism* [*G-flow isomorphism*]; note that a G-flow epimorphism is necessarily a topological quotient map, and an inverse of a G-flow isomorphism is necessarily a G-flow isomorphism itself.

Note that if $\Phi : \mathrm{EL}(X) \to \mathrm{EL}(Y)$ is a G-flow and semigroup epimorphism, then $\Phi(g_X) = g_Y$ for every $g \in G$. (Here we write g_X to stress that we consider g as an element of $\mathrm{EL}(X)$, and similarly for g_Y.) Indeed, for the neutral $e \in G$ we have that $e_X = \mathrm{id}_X$ and $e_Y = \mathrm{id}_Y$ are neutrals in $\mathrm{EL}(X)$ and $\mathrm{EL}(Y)$, respectively. Since Φ is a surjective semigroup homomorphism, we easily see that $\Phi(e_X)$ is neutral in $\mathrm{EL}(Y)$, so $\Phi(e_X) = e_Y$, as the neutral in $\mathrm{EL}(Y)$ is unique. Now, for each $g \in G$ we have $\Phi(g_X) = \phi(g_X e_X) = g_Y \Phi(e_X) = g_Y e_Y = g_Y$, because Φ is a G-flow homomorphism.

Fact 2.2. *Let (G, X) and (G, Y) be two G-flows, and let $\Phi : \mathrm{EL}(X) \to \mathrm{EL}(Y)$ be a G-flow and semigroup epimorphism. Let $\mathcal{M}$ be a minimal left ideal of $\mathrm{EL}(X)$ and $u \in \mathcal{J}(\mathcal{M})$. Then:*

(i) $\mathcal{M}' := \Phi[\mathcal{M}]$ is a minimal left ideal of $\mathrm{EL}(Y)$ and $u' := \Phi(u) \in \mathcal{J}(\mathcal{M}')$;
(ii) $\Phi_{\restriction u\mathcal{M}} : u\mathcal{M} \to u'\mathcal{M}'$ is a group epimorphism and a quotient map in the τ-topologies.

Proof. This is basically the argument from the proof of [29, Proposition 5.41].

(i) is straightforward. For (ii), $\phi := \Phi_{\restriction u\mathcal{M}} : u\mathcal{M} \to u'\mathcal{M}'$ is clearly a group epimorphism. For the proof of the second assertion we have to recall some basic facts about sets $a \circ B$:

(F1) for $A \subseteq u\mathcal{M}$, $\mathrm{cl}_\tau(A) = u(u \circ A)$;
(F2) for $a \in \mathrm{EL}(X)$ and $B \subseteq \mathrm{EL}(X)$, $aB \subseteq a \circ B$;
(F3) for $a, b \in \mathrm{EL}(X)$ and $C \subseteq \mathrm{EL}(X)$, $a \circ (b \circ C) \subseteq (ab) \circ C$;

For the proof of these facts see e.g. [29, Fact A.25, Fact A.32].

We first prove that ϕ is τ-continuous. Let $F' \subseteq u'\mathcal{M}'$ be a τ-closed set, and we will prove that $F := \phi^{-1}[F']$ is τ-closed. Take $\eta \in \mathrm{cl}_\tau(F) = u\mathcal{M} \cap (u \circ F)$; we have to prove that $\eta \in F$. There exists a net $(g_i f_i)_i$ such that $g_i \in G$, $f_i \in F$, $\lim_i g_i = u$ and $\lim_i g_i f_i = \eta$. By the assumptions on Φ and the paragraph preceding Fact 2.2, we have: $u' = \Phi(u) = \Phi(\lim_i g_i) = \lim_i \Phi(g_i) = \lim_i g_i$ and $\phi(\eta) = \Phi(\eta) = \Phi(\lim_i g_i f_i) = \lim_i \Phi(g_i f_i) = \lim_i g_i \Phi(f_i) = \lim_i g_i \phi(f_i)$. Since $\phi(f_i) \in \phi[F] = \phi[\phi^{-1}[F']] = F'$, we obtain $\phi(\eta) \in u'\mathcal{M}' \cap (u' \circ F') = \mathrm{cl}_\tau(F') = F'$. Thus, $\eta \in \phi^{-1}[F'] = F$.

To complete the proof that ϕ is a τ-quotient map, we need to take any $F' \subseteq u'\mathcal{M}'$ such that $F := \phi^{-1}[F']$ is τ-closed and prove that F' is τ-closed. Take $\eta' \in \mathrm{cl}_\tau(F') = u'\mathcal{M}' \cap (u' \circ F')$; we have to show that $\eta' \in F'$. There exists a net $(g_i f_i')_i$ such that $g_i \in G$, $f_i' \in F'$, $\lim_i g_i = u'$ and $\lim_i g_i f_i' = \eta'$. Take $f_i \in F$ such that $\phi(f_i) = f_i'$. By considering subnets and using compactness, we may assume that $\lim_i g_i = v$ and $\lim_i g_i f_i = \eta$ in $\mathrm{EL}(X)$. By the assumptions on Φ, we get $\Phi(v) = u'$, $\Phi(\eta) = \eta'$, and $\Phi(u\eta) = \Phi(u)\Phi(\eta) = u'\eta' = \eta'$.

Since $\eta \in v \circ F$, we have $u\eta \in u(v \circ F)$. Note that we have:

$$u(v \circ F) = uu(v \circ (uF)) \subseteq u(u \circ (v \circ (u \circ F))) \subseteq u((uvu) \circ F) = u(w \circ F),$$

where the first equality holds as u is an idempotent and $F \subseteq u\mathcal{M}$, the first inclusion holds by applying (F2) twice, the second inclusion holds by applying (F3) twice, and in the last equality we just put $w := uvu \in u\mathcal{M}$. We further have:

$$u(w \circ F) = uww^{-1}(w \circ F) \subseteq uw((w^{-1}w) \circ F) = wu(u \circ F) = wF,$$

where w^{-1} is calculated in $u\mathcal{M}$, the first equality is by idempotency of u, the inclusion holds by combining (F2) and (F3), the second equality holds since $uw = w = wu$, and the last equality holds because $u(u \circ F) = \mathrm{cl}_\tau(F)$ by (F1) and F is τ-closed by assumption.

Therefore, $u\eta \in wF = uvuF \subseteq u\mathcal{M}$ and $\phi[uvuF] = \Phi[uvuF] = u'u'u'\Phi[F] = u'\phi[F] = u'F' = F'$, so $\eta' = \Phi(u\eta) = \phi(u\eta) \in \phi[uvuF] = F'$. □

Corollary 2.3. *Let (G, X) and (G, Y) be two G-flows, and let $\Phi : \mathrm{EL}(X) \to \mathrm{EL}(Y)$ be a G-flow and semigroup isomorphism. Let $\mathcal{M}$ be a minimal left ideal of $\mathrm{EL}(X)$ and $u \in \mathcal{J}(\mathcal{M})$. Then $\Phi_{\restriction u\mathcal{M}} : u\mathcal{M} \to \Phi(u)\Phi[\mathcal{M}]$ is a group isomorphism and a homeomorphism (in the τ-topologies).* □

Natural ways of obtaining a G-flow and semigroup epimorphism $\mathrm{EL}(X) \to \mathrm{EL}(Y)$ is to induce it from a G-flow epimorphism $X \to Y$ or a G-flow monomorphism $Y \to X$. This is explained in the next fact whose proof is left as a standard exercise.

Fact 2.4. *Let (G, X) and (G, Y) be G-flows.*

(i) *If $f : X \to Y$ is a G-flow epimorphism, then it induces a G-flow and semigroup epimorphism $\hat{f} : \mathrm{EL}(X) \to \mathrm{EL}(Y)$ given by:*

$$\hat{f}(\eta)(y) = f(\eta(x))$$

for $\eta \in \mathrm{EL}(X)$ and $y \in Y$, where $x \in X$ is any element such that $f(x) = y$.

(ii) *If $f : Y \to X$ is a G-flow monomorphism, then it induces a G-flow and semigroup epimorphism $\hat{f} : \mathrm{EL}(X) \to \mathrm{EL}(Y)$ given by:*

$$\hat{f}(\eta)(y) = f^{-1}(\eta(f(y)))$$

for $\eta \in \mathrm{EL}(X)$ and $y \in Y$, where f^{-1} is the inverse $f^{-1} : \mathrm{Im}(f) \to Y$ of f. □

Corollary 2.5. *Let $f : X \to Y$ be a G-flow epimorphism and let $\hat{f} : \mathrm{EL}(X) \to \mathrm{EL}(Y)$ be the induced epimorphism given by Fact 2.4(i). Then, for any $\eta \in \mathrm{EL}(X)$, $\mathrm{Im}(\hat{f}(\eta)) = f[\mathrm{Im}(\eta)]$. Thus, if $\mathrm{Im}(\eta)$ is finite, so is $\mathrm{Im}(\hat{f}(\eta))$.* □

Corollary 2.6. *If (G, Y) is a subflow of (G, X) and there is an element $\eta \in \mathrm{EL}(X)$ with $\mathrm{Im}(\eta) \subseteq Y$, then the Ellis groups of the flows X and Y are topologically isomorphic.*

Proof. Let $f : Y \to X$ be the identity map. Then the map $\hat{f} : \mathrm{EL}(X) \to \mathrm{EL}(Y)$ from Fact 2.4(ii) is just the restriction to Y. Let $\mathcal{M}$ be a minimal left ideal of $\mathrm{EL}(X)$. Replacing η by any element of $\eta\mathcal{M}$, we can assume that $\eta \in \mathcal{M}$. Let $u \in \mathcal{J}(\mathcal{M})$ be such that $\eta \in u\mathcal{M}$. Then $\mathrm{Im}(u) = \mathrm{Im}(\eta) \subseteq Y$. Let $\mathcal{M}' = \hat{f}[\mathcal{M}]$ and $u' = \hat{f}(u)$. By Fact 2.2, $u'\mathcal{M}'$ is the Ellis group of (G, Y) and $\hat{f}_{\restriction u\mathcal{M}} : u\mathcal{M} \to u'\mathcal{M}'$ is an epimorphism and a topological quotient map. Finally, the penultimate sentence of the proof of Fact 2.1 shows that $\hat{f}_{\restriction u\mathcal{M}}$ is bijective. □

We further consider an inverse system of G-flows $((G, X_i))_{i \in I}$, where I is a directed set, and for each $i < j$ in I a G-flow epimorphism $\pi_{i,j} : X_j \to X_i$ is given. Then $X := \varprojlim_{i \in I} X_i$ is a compact Hausdorff space and G acts naturally and continuously on X. Denote by $\pi_i : X \to X_i$ the natural projections; they are G-flow epimorphisms. By Fact 2.4, we have G-flow and semigroup epimorphisms $\hat{\pi}_{i,j} : \mathrm{EL}(X_j) \to \mathrm{EL}(X_i)$ for $i < j$ and $\hat{\pi}_i : \mathrm{EL}(X) \to \mathrm{EL}(X_i)$. It turns out that the previous inverse limit construction transfers to Ellis semigroups, and furthermore to Ellis groups. We state this in the following fact; for the proof see [29, Lemma 6.42].

Fact 2.7. *Fix the notation from the previous paragraph.*

The family $(\mathrm{EL}(X_i))_{i \in I}$, together with the mappings $\hat{\pi}_{i,j}$, is an inverse system of semigroups and of G-flows. There exists a natural G-flow and semigroup isomorphism $\mathrm{EL}(X) \cong \varprojlim_{i \in I} \mathrm{EL}(X_i)$, and after identifying $\mathrm{EL}(X)$ with $\varprojlim_{i \in I} \mathrm{EL}(X_i)$, the natural projections of this inverse limit are just the $\hat{\pi}_i$'s.

For every minimal left ideal $\mathcal{M}$ of $\mathrm{EL}(X)$, each $\mathcal{M}_i := \hat{\pi}_i[\mathcal{M}]$ is a minimal left ideal of $\mathrm{EL}(X_i)$ and $\mathcal{M} = \varprojlim_{i \in I} \mathcal{M}_i$. Also, if $u \in \mathcal{J}(\mathcal{M})$, then $u_i := \hat{\pi}_i(u) \in \mathcal{J}(\mathcal{M}_i)$. Furthermore, $u\mathcal{M} = \varprojlim_{i \in I} u_i\mathcal{M}_i$ and the τ-topology on $u\mathcal{M}$ coincides with the inverse limit topology induced from the τ-topologies on the $u_i\mathcal{M}_i$'s. □

The material in the rest of this subsection will be needed in the analysis of Examples 6.11 and 6.12.

Recall that a G-ambit is a G-flow (G, X, x_0) with a distinguished point x_0 with dense orbit. It is well-known that for any discrete group G, $(G, \beta G, \mathcal{U}_e)$ is the universal G-ambit, where $\mathcal{U}_e$ is the principal ultrafilter concentrated on e; we will identify e with $\mathcal{U}_e$. This easily yields a unique left continuous semigroup operation on βG extending the action of G on βG, and for any G-flow (G, X) this also yields a unique left continuous action $*$ of the semigroup βG on (G, X) extending the action of G. Thus, there is a unique G-flow and semigroup epimorphism $\beta G \to \mathrm{EL}(X)$ (mapping e to the identity): it is given by $p \mapsto l_p$, where $l_p(x) := p * x$. Applying this to $X := \beta G$, we get an isomorphism $\beta G \cong \mathrm{EL}(\beta G)$. For some details on this see [8, Chapter 1].

It is also well-known (see [9, Exercise 1.25]) that for the Bernoulli shift $X := 2^G$ the above map $\beta G \to \mathrm{EL}(X)$ is an isomorphism of flows and of semigroups, i.e. $\beta G \cong \mathrm{EL}(2^G)$.

Proposition 2.8. *Let (G, X) be a G-flow for which there is a G-flow epimorphism $\varphi : X \to 2^G$. Then,*

(i) $\mathrm{EL}(X) \cong \beta G$ *as G-flows and as semigroups.*

(ii) The Ellis group of X is topologically isomorphic with the Ellis group of βG.

Proof. (i) By Fact 2.4(i), we have the induced G-flow and semigroup epimorphism $\hat{\varphi} : \mathrm{EL}(X) \to \mathrm{EL}(2^G)$. On the other hand, by the above discussion, there is a unique G-flow and semigroup epimorphism $\psi : \beta G \to \mathrm{EL}(X)$. So the composition $\hat{\varphi} \circ \psi : \beta G \to \mathrm{EL}(2^G)$ is a G-flow and semigroup epimorphism. But such an epimorphism is clearly unique, and, by the above discussion, we know that it is an isomorphism. Therefore, ψ must be an isomorphism, too.
(ii) follows from (i) and Corollary 2.3. □

Recall that the Bohr compactification of a topological group G is a unique (up to isomoprhism) universal object in the category of group compactifications of G, i.e. continuous homomorphisms $G \to H$ with dense image, where H is a compact Hausdorff group. We often identify the Bohr compactification with the target compact group, and denote it by bG. For a discrete group G, $\mathcal{M}$ a minimal left ideal in βG and $u \in \mathcal{M}$ an idempotent, the quotient $u\mathcal{M}/H(u\mathcal{M})$ turns out to be the generalized Bohr compactification of G in the terminology from [8]. There is always a continuous surjection from the generalized Bohr compactification to the Bohr compactification, which can be nicely seen model-theoretically in a more general context: it is given by the composition $\pi \circ \bar{f}$ in the notation from formula (0.2) of [17]. We will need the following result, which is Corollary 4.3 of [8] (and also appears in a more general context in Corollary 0.4 of [17]). We do not recall here the notion of strongly amenable group. We only need to know that abelian groups are strongly amenable. For more details see [8].

Fact 2.9. *If a discrete group G is strongly amenable (e.g. G is abelian), then the generalized Bohr compactification of G and the Bohr compactification of G coincide.* □

We will also need the following consequence of the presentation of bG (for an abelian discrete group G) as the "double Pontryagin dual", which can for example be found in [33]; a short proof based on Pontryagin duality is given in [6, Section 1].

Fact 2.10. *Let G be a discrete abelian group. Then, bG is profinite iff G is of finite exponent.* □

2.3. Structural Ramsey theory.

Recall that a first order structure K is called a *Fraïssé structure* if it is countable, locally finite (finitely generated substructures of K are finite; although this property is not always taken as a part of the definition) and ultrahomogeneous (every isomorphism between finite substructures of K lifts to an automorphism of K). Every Fraïssé structure K is uniquely determined by its *age*, $\mathrm{Age}(K)$, i.e. the class of all finite structures which are embeddable in K. The age of a Fraïssé structure is a *Fraïssé class*, i.e. a class of finite structures which is closed under isomorphisms, countable up to isomorphism, and satisfies *hereditary*, *joint embedding* and *amalgamation property*. On the other hand, for every Fraïssé class $\mathcal{C}$ of first order structures there exists a unique Fraïssé structure, called the *Fraïssé limit* of $\mathcal{C}$, whose age is exactly $\mathcal{C}$.

Structural Ramsey theory, invented by Nešetřil and Rödl in the 1970s, investigates combinatorial properties of Fraïssé classes (and, more generally, categories), i.e. classes of the form $\mathrm{Age}(K)$, where K is a Fraïssé structure. Originally, these structural combinatorial properties are given by colorings of isomorphic copies of A in B, where A and B are members of the age such that A is embeddable in B. We follow the approach from [35]. For $A, B \in \mathrm{Age}(K)$, $\mathrm{Emb}(A, B)$ stands for the set of all embeddings $A \to B$. A Fraïssé structure K (or $\mathrm{Age}(K)$) has *separately finite embedding Ramsey degree* if for every $A \in \mathrm{Age}(K)$ there exists $l < \omega$ such that for every $B \in \mathrm{Age}(K)$ with $\mathrm{Emb}(A, B) \neq \emptyset$ and every $r < \omega$ there exists $C \in \mathrm{Age}(K)$ such that for every coloring $c : \mathrm{Emb}(A, C) \to r$ there exists $f \in \mathrm{Emb}(B, C)$ with $\#c[f \circ \mathrm{Emb}(A, B)] \leqslant l$. We added the word "separately" to emphasize that l depends on $\bar{a}$. If in the previous definition l can be chosen to be 1 for every $\bar{a}$, we obtain the notion of the *embedding Ramsey property*. By [35, Proposition 4.4], K has separately finite embedding Ramsey degree iff it has separately finite structural Ramsey degree (defined by using colorings of isomorphic copies of A in B in place of embeddings). Moreover, by [35, Corollary 4.5], K has the embedding Ramsey property iff it has the structural Ramsey property and all structures in $\mathrm{Age}(K)$ are rigid (have trivial automorphism groups).

Both the property of having separately finite embedding Ramsey degree and the embedding Ramsey property for a Fraïssé structure K can be alternatively defined as follows. For finite $\bar{a} \subseteq K$ and $C \subseteq K$ denote by $\binom{C}{\bar{a}}^{\mathrm{qf}}$ the set of all $\bar{a}' \subseteq C$ such that $\bar{a}' \equiv^{\mathrm{qf}} \bar{a}$ ($\bar{a}'$ and $\bar{a}$ have the same quantifier-free type).

Fact 2.11. *A Fraïssé structure K has separately finite embedding Ramsey degree iff for every finite $\bar{a} \subseteq K$ there exists $l < \omega$ such that for any finite $\bar{b} \subseteq K$ containing $\bar{a}$ and $r < \omega$ there exists a finite $C \subseteq K$ such that for every coloring $c : \binom{C}{\bar{a}}^{\mathrm{qf}} \to r$ there exists $\bar{b}' \in \binom{C}{\bar{b}}^{\mathrm{qf}}$ with $\#c[\binom{\bar{b}'}{\bar{a}}^{\mathrm{qf}}] \leqslant l$.*

The same holds for the embedding Ramsey property and $l = 1$.

Proof. The key observation here is that if we take any enumeration $\bar{a}$ of a structure A, then there exists a natural correspondence between $\mathrm{Emb}(A, B)$ and $\binom{B}{\bar{a}}^{\mathrm{qf}}$ given by $f \mapsto$

$f(\bar{a})$. Furthermore, consider any finite $\bar{a} \subseteq K$, and fix an enumeration $e(\bar{a})$ of the (finite) structure $\langle \bar{a} \rangle$ generated by $\bar{a}$. Then, for any $\bar{a}' \equiv^{\mathrm{qf}} \bar{a}$ the unique isomorphism $\langle \bar{a} \rangle \to \langle \bar{a}' \rangle$ extending the map $\bar{a} \to \bar{a}'$ induces an enumeration $e(\bar{a}')$ of $\langle \bar{a}' \rangle$. So, for a substructure $C \subseteq K$ we have a natural correspondence between $\binom{C}{\bar{a}}^{\mathrm{qf}}$ and $\binom{C}{e(\bar{a})}^{\mathrm{qf}}$ given by $\bar{a}' \mapsto e(\bar{a}')$. Now, it is easy to see that the definitions of separately finite embedding Ramsey degree and the embedding Ramsey property transfer via the described correspondences. $\square$

2.4. **Flows in model theory.**

As mentioned in the introduction, the methods of topological dynamics were introduced to model theory by Newelski over ten years ago, and since then this approach has gained a lot of attention which resulted in some deep applications (e.g. to strong types in [19, 21]). Flows in model theory occur naturally in two ways. One way is to consider the action of a definable group on various spaces of types concentrated on this group. The other one is to consider the action of the automorphism group of a model on various spaces of types. In this paper, we consider the second situation. We are mostly interested in the flow $(\mathrm{Aut}(\mathfrak{C}), S_{\bar{c}}(\mathfrak{C}))$, where $\mathfrak{C}$ is a monster model of a theory T, $\bar{c}$ is an enumeration of $\mathfrak{C}$, $S_{\bar{c}}(\mathfrak{C})$ is the space of all types over $\mathfrak{C}$ extending $\mathrm{tp}(\bar{c})$, and the action of $\mathrm{Aut}(\mathfrak{C})$ on $S_{\bar{c}}(\mathfrak{C})$ is the obvious one.

The investigation of this flow in concrete examples is not easy, thus we need some reductions of it to flows which are easier to deal with. Some reductions of this kind appeared in Section 2 of [16], but here we need more specific ones.

Let $\bar{d}$ be a tuple of all elements of $\mathfrak{C}$ in which each element of $\mathfrak{C}$ is repeated infinitely many times. It is clear that $S_{\bar{d}}(\mathfrak{C})$ and $S_{\bar{c}}(\mathfrak{C})$ are isomorphic $\mathrm{Aut}(\mathfrak{C})$-flows. Hence, their Ellis semigroups are isomorphic as semigroups and as $\mathrm{Aut}(\mathfrak{C})$-flows. So, in fact, one can work with $S_{\bar{d}}(\mathfrak{C})$ in place of $S_{\bar{c}}(\mathfrak{C})$ whenever it is convenient.

Let us fix some notation. Let $\bar{x}$ be a tuple of variables corresponding to $\bar{d}$. For $\bar{x}' \subseteq \bar{x}$, $\bar{z}$ with $|\bar{z}| = |\bar{x}'|$, and for $p(\bar{x}') \in S_{\bar{x}'}(\mathfrak{C})$ we denote by $p[\bar{x}'/\bar{z}]$ the type from $S_{\bar{z}}(\mathfrak{C})$ obtained by replacing the variables $\bar{x}'$ by $\bar{z}$. Denote by $\bar{z}$ a sequence $(z_i)_{i<n}$ of variables, where $n \leqslant \omega$. We discuss connections between the $\mathrm{Aut}(\mathfrak{C})$-flows $S_{\bar{d}}(\mathfrak{C})$ and $S_{\bar{z}}(\mathfrak{C})$. Let $\mathfrak{C}^* \succ \mathfrak{C}$ be a bigger monster model.

Lemma 2.12. *Let $\Phi : \mathrm{EL}(S_{\bar{d}}(\mathfrak{C})) \to \mathrm{EL}(S_{\bar{z}}(\mathfrak{C}))$ be defined by $\Phi(\eta) := \hat{\eta}$, where $\hat{\eta}$ is given by:*

$$\hat{\eta}(p(\bar{z})) = \eta(q(\bar{x}))_{\restriction \bar{x}'}[\bar{x}'/\bar{z}],$$

where $p(\bar{z}) \in S_{\bar{z}}(\mathfrak{C})$, and $\bar{x}' \subseteq \bar{x}$ and $q(\bar{x}) \in S_{\bar{d}}(\mathfrak{C})$ are such that $q(\bar{x})_{\restriction \bar{x}'}[\bar{x}'/\bar{z}] = p(\bar{z})$. Then Φ is a well-defined epimorphism of $\mathrm{Aut}(\mathfrak{C})$-flows and of semigroups.

Proof. Let us check the correctness of the definition. First, for $p(\bar{z}) \in S_{\bar{z}}(\mathfrak{C})$ take $\bar{\alpha}^* \models p$ in $\mathfrak{C}^*$ and $\bar{\alpha} \subseteq \mathfrak{C}$ such that $\bar{\alpha} \equiv \bar{\alpha}^*$. Further, take $\sigma \in \mathrm{Aut}(\mathfrak{C}^*)$ mapping $\sigma(\bar{\alpha}) = \bar{\alpha}^*$. Since each element of $\mathfrak{C}$ is repeated infinitely many times in $\bar{d}$, we can find $\bar{x}' \subseteq \bar{x}$ such that $\bar{d}_{\restriction \bar{x}'} = \bar{\alpha}$. Then, for $q(\bar{x}) := \mathrm{tp}(\sigma(\bar{d})/\mathfrak{C})$ we have $q(\bar{x})_{\restriction \bar{x}'}[\bar{x}'/\bar{z}] = p(\bar{z})$, so for a given $p(\bar{z})$ the desired $\bar{x}'$ and $q(\bar{x})$ exist. Moreover, if $\bar{x}', \bar{x}'' \subseteq \bar{x}$ and $q'(\bar{x}), q''(\bar{x}) \in S_{\bar{d}}(\mathfrak{C})$ are such that $q'(\bar{x})_{\restriction \bar{x}'}[\bar{x}'/\bar{z}] = q''(\bar{x})_{\restriction \bar{x}''}[\bar{x}''/\bar{z}] = p(\bar{z})$, we have $\eta(q'(\bar{x}))_{\restriction \bar{x}'}[\bar{x}'/\bar{z}] = \eta(q''(\bar{x}))_{\restriction \bar{x}''}[\bar{x}''/\bar{z}]$. Otherwise, we can find a formula $\phi(\bar{z}, \bar{a})$ such that $\phi(\bar{x}', \bar{a}) \in \eta(q'(\bar{x}))$ and $\neg\phi(\bar{x}'', \bar{a}) \in$

$\eta(q''(\bar{x}))$. This is an open condition on η, so we can find $\sigma \in \mathrm{Aut}(\mathfrak{C})$ such that $\phi(\bar{x}', \bar{a}) \in \sigma(q'(\bar{x}))$ and $\neg\phi(\bar{x}'', \bar{a}) \in \sigma(q''(\bar{x}))$, i.e. $\phi(\bar{x}', \sigma^{-1}(\bar{a})) \in q'(\bar{x})$ and $\neg\phi(\bar{x}'', \sigma^{-1}(\bar{a})) \in q''(\bar{x})$. But then $\phi(\bar{z}, \sigma^{-1}(\bar{a})) \in q'(\bar{x})_{\restriction \bar{x}'}[\bar{x}'/\bar{z}]$ and $\neg\phi(\bar{z}, \sigma^{-1}(\bar{a})) \in q''(\bar{x})_{\restriction \bar{x}''}[\bar{x}''/\bar{z}]$, contradicting our choice of $\bar{x}', \bar{x}'', q'(\bar{x})$ and $q''(\bar{x})$.

To finish the proof of correctness, we should check that $\hat{\eta} \in \mathrm{EL}(S_{\bar{z}}(\mathfrak{C}))$ for every $\eta \in \mathrm{EL}(S_{\bar{d}}(\mathfrak{C}))$. Take an open basis neighbourhood $U := \{\tau \in S_{\bar{z}}(\mathfrak{C})^{S_{\bar{z}}(\mathfrak{C})} \mid \varphi_i(\bar{z}, \bar{a}) \in \tau(p_i(\bar{z}))$ for $i < k\}$ of $\hat{\eta}$ in $S_{\bar{z}}(\mathfrak{C})^{S_{\bar{z}}(\mathfrak{C})}$ (where $\varphi_i(\bar{z}, \bar{a})$ are formulae and $p_i(\bar{z}) \in S_{\bar{z}}(\mathfrak{C})$ for $i < k$). Take $\bar{x}_i \subseteq \bar{x}$ and $q_i(\bar{x}) \in S_{\bar{d}}(\mathfrak{C})$ such that $p_i(\bar{z}) = q_i(\bar{x})_{\restriction \bar{x}_i}[\bar{x}_i/\bar{z}]$. Then $\varphi_i(\bar{x}_i, \bar{a}) \in \eta(q_i(\bar{x}))$, which is an open condition on η, so we can find $\sigma \in \mathrm{Aut}(\mathfrak{C})$ such that $\varphi_i(\bar{x}_i, \sigma^{-1}(\bar{a})) \in q_i(\bar{x})$ for all $i < k$. Hence, $\varphi_i(\bar{z}, \sigma^{-1}(\bar{a})) \in q_i(\bar{x})_{\restriction \bar{x}_i}[\bar{x}_i/\bar{z}] = p_i(\bar{z})$, and we get $\varphi_i(\bar{z}, \bar{a}) \in \sigma(p_i(\bar{z}))$ for all $i < k$, i.e. $\sigma \in U$. This finishes the proof of correctness.

Showing that Φ is an epimorphism of $\mathrm{Aut}(\mathfrak{C})$-flows and of semigroups is left as an exercise. □

Having in mind that the flows $S_{\bar{d}}(\mathfrak{C})$ and $S_{\bar{c}}(\mathfrak{C})$ are isomorphic, we get:

Corollary 2.13. *There exists an* $\mathrm{Aut}(\mathfrak{C})$*-flow and semigroup epimorphism from* $\mathrm{EL}(S_{\bar{c}}(\mathfrak{C}))$ *to* $\mathrm{EL}(S_{\bar{z}}(\mathfrak{C}))$*. In particular, such an epimorphism exists from* $\mathrm{EL}(S_{\bar{c}}(\mathfrak{C}))$ *to* $\mathrm{EL}(S_n(\mathfrak{C}))$ *for all* $n < \omega$*.* □

Corollary 2.14. *If* $\bar{z} = (z_i)_{i<\omega}$*, then* $\mathrm{EL}(S_{\bar{c}}(\mathfrak{C}))$ *and* $\mathrm{EL}(S_{\bar{z}}(\mathfrak{C}))$ *are* $\mathrm{Aut}(\mathfrak{C})$*-flow and semigroup isomorphic. Consequently, the Ellis groups of the flows* $(\mathrm{Aut}(\mathfrak{C}), S_{\bar{c}}(\mathfrak{C}))$ *and* $(\mathrm{Aut}(\mathfrak{C}), S_{\bar{z}}(\mathfrak{C}))$ *are* τ*-homeomorphic and isomorphic.*

Proof. It is enough to prove that in this case Φ from Lemma 2.12 is injective. Let $\eta_1 \neq \eta_2 \in \mathrm{EL}(S_{\bar{d}}(\mathfrak{C}))$, and take $q(\bar{x})$ and $\varphi(\bar{x}, \bar{a})$ such that $\varphi(\bar{x}, \bar{a}) \in \eta_1(q(\bar{x}))$ and $\neg\varphi(\bar{x}, \bar{a}) \in \eta_2(q(\bar{x}))$. Take $\bar{x}' \subseteq \bar{x}$ such that $|\bar{x}'| = \omega$ and all variables occurring in $\varphi(\bar{x}, \bar{a})$ are in $\bar{x}'$, so we may write $\varphi(\bar{x}, \bar{a})$ as $\varphi'(\bar{x}', \bar{a})$. Let $p(\bar{z}) = q(\bar{x})_{\restriction \bar{x}'}[\bar{x}'/\bar{z}]$. By the definition of $\hat{\eta}_1$ and $\hat{\eta}_2$, we have $\varphi'(\bar{z}, \bar{a}) \in \hat{\eta}_1(p(\bar{z}))$ and $\neg\varphi'(\bar{z}, \bar{a}) \in \hat{\eta}_2(p(\bar{z}))$. Thus, $\hat{\eta}_1 \neq \hat{\eta}_2$. □

If $|\bar{z}| = n < \omega$, then in general we do not have injectivity of Φ, but we may distinguish a sufficient condition on T for which Φ is injective for some n. We say that a theory T is *m-ary* (for some $m < \omega$) if every L-formula is equivalent modulo T to a Boolean combination of L-formulae with at most m free variables.

Corollary 2.15. *If* T *is* $(m+1)$*-ary, then* $\mathrm{EL}(S_{\bar{c}}(\mathfrak{C}))$ *and* $\mathrm{EL}(S_m(\mathfrak{C}))$ *are* $\mathrm{Aut}(\mathfrak{C})$*-flow and semigroup isomorphic. Consequently, the Ellis groups of the flows* $(\mathrm{Aut}(\mathfrak{C}), S_{\bar{c}}(\mathfrak{C}))$ *and* $(\mathrm{Aut}(\mathfrak{C}), S_m(\mathfrak{C}))$ *are* τ*-homeomorphic and isomorphic.*

Proof. We need to prove that Φ is injective in the case $|\bar{z}| = m$. Let $\eta_1 \neq \eta_2 \in \mathrm{EL}(S_{\bar{d}}(\mathfrak{C}))$ and let $q(\bar{x}) \in S_{\bar{d}}(\mathfrak{C})$ be such that $\eta_1(q(\bar{x})) \neq \eta_2(q(\bar{x}))$. By $(m+1)$-arity of the theory, we can find a formula $\varphi(\bar{x}'', \bar{y})$ with $\bar{x}'' \subseteq \bar{x}$ and $|\bar{x}''| + |\bar{y}| \leqslant m + 1$, and $\bar{a}$ such that $\varphi(\bar{x}'', \bar{a}) \in \eta_1(q(\bar{x}))$ and $\neg\varphi(\bar{x}'', \bar{a}) \in \eta_2(q(\bar{x}))$. Note that $|\bar{y}| \geqslant 1$, as otherwise $\varphi(\bar{x}'', \bar{a})$ is an L-formula (i.e. without parameters), so it belongs to $\eta_1(q(\bar{x}))$ iff it belongs to $\eta_2(q(\bar{x}))$. So $|\bar{x}''| \leqslant m$, and we can write $\varphi(\bar{x}'', \bar{a})$ as $\varphi'(\bar{x}', \bar{a})$ for some $\bar{x}' \subseteq \bar{x}$ such that $|\bar{x}'| = m$. Now, the final step of the proof of Corollary 2.14 goes through. □

The next remark is a general observation on flows.

Remark 2.16. Let (G,X) and (G,Y) be G-flows. Denote by $\mathrm{Inv}(X)$ and $\mathrm{Inv}(Y)$ the sets of all points fixed by G in X and Y, respectively. Assume that $F:\mathrm{EL}(X)\to\mathrm{EL}(Y)$ is a G-flow and semigroup epimorphism. Then for any $\eta\in\mathrm{EL}(X)$ we have: $\mathrm{Im}(\eta)\subseteq\mathrm{Inv}(X)$ iff $\mathrm{Im}(\eta)=\mathrm{Inv}(X)$, and these equivalent conditions imply $\mathrm{Im}(F(\eta))=\mathrm{Inv}(Y)$. Thus, if F is an isomorphism, then $\mathrm{Im}(\eta)=\mathrm{Inv}(X)$ iff $\mathrm{Im}(F(\eta))=\mathrm{Inv}(Y)$.

Proof. The equivalence is clear, since each element of $\mathrm{Inv}(X)$ is fixed by any $\eta\in EL(X)$. For the rest, note that for any $\eta\in\mathrm{EL}(X)$: $\mathrm{Im}(\eta)\subseteq\mathrm{Inv}(X)$ iff $(\forall g\in G)(g\eta=\eta)$. Assume $\mathrm{Im}(\eta)\subseteq\mathrm{Inv}(X)$. Then, for every $g\in G$ we have $gF(\eta)=F(g\eta)=F(\eta)$, which means that $\mathrm{Im}(F(\eta))\subseteq\mathrm{Inv}(Y)$. □

By Corollary 2.15 and Remark 2.16, we get

Corollary 2.17. *If T is $(m+1)$-ary, then the existence of $\eta\in\mathrm{EL}(S_{\bar c}(\mathfrak{C}))$ with $\mathrm{Im}(\eta)=\mathrm{Inv}_{\bar c}(\mathfrak{C})$ is equivalent to the existence of $\eta'\in\mathrm{EL}(S_m(\mathfrak{C}))$ with $\mathrm{Im}(\eta')=\mathrm{Inv}_m(\mathfrak{C})$ (where $\mathrm{Inv}_m(\mathfrak{C})$ is the set of all invariant types in $S_m(\mathfrak{C})$).* □

We will need one more consequence of the above investigations. Let us consider the following situation. Let $L\subseteq L^*$ be two languages, let T^* be a complete L^*-theory and $T:=T^*_{\restriction L}$. Assume that $\mathfrak{C}^*$ is a monster of T^* such that $\mathfrak{C}:=\mathfrak{C}^*_{\restriction L}$ is a monster of T. Let $\bar c$ be an enumeration of $\mathfrak{C}$ (and $\mathfrak{C}^*$). We can treat $S_{\bar c}(\mathfrak{C})$ as an $\mathrm{Aut}(\mathfrak{C}^*)$-flow, and when we do that (in this section) we write $S^*_{\bar c}(\mathfrak{C})$ in place of $S_{\bar c}(\mathfrak{C})$; similarly for $S^*_{\bar z}(\mathfrak{C})$. Note that $\mathrm{Aut}(\mathfrak{C}^*)\leqslant\mathrm{Aut}(\mathfrak{C})$, and so $\mathrm{EL}(S^*_{\bar c}(\mathfrak{C}))\subseteq\mathrm{EL}(S_{\bar c}(\mathfrak{C}))$ and $\mathrm{EL}(S^*_{\bar z}(\mathfrak{C}))\subseteq\mathrm{EL}(S_{\bar z}(\mathfrak{C}))$.

Lemma 2.18. *Take the previous notation.*

(i) There is an $\mathrm{Aut}(\mathfrak{C}^*)$-flow and semigroup epimorphism $\Psi:\mathrm{EL}(S_{\bar c}(\mathfrak{C}^*))\to\mathrm{EL}(S^*_{\bar c}(\mathfrak{C}))$.

(ii) If there is $\eta^*\in\mathrm{EL}(S_{\bar c}(\mathfrak{C}^*))$ such that $\mathrm{Im}(\eta^*)\subseteq\mathrm{Inv}_{\bar c}(\mathfrak{C}^*)$ (equiv. $=\mathrm{Inv}_{\bar c}(\mathfrak{C}^*)$), then there is $\eta\in\mathrm{EL}(S_{\bar c}(\mathfrak{C}))$ such that $\mathrm{Im}(\eta)=\mathrm{Inv}^*_{\bar c}(\mathfrak{C})$, where $\mathrm{Inv}_{\bar c}(\mathfrak{C}^*)\subseteq S_{\bar c}(\mathfrak{C}^*)$ and $\mathrm{Inv}^*_{\bar c}(\mathfrak{C})\subseteq S^*_{\bar c}(\mathfrak{C})$ are the subsets of all $\mathrm{Aut}(\mathfrak{C}^*)$-invariant types in $S_{\bar c}(\mathfrak{C}^*)$ and $S^*_{\bar c}(\mathfrak{C})$, respectively.

(iii) There is an $\mathrm{Aut}(\mathfrak{C}^*)$-flow and semigroup epimorphism $\Psi_{\bar z}:\mathrm{EL}(S_{\bar c}(\mathfrak{C}^*))\to\mathrm{EL}(S^*_{\bar z}(\mathfrak{C}))$.

(iv) If there is $\eta^*\in\mathrm{EL}(S_{\bar c}(\mathfrak{C}^*))$ such that $\mathrm{Im}(\eta^*)\subseteq\mathrm{Inv}_{\bar c}(\mathfrak{C}^*)$ (equiv. $=\mathrm{Inv}_{\bar c}(\mathfrak{C}^*)$), then there is $\eta\in\mathrm{EL}(S_{\bar z}(\mathfrak{C}))$ such that $\mathrm{Im}(\eta)=\mathrm{Inv}^*_{\bar z}(\mathfrak{C})$, where $\mathrm{Inv}^*_{\bar z}(\mathfrak{C})\subseteq S^*_{\bar z}(\mathfrak{C})$ is the subset of all $\mathrm{Aut}(\mathfrak{C}^*)$-invariant types in $S^*_{\bar z}(\mathfrak{C})$.

Proof. (i) Let $\bar z$ be an infinite tuple of variables indexed by ω. By Corollary 2.14, we have an $\mathrm{Aut}(\mathfrak{C}^*)$-flow and semigroup isomorphism $\Phi^*:\mathrm{EL}(S_{\bar c}(\mathfrak{C}^*))\to\mathrm{EL}(S_{\bar z}(\mathfrak{C}^*))$, and an $\mathrm{Aut}(\mathfrak{C})$-flow and semigroup isomorphism $\Phi:\mathrm{EL}(S_{\bar c}(\mathfrak{C}))\to\mathrm{EL}(S_{\bar z}(\mathfrak{C}))$. Since $\mathrm{Aut}(\mathfrak{C}^*)\leqslant\mathrm{Aut}(\mathfrak{C})$, we easily get that $\Phi_{\restriction\mathrm{EL}(S^*_{\bar c}(\mathfrak{C}))}$ is an $\mathrm{Aut}(\mathfrak{C}^*)$-flow and semigroup isomorphism $\mathrm{EL}(S^*_{\bar c}(\mathfrak{C}))\to\mathrm{EL}(S^*_{\bar z}(\mathfrak{C}))$. We also have an $\mathrm{Aut}(\mathfrak{C}^*)$-flow epimorphism $S_{\bar z}(\mathfrak{C}^*)\to S^*_{\bar z}(\mathfrak{C})$ given by the restriction of the language; hence, by Fact 2.4, there exists an $\mathrm{Aut}(\mathfrak{C}^*)$-flow and semigroup epimorphism $\Theta:\mathrm{EL}(S_{\bar z}(\mathfrak{C}^*))\to\mathrm{EL}(S^*_{\bar z}(\mathfrak{C}))$. Then $\Psi:=\Phi^{-1}_{\restriction\mathrm{EL}(S^*_{\bar c}(\mathfrak{C}))}\circ\Theta\circ\Phi^*$ is the desired $\mathrm{Aut}(\mathfrak{C}^*)$-flow and semigroup epimorphism.

(ii) Since $\mathrm{EL}(S^*_{\bar c}(\mathfrak{C}))\subseteq\mathrm{EL}(S_{\bar c}(\mathfrak{C}))$, (ii) follows from (i) and Remark 2.16 applied to $X:=S_{\bar c}(\mathfrak{C}^*)$, $Y:=S^*_{\bar c}(\mathfrak{C})$, and $G:=\mathrm{Aut}(\mathfrak{C}^*)$.

(iii) $\Psi_{\bar{z}} := \Theta \circ \Phi^*$ from the proof of (i) does the job (but here $\Phi^* : \mathrm{EL}(S_{\bar{c}}(\mathfrak{C}^*)) \to \mathrm{EL}(S_{\bar{z}}(\mathfrak{C}^*))$ is an epimorphism provided by Corollary 2.13).

(iv) follows from (iii) and Remark 2.16. □

We now describe a natural way of presenting $S_{\bar{c}}(\mathfrak{C})$ as an inverse limit of $\mathrm{Aut}(\mathfrak{C})$ flows, which refines the usual presentation as the inverse limit of complete Δ-types and which is one of the key tools in this paper.

For any $\bar{a} \subseteq \mathfrak{C}^* \succ \mathfrak{C}$, $\Delta = \{\varphi_0(\bar{x}, \bar{y}), \dots, \varphi_{k-1}(\bar{x}, \bar{y})\}$ where $|\bar{x}| = |\bar{a}|$, and $\bar{p} = \{p_0, \dots, p_{m-1}\} \subseteq S_{\bar{y}}(T)$, by $\mathrm{tp}_\Delta(\bar{a}/\bar{p})$ we mean the Δ-type of $\bar{a}$ over $\bigsqcup_{j<m} p_j(\mathfrak{C})$, i.e. the set of all formulae of the form $\varphi_i(\bar{x}, \bar{b})^\epsilon$ such that $\epsilon \in 2$, $\bar{b}$ realizes one of p_j's and $\models \varphi_i(\bar{a}, \bar{b})^\epsilon$. (Here, as usual, φ^0 denotes $\neg\varphi$, and φ^1 denotes φ.)

For $\Delta = \{\varphi_0(\bar{x}, \bar{y}), \dots, \varphi_{k-1}(\bar{x}, \bar{y})\}$, where $\bar{x}$ is reserved for $\bar{c}$, and $\bar{p} = \{p_0, \dots, p_{m-1}\} \subseteq S_{\bar{y}}(T)$, by $S_{\bar{c},\Delta}(\bar{p})$ we denote the space of all complete Δ-types over $\bigsqcup_{j<m} p_j(\mathfrak{C})$ consistent with $\mathrm{tp}(\bar{c})$; equivalently:

$$S_{\bar{c},\Delta}(\bar{p}) = \{\mathrm{tp}_\Delta(\bar{c}^*/\bar{p}) \mid \bar{c}^* \subseteq \mathfrak{C}^* \text{ and } \bar{c}^* \equiv \bar{c}\}.$$

In the usual way, we endow $S_{\bar{c},\Delta}(\bar{p})$ with a topology, turning it into a 0-dimensional, compact, Hausdorff space. Moreover, it is naturally an $\mathrm{Aut}(\mathfrak{C})$-flow.

Let $\mathcal{F}$ be the family of all pairs $(\Delta = \{\varphi_i\}_{i<k}, \bar{p} = \{p_j\}_{j<m})$ as above. We order $\mathcal{F}$ naturally by:

$$(\Delta = \{\varphi_i(\bar{x}, \bar{y})\}_{i<k}, \bar{p} = \{p_j\}_{j<m}) \leqslant (\Delta' = \{\varphi'_i(\bar{x}, \bar{y}')\}_{i<k'}, \bar{p}' = \{p'_j\}_{j<m'})$$

if $\bar{y} \subseteq \bar{y}'$, $\{\varphi_i(\bar{x}, \bar{y})\}_{i<k} \subseteq \{\varphi'_i(\bar{x}, \bar{y}')\}_{i<k'}$ by using dummy variables, and $\{p_j\}_{j<m} \subseteq \{p_{j'\restriction\bar{y}}\}_{j<m'}$, where $p_{j'\restriction\bar{y}}$ denotes the appropriate restriction of variables. It is not hard to see that $\mathcal{F}$ is actually directed by $\leqslant$, and that for pairs $t = (\Delta, \bar{p})$ and $t' = (\Delta', \bar{p}')$ in $\mathcal{F}$, if $t \leqslant t'$, we have an $\mathrm{Aut}(\mathfrak{C})$-flow epimorphism given by the restriction:

$$\pi_{t,t'} : S_{\bar{c},\Delta'}(\bar{p}') \to S_{\bar{c},\Delta}(\bar{p}).$$

Therefore, we have an inverse system of $\mathrm{Aut}(\mathfrak{C})$-flows $((\mathrm{Aut}(\mathfrak{C}), S_{\bar{c},\Delta}(\bar{p})))_{(\Delta,\bar{p})\in\mathcal{F}}$, and we clearly have:

Lemma 2.19. *$S_{\bar{c}}(\mathfrak{C}) \cong \varprojlim_{(\Delta,\bar{p})\in\mathcal{F}} S_{\bar{c},\Delta}(\bar{p})$ as $\mathrm{Aut}(\mathfrak{C})$-flows.* □

The usual $\mathrm{Aut}(\mathfrak{C})$-flow $S_{\bar{c},\Delta}(\mathfrak{C})$ of complete Δ-types over the whole $\mathfrak{C}$ consistent with $\mathrm{tp}(\bar{c})$ clearly projects onto the flow $S_{\bar{c},\Delta}(\bar{p})$. We also have $S_{\bar{c}}(\mathfrak{C}) \cong \varprojlim_{\Delta\in\mathcal{F}} S_{\bar{c},\Delta}(\mathfrak{C})$, but this presentation is not sufficient to be used in our main results which is illustrated by Example 6.10. Let us also remark that we could define $S_{\bar{c},\Delta}(\bar{p})$ for tuples (rather than sets) Δ, $\bar{p}$ of the same length (i.e. $k = m$), and for each $i < k$ allowing in $\varphi_i(\bar{x}, \bar{b})^\epsilon$ only parameters $\bar{b} \models p_i$. Note that $S_{\bar{c},\Delta}(\bar{p})$ defined earlier coincides with $S_{\bar{c},\Delta'}(\bar{p}')$ defined in the previous sentence (for some finite Δ' and $\bar{p}'$). In fact, the whole theory developed in this paper would work with some minor adjustments in the proofs for this modified definition of $S_{\bar{c},\Delta}(\bar{p})$.

2.5. **Contents and strong heirs.**

The following definition is given in [16].

Definition 2.20 ([16, Definition 3.1])**.** Fix $A \subseteq B$.

a) For $p(\bar{x}) \in S(B)$, the *content* of p over A is defined as:

$$\mathrm{ct}_A(p) = \{(\varphi(\bar{x}, \bar{y}), q(\bar{y})) \in L(A) \times S(A) \mid \varphi(\bar{x}, \bar{b}) \in p(\bar{x}) \text{ for some } \bar{b} \models q\}.$$

b) The content of a sequence of types $p_0(\bar{x}), \dots, p_{n-1}(\bar{x}) \in S(B)$ over A, $\mathrm{ct}_A(p_0, \dots, p_{n-1})$, is defined as the set of all $(\varphi_0(\bar{x}, \bar{y}), \dots, \varphi_{n-1}(\bar{x}, \bar{y}), q(\bar{y})) \in L(A)^n \times S(A)$ such that $\varphi_i(\bar{x}, \bar{b}) \in p_i$ for every $i < n$ and some $\bar{b} \models q$.

If $A = \emptyset$, we write just $\mathrm{ct}(p)$ and $\mathrm{ct}(p_0, \dots, p_{n-1})$.

A fundamental connection between contents and the Ellis semigroup is given by the following fact.

Fact 2.21 ([16, Proposition 3.5])**.** *Let $\pi(\bar{x})$ be a type over $\emptyset$, and $(p_0, \dots, p_{n-1})$ and $(q_0, \dots, q_{n-1})$ sequences of types from $S_\pi(\mathfrak{C})$. Then $\mathrm{ct}(q_0, \dots, q_{n-1}) \subseteq \mathrm{ct}(p_0, \dots, p_{n-1})$ iff there exists $\eta \in \mathrm{EL}(S_\pi(\mathfrak{C}))$ such that $\eta(p_i) = q_i$ for every $i < n$.* □

As was explained in [16], contents expand the concept of fundamental order of Lascar and Poizat, hence an analogous notion of heir can be defined.

Definition 2.22 ([16, Definition 3.2])**.** Let $M \subseteq A$ and $p(\bar{x}) \in S(A)$. $p(\bar{x})$ is a *strong heir* over M if for every finite $\bar{m} \subseteq M$ and $\varphi(\bar{x}, \bar{a}) \in p(\bar{x})$, where $\bar{a} \subseteq A$ is finite and $\varphi(\bar{x}, \bar{y}) \in L(M)$, there is $\bar{a}' \subseteq M$ such that $\varphi(\bar{x}, \bar{a}') \in p(\bar{x})$ and $\mathrm{tp}(\bar{a}'/\bar{m}) = \mathrm{tp}(\bar{a}/\bar{m})$.

The notion of strong coheir is defined as well, but we will not need it in this paper. The fundamental fact around strong heirs is the following.

Fact 2.23 ([16, Lemma 3.3])**.** *Let $M \subseteq A$ be such that M is $\aleph_0$-saturated. Then every $p(\bar{x}) \in S(M)$ has an extension $p'(\bar{x}) \in S(A)$ which is a strong heir over M.* □

2.6. **Amenability of a theory.**

Amenable and extremely amenable theories were introduced and studied by Hrushovski, Krupiński and Pillay in [13]. We will not give the original definitions but rather the characterizations which we will use in this paper. For the details, the reader should consult [13, Section 4].

Definition 2.24. Let T be a theory.

a) A theory T is *amenable* if every finitary type $p \in S(T)$ is *amenable*, i.e. if there exists an invariant, (regular) Borel probability measure on $S_p(\mathfrak{C})$.
b) A theory T is *extremely amenably* if every finitary type $p \in S(T)$ is *extremely amenable*, i.e. if there exists an invariant type in $S_p(\mathfrak{C})$.

In fact, in the above definitions we can remove the adjective "finitary", and we get the same notions. We also get the same notions if we use only $p := \mathrm{tp}(\bar{c})$ (where $\bar{c}$ is an enumeration of $\mathfrak{C}$), e.g. T is extremely amenable iff there is an invariant type in $S_{\bar{c}}(\mathfrak{C})$.

These definitions do not depend on the choice of the monster model $\mathfrak{C}$, i.e. they are indeed properties of the theory T. In fact, it is enough to assume only that $\mathfrak{C}$ is $\aleph_0$-saturated and strongly $\aleph_0$-homogeneous.

One should also recall that a regular, Borel probability measure on the space $S_p(\mathfrak{C})$ (or on any 0-dimensional compact space) is the same thing as a Keisler measure, i.e. finitely additive probability measure on the Boolean algebra of all clopen sets. All such measures form a compact subspace $\mathfrak{M}_p$ of $[0,1]^{\text{clopens}}$ equipped with the product topology.

3. Some general criteria for profiniteness of the Ellis group

Let us consider the $\mathrm{Aut}(\mathfrak{C})$-flow $(\mathrm{Aut}(\mathfrak{C}), S_{\bar{c}}(\mathfrak{C}))$, where $\bar{c}$ is an enumeration of the monster model $\mathfrak{C}$, and $S_{\bar{c}}(\mathfrak{C})$ is the space of all global types extending $\mathrm{tp}(\bar{c})$. Let $\mathcal{M}$ be any minimal left ideal of $\mathrm{EL}(S_{\bar{c}}(\mathfrak{C}))$ and $u \in \mathcal{J}(\mathcal{M})$. In [19], it is proved that there is a sequence of quotient topological maps and group epimorphisms:

$$u\mathcal{M} \to u\mathcal{M}/H(u\mathcal{M}) \to \mathrm{Gal}_{\mathrm{L}}(T) \to \mathrm{Gal}_{\mathrm{KP}}(T),$$

where $u\mathcal{M}$ is equipped with the τ-topology, $u\mathcal{M}/H(u\mathcal{M})$ with the corresponding quotient topology, and $\mathrm{Gal}_{\mathrm{L}}(T)$ and $\mathrm{Gal}_{\mathrm{KP}}(T)$ with the topologies described in Subsection 2.1. Moreover, one can check that this is also true if we consider the $\mathrm{Aut}(\mathfrak{C})$-flow $(\mathrm{Aut}(\mathfrak{C}), \mathbb{M})$, where $\mathbb{M}$ is a minimal subflow of $S_{\bar{c}}(\mathfrak{C})$, or the $\mathrm{Aut}(\mathfrak{C})$-flow $(\mathrm{Aut}(\mathfrak{C}), S_{\bar{m}}(\mathfrak{C}))$, where $\bar{m}$ is an enumeration of a small model, instead of $(\mathrm{Aut}(\mathfrak{C}), S_{\bar{c}}(\mathfrak{C}))$.

Recall that the Kim-Pillay strong types and Shelah strong types coincide iff $\mathrm{Gal}_{\mathrm{KP}}(T)$ is profinite. The following easy consequence of the aforementioned result of [19] gives us a sufficient condition for profiniteness of $\mathrm{Gal}_{\mathrm{KP}}(T)$.

Proposition 3.1. *Under the above notation, $\mathrm{Gal}_{\mathrm{KP}}(T)$ is profinite if $u\mathcal{M}/H(u\mathcal{M})$ is profinite. Furthermore, $u\mathcal{M}/H(u\mathcal{M})$ is profinite if $u\mathcal{M}$ is 0-dimensional.*

Proof. Both assertions are the consequences of a more general fact: If G is a compact semitopological group, H is a Hausdorff topological group, and $f : G \to H$ is a quotient topological map and group epimorphism, then H is profinite if G is 0-dimensional.

To prove it, first note that H is compact as a continuous image of a compact space G, so we have to justify 0-dimensionality of H. It is enough to prove that the image of a clopen set in G is clopen in H, as then the images of a basis consisting of clopens in G form a basis consisting of clopens in H. Let $C \subseteq G$ be a clopen. Since f is a continuous map from a compact space to a Hausdorff space, it is closed, so $f[C]$ is closed. To see that $f[C]$ is open, it is enough to prove that $f^{-1}[f[C]]$ is open, as f is a quotient topological map. Since f is a group homomorphism, $f^{-1}[f[C]] = C\ker(f) = \bigcup_{a\in\ker(f)} Ca$. Also, all sets Ca are open, because G is a semitopological group. Therefore, $f^{-1}[f[C]]$ is open, and we are done. □

In the next remark, one can work with an arbitrary flow (G, X), a minimal left ideal $\mathcal{M} \lhd \mathrm{EL}(X)$ and an idempotent $u \in \mathcal{J}(\mathcal{M})$.

Remark 3.2. a) If $u\mathcal{M}/H(u\mathcal{M})$ is profinite, then $u\mathcal{M}$ is profinite iff it is Hausdorff.
b) $u\mathcal{M}$ is 0-dimensional iff it is profinite.

Proof. (i) ($\Rightarrow$) is trivial. The converse holds, as $u\mathcal{M}$ is Hausdorff iff $H(u\mathcal{M})$ is trivial; indeed, this implies that if $u\mathcal{M}$ is Hausdorff, then $u\mathcal{M} = u\mathcal{M}/H(u\mathcal{M})$ is profinite.

(ii) ($\Leftarrow$) is trivial. For ($\Rightarrow$) note that $H(u\mathcal{M}) = \bigcap_U \mathrm{cl}_\tau(U)$, where the intersection is taken over all τ-open neighbourhoods U of u. So $H(u\mathcal{M}) \subseteq \bigcap_U U$, where the intersection is taken over all τ-clopen neighbourhoods of u. Since $u\mathcal{M}$ is T_1 and 0-dimensional, the previous intersection is just $\{u\}$, so $H(u\mathcal{M})$ is trivial, i.e. $u\mathcal{M}$ is Hausdorff. Hence, $u\mathcal{M} = u\mathcal{M}/H(u\mathcal{M})$ is a compact, Hausdorff, 0-dimensional topological group (see Subsection 2.2), so it is profinite. □

By Proposition 3.1 and Remark 3.2, for profiniteness of $\mathrm{Gal}_{\mathrm{KP}}(T)$ it is enough to prove profiniteness (equiv. 0-dimensionality) of either $u\mathcal{M}$ or $u\mathcal{M}/H(u\mathcal{M})$. We now investigate these questions in a general setting of arbitrary flows. So, let us fix a flow (G, X), a minimal left ideal $\mathcal{M} \lhd \mathrm{EL}(X)$ and an idempotent $u \in \mathcal{J}(\mathcal{M})$. We consider the closure $\mathrm{cl}(u\mathcal{M})$ of the Ellis group $u\mathcal{M}$ in the topology inherited from $\mathrm{EL}(X)$.

Fact 3.3 ([19, Lemma 3.1])**.** *The map $F : \mathrm{cl}(u\mathcal{M}) \to u\mathcal{M}/H(u\mathcal{M})$ given by $F(x) = ux/H(u\mathcal{M})$ is continuous.* □

Remark 3.4. The map F from Fact 3.3 is a semigroup homomorphism and a closed quotient topological map.

Proof. F is a homomorphism, as $F(x)F(y) = (ux/H(u\mathcal{M}))(uy/H(u\mathcal{M})) = uxuy/H(u\mathcal{M}) = uxy/H(u\mathcal{M}) = F(xy)$ (where we used that $xu = x$). The second part follows from the fact that $\mathrm{cl}(u\mathcal{M})$ is compact, $u\mathcal{M}/H(u\mathcal{M})$ is Hausdorff and F is a continuous surjection. □

Lemma 3.5. *$\mathrm{cl}(u\mathcal{M})$ is a union of Ellis groups. In particular, it is a semigroup.*

Proof. Note that for every $\eta \in \mathcal{M}$, $\eta\mathcal{M}$ is an Ellis group. Namely, $\eta \in v\mathcal{M}$ for some $v \in \mathcal{J}(\mathcal{M})$. Thus, $\eta\mathcal{M} \subseteq v\mathcal{M}$, but also $v\mathcal{M} \subseteq \eta\mathcal{M}$, because $v = \eta\eta^{-1}$, where η^{-1} is the inverse of η in the Ellis group $v\mathcal{M}$.

Let $\eta_0 \in \mathrm{cl}(u\mathcal{M})$. By the first paragraph, it suffices to prove that $\eta_0\mathcal{M} \subseteq \mathrm{cl}(u\mathcal{M})$. Since $\eta_0 u = \eta_0$, we have $\eta_0 u\mathcal{M} = \eta_0\mathcal{M}$. Take any $\eta \in \eta_0\mathcal{M}$. Then $\eta = \eta_0\eta'$ for some $\eta' \in u\mathcal{M}$. Since $\eta_0 \in \mathrm{cl}(u\mathcal{M})$, we have that η_0 is the limit point of a net $(\eta_0^i)_i \subseteq u\mathcal{M}$. By left continuity, $\eta = \eta_0\eta' = \lim_i \eta_0^i\eta'$, so $\eta \in \mathrm{cl}(u\mathcal{M})$, as $\eta_0^i\eta' \in u\mathcal{M}$ for all i's. □

By the previous lemma, let us fix $\mathcal{V} \subseteq \mathcal{J}(\mathcal{M})$ such that $\mathrm{cl}(u\mathcal{M}) = \bigcup\{v\mathcal{M} \mid v \in \mathcal{V}\}$. By $C(X)$ we will denote the set of all continuous functions from X to X.

Lemma 3.6. *Let $U \subseteq \mathrm{cl}(u\mathcal{M})$ be open. Then:*

(i) for every $A \subseteq \mathrm{cl}(u\mathcal{M})$, UA is an open subset of $\mathrm{cl}(u\mathcal{M})$;

(ii) for every $\eta \in \mathrm{cl}(u\mathcal{M}) \cap C(X)$, $f_\eta^{-1}[U]$ is an open subset of $\mathrm{cl}(u\mathcal{M})$, where $f_\eta : \mathrm{cl}(u\mathcal{M}) \to \mathrm{cl}(u\mathcal{M})$ is given by $\tau \mapsto \eta\tau$.

Proof. (i) By the previous lemma, UA is contained in $\mathrm{cl}(u\mathcal{M})$. Since multiplication in the Ellis semigroup is left continuous, its restriction to $\mathrm{cl}(u\mathcal{M})$ is left continuous as well, i.e. the map $g_a : \mathrm{cl}(u\mathcal{M}) \to \mathrm{cl}(u\mathcal{M})$ given by $x \mapsto xa$ is continuous for any $a \in \mathrm{cl}(u\mathcal{M})$. Note that g_a has a continuous inverse $g_{a^{-1}}$, where a^{-1} is the inverse of a in the Ellis group $a\mathcal{M}$ (with the neutral $v \in \mathcal{J}(\mathcal{M})$, where $a \in v\mathcal{M}$). Thus, g_a is a homeomorphism of $\mathrm{cl}(u\mathcal{M})$, so Ua is an open subset of $\mathrm{cl}(u\mathcal{M})$, and therefore UA is an open subset of $\mathrm{cl}(u\mathcal{M})$ as a union of open subsets.

(ii) Since $\eta \in C(X)$, the mapping $\mathrm{EL}(X) \to \mathrm{EL}(X)$ given by $\tau \mapsto \eta\tau$ is continuous, and so is its restriction to $\mathrm{cl}(u\mathcal{M})$. $\square$

Lemma 3.7. *Let F be as in Fact 3.3 and f_η as in Lemma 3.6(ii). For any $B \subseteq \mathrm{cl}(u\mathcal{M})$, $F^{-1}[F[B]] = \bigcup_{\substack{b\in B\\ v\in\mathcal{V}}} f_v^{-1}(b)\ker(F)$ holds, where $\ker(F) := \{\eta \in \mathrm{cl}(u\mathcal{M}) \mid F(\eta) = u/H(u\mathcal{M})\}$.*

Proof. First note that $\mathcal{V} \subseteq \ker(F)$, as $F(v) = uv/H(u\mathcal{M}) = u/H(u\mathcal{M})$ for $v \in \mathcal{V}$. Also, $F_{\restriction v\mathcal{M}} : v\mathcal{M} \to u\mathcal{M}/H(u\mathcal{M})$ is a group homomorphism for every $v \in \mathcal{V}$.

($\subseteq$) Let $x \in F^{-1}[F[B]]$, and let $b \in B$ be such that $F(x) = F(b)$. Let $v, v' \in \mathcal{V}$ be such that $b \in v\mathcal{M}$ and $x \in v'\mathcal{M}$. Since $vv'\mathcal{M} = v\mathcal{M}$, write $b = vc$ for $c \in v'\mathcal{M}$. Now, $F(x) = F(b) = F(vc) = F(v)F(c) = F(c)$, so $x \in c\ker(F)$, as $x, c \in v'\mathcal{M}$ and $F_{\restriction v'\mathcal{M}}$ is a group homomorphism. Since $c \in f_v^{-1}(b)$, we conclude that $x \in f_v^{-1}(b)\ker(F)$.

($\supseteq$) Let $x \in f_v^{-1}(b)\ker(F)$ for $b \in B$ and $v \in \mathcal{V}$, and write $x = yk$ for $y \in f_v^{-1}(b)$ and $k \in \ker(F)$. Then $vx = vyk = bk$, so $F(v)F(x) = F(b)F(k)$, i.e. $F(x) = F(b)$. Thus, $x \in F^{-1}[F[B]]$. $\square$

In the following proposition, we distinguish several sufficient conditions for profiniteness of $u\mathcal{M}/H(u\mathcal{M})$. We keep the notation from the previous considerations.

Proposition 3.8. *(i) If X is 0-dimensional and $u\mathcal{M}$ is closed in $\mathrm{EL}(X)$, then $u\mathcal{M}/H(u\mathcal{M})$ is profinite.*

(ii) If there is a continuous $u \in \mathcal{J}(\mathcal{M})$, then $u\mathcal{M}$ is closed.

(iii) If there is $u \in \mathcal{J}(\mathcal{M})$ (or equivalently, in $\mathcal{M}$) with $\mathrm{Im}(u)$ closed, then $u\mathcal{M}$ is closed.

Proof. (i) Assume that X is 0-dimensional and $u\mathcal{M}$ is closed, i.e. $\mathrm{cl}(u\mathcal{M}) = u\mathcal{M}$. Then X^X is 0-dimensional, so $\mathrm{EL}(X)$ and $\mathrm{cl}(u\mathcal{M})$ are 0-dimensional, too. Consider $F : u\mathcal{M} \to u\mathcal{M}/H(u\mathcal{M})$. Since F is a continuous surjection, it is enough to prove that $F[U]$ is clopen for any clopen $U \subseteq u\mathcal{M}$ (in the topology inherited from $\mathrm{EL}(X)$). By Remark 3.4, F is a closed map, so $F[U]$ is closed. To see that $F[U]$ is open, it is enough to show that $F^{-1}[F[U]]$ is open, as F is a quotient topological map by Remark 3.4. By Lemma 3.7, $F^{-1}[F[U]] = \bigcup_{v\in\mathcal{V}} f_v^{-1}[U]\ker(F)$. Since $\mathrm{cl}(u\mathcal{M}) = u\mathcal{M}$, we have $\mathcal{V} = \{u\}$, so $F^{-1}[F[U]] = f_u^{-1}[U]\ker(F)$. The map f_u is the identity map on $u\mathcal{M}$, thus $F^{-1}[F[U]] = U\ker(F)$ and it is open by Lemma 3.6(i).

(ii) If $u \in \mathcal{M}$ is continuous, then the map on $h : \mathrm{EL}(X) \to \mathrm{EL}(X)$ given by $\eta \mapsto u\eta$ is continuous. Since $\mathrm{EL}(X)$ is compact and Hausdorff, h is closed, so $u\mathcal{M} = h[\mathcal{M}]$ is closed.

(iii) Let $u \in \mathcal{J}(\mathcal{M})$ be such that $\mathrm{Im}(u)$ is closed, and let $\eta \in \mathrm{cl}(u\mathcal{M})$ and $x \in X$. We claim that $\eta(x) \in \mathrm{Im}(u)$. Since $\mathrm{Im}(u)$ is closed, it is enough to see that $\eta(x) \in \mathrm{cl}(\mathrm{Im}(u))$. Let $U \subseteq X$ be any open neighbourhood of $\eta(x)$. Then η belongs to the open set $\{\tau \in \mathrm{EL}(X) \mid \tau(x) \in U\}$ in $\mathrm{EL}(X)$, so we can find $\eta' \in \mathcal{M}$ such that $u\eta'(x) \in U$, since $\eta \in \mathrm{cl}(u\mathcal{M})$. But then $u\eta'(x) \in U \cap \mathrm{Im}(u)$, so U meets $\mathrm{Im}(u)$, and thus $\eta(x) \in \mathrm{cl}(\mathrm{Im}(u))$. Since u acts trivially on $\mathrm{Im}(u)$ (as u is an idempotent), we have $u\eta(x) = \eta(x)$ for all $x \in X$, so $\eta = u\eta \in u\mathcal{M}$. Therefore, $\mathrm{cl}(u\mathcal{M}) = u\mathcal{M}$, i.e. $u\mathcal{M}$ is closed.

For the "equivalently" part note that if $\eta \in \mathcal{M}$ is such that $\mathrm{Im}(\eta)$ is closed, then for the unique $u \in \mathcal{J}(\mathcal{M})$ with $\eta \in u\mathcal{M}$ we have that $\mathrm{Im}(u) = \mathrm{Im}(\eta)$ is closed. $\square$

We do not know whether the converse of Proposition 3.8(iii) holds in general, but that is true if in addition we assume that the flow is minimal.

Proposition 3.9. *If (G,X) is a minimal G-flow (i.e. the orbit of each element $x \in X$ is dense in X), then there is $u \in \mathcal{J}(\mathcal{M})$ with $\mathrm{Im}(u)$ closed iff $u\mathcal{M}$ is closed.*

Proof. ($\Rightarrow$) is Proposition 3.8(iii). For the converse assume that $u\mathcal{M}$ is closed. For any $S \subseteq \mathrm{EL}(X)$ and $x \in X$ put $S(x) := \{\eta(x) \mid \eta \in S\}$. Since the evaluation map $\mathrm{EL}(X) \to X$ given by $\eta \mapsto \eta(x)$ is continuous and both spaces are compact Hausdorff, we see that if S is closed in $\mathrm{EL}(X)$, then $S(x)$ is closed in X. Therefore, by minimality of (G,X), if S is a closed left ideal, then $S(x) = X$. In particular, $\mathcal{M}(x) = X$.

Fix any $x \in X$. Since $u\mathcal{M}$ is closed, $u\mathcal{M}(x)$ is closed. We prove that $\mathrm{Im}(u) \subseteq u\mathcal{M}(x)$. Let $y \in \mathrm{Im}(u)$; then $u(y) = y$ (as u is idempotent). Since $\mathcal{M}(x) = X$, we can find $\eta \in \mathcal{M}$ such that $\eta(x) = y$. Then $u\eta \in u\mathcal{M}$ and $u\eta(x) = u(y) = y$, so $y \in u\mathcal{M}(x)$. Hence, indeed $\mathrm{Im}(u) \subseteq u\mathcal{M}(x)$. Since $u\mathcal{M}(x)$ is closed, we have $\mathrm{cl}(\mathrm{Im}(u)) \subseteq u\mathcal{M}(x)$. On the other hand, obviously $u\mathcal{M}(x) \subseteq \mathrm{Im}(u)$, so $\mathrm{cl}(\mathrm{Im}(u)) = \mathrm{Im}(u)$, i.e. $\mathrm{Im}(u)$ is closed. □

Proposition 3.8 can be applied to some more concrete situations, e.g. see Remark 5.3 and Proposition 5.6 as well as various examples in Section 6. However, the methods developed in the following sections yield (under some assumptions) a stronger conclusion that $u\mathcal{M}$ (rather than $u\mathcal{M}/H(u\mathcal{M})$) is profinite. Proposition 3.8 can also be used to deduce profiniteness of $u\mathcal{M}/H(u\mathcal{M})$ for 0-dimensional WAP flows, which we discuss below.

For a G-flow (G,X), let us denote by $C(X,\mathbb{C})$ the set of continuous complex-valued functions on X. The formula $(gf)(x) := f(g^{-1}x)$ defines an action of G on $C(X,\mathbb{C})$. Recall that a function $f \in C(X,\mathbb{C})$ is said to be *WAP (weakly almost periodic)* if its G-orbit is relatively compact in the weak topology on $C(X,\mathbb{C})$, or equivalently in the topology of pointwise convergence on $C(X,\mathbb{C})$. A flow (G,X) is WAP if every $f \in C(X,\mathbb{C})$ is WAP. It is a fact that a flow (G,X) is WAP iff every element of $\mathrm{EL}(X)$ is continuous. More details can be found in [5].

Corollary 3.10. *If (G,X) is WAP and X is 0-dimensional, then $u\mathcal{M}/H(u\mathcal{M})$ is profinite.*

Proof. By WAP, any $u \in \mathcal{J}(\mathcal{M})$ is continuous, so by Proposition 3.8(ii), $u\mathcal{M}$ is closed, so the conclusion follows by Proposition 3.8(i). □

In fact, using [5], one can strengthen the conclusion of the last corollary to saying that $u\mathcal{M}$ is profinite. Namely, by Proposition II.5 of [5], $\mathcal{M}$ is a compact, Hausdorff topological group, so $\mathcal{M} = u\mathcal{M}$. Then profiniteness of $u\mathcal{M}$ follows from the following two lemmas. Indeed, by these lemmas, the τ-topology on $u\mathcal{M} = \mathcal{M}$ coincides with the topology on $\mathcal{M}$ inherited from $EL(X)$ which is profinite by 0-dimensionality of X.

Lemma 3.11. *For any flow (G,X), and $A \subseteq u\mathcal{M}$, the τ-closure $\mathrm{cl}_\tau(A)$ can be described as the set of all limits contained in $u\mathcal{M}$ of nets $(\eta_i a_i)_i$ such that $\eta_i \in \mathcal{M}$, $a_i \in A$ and $\lim_i \eta_i = u$.*

Proof. Consider $a \in \mathrm{cl}_\tau(A)$. Then, by the definition of the τ-topology, there are nets $(g_i)_i \subseteq G$ and $(a_i)_i \subseteq A$ such that $\lim_i g_i = u$ and $\lim_i g_i a_i = a$. Note that $ua_i = a_i$, as

$a_i \in A \subseteq u\mathcal{M}$. Put $\eta_i := g_i u \in \mathcal{M}$ for all i. By left continuity, we have that $\lim_i \eta_i = \lim_i g_i u = (\lim_i g_i)u = uu = u$. Furthermore, $\lim_i \eta_i a_i = \lim_i g_i u a_i = \lim_i g_i a_i = a$.

Conversely, consider any $a \in u\mathcal{M}$ for which there are nets $(\eta_i)_i \subseteq \mathcal{M}$ and $(a_i)_i \subseteq A$ such that $\lim_i \eta_i = u$ and $\lim_i \eta_i a_i = a$. Since each η_i can be approximated by elements of G and the semigroup operation is left continuous, one can find a subnet $(a'_j)_j$ of $(a_i)_i$ and a net $(g_j)_j \subseteq G$ such that $\lim_j g_j = u$ and $\lim_j g_j a'_j = a$, which means that $a \in \mathrm{cl}_\tau(A)$. □

Lemma 3.12. *If E is a topological group, then for any $A \subseteq E$, the closure $\mathrm{cl}(A)$ of A can be described as the set of all limits of nets $(\eta_i a_i)_i$ such that $\eta_i \in E$, $a_i \in A$ and $\lim_i \eta_i = e$ (where e is the neutral element of E).*

Proof. Take $a \in \mathrm{cl}(A)$. Then there is a net $(a_i)_i \subseteq A$ converging to a, and it is enough to put $\eta_i := e$ for all i, because then $\lim \eta_i = e$ and $\lim_i \eta_i a_i = \lim_i a_i = a$.

Conversely, consider any $a \in E$ for which there are nets $(\eta_i)_i \subseteq E$ and $(a_i)_i \subseteq A$ such that $\lim_i \eta_i = e$ and $\lim_i \eta_i a_i = a$. We need to show that $a \in \mathrm{cl}(A)$.

Take any open neighborhood U of a. Since $ea = a$ and the group operation in E is jointly continuous, we have open neighborhoods V_1 and V_2 of e and a, respectively, such that $V_1 V_2 \subseteq U$. Furthermore, we may assume that $(V_1)^{-1} = V_1$. Since $\lim_i \eta_i = e$ and $\lim_i \eta_i a_i = a$, there is an index i_0 such that $\eta_{i_0} \in V_1$ and $\eta_{i_0} a_{i_0} \in V_2$. As $V_1 = (V_1)^{-1}$, $\eta_{i_0}^{-1} \in V_1$. Using $V_1 V_2 \subseteq U$, we conclude that $a_{i_0} = \eta_{i_0}^{-1}(\eta_{i_0} a_{i_0}) \in U$, and so $U \cap A \neq \emptyset$. Therefore, $a \in \mathrm{cl}(A)$. □

It is nowadays folklore that in a model-theoretic context, WAP corresponds to stability. In particular, one can check that T is stable iff $(\mathrm{Aut}(\mathfrak{C}), S_{\bar{c}}(\mathfrak{C}))$ is WAP. Thus, as a conclusion of the previous result, we get that the Ellis group of a stable theory is profinite. (This was already computed directly in [16, Section 6].) This implies the well-known fact that in stable theories $\mathrm{Gal}_{KP}(T)$ is profinite. More generally, one can easily show directly that whenever T satisfies the independence theorem (for forking) over $\mathrm{acl}^{eq}(\emptyset)$ and $\emptyset$ is an extension base, then $\mathrm{Gal}_{KP}(T)$ is profinite. On the other hand, even supersimplicity of T does not imply that the Ellis group is profinite, as we will see in Example 6.12.

4. Definable structural Ramsey theory and topological dynamics

As mentioned in the introduction, in order to find interactions between Ramsey-like properties of a given theory T and dynamical properties of T, one has to impose the appropriate definability conditions on colorings.

For a finite tuple $\bar{a}$ and a subset $C \subseteq \mathfrak{C}$, by $\binom{C}{\bar{a}}$ we denote the set of all realizations of $\mathrm{tp}(\bar{a})$ in C:

$$\binom{C}{\bar{a}} := \{\bar{a}' \in C^{|\bar{a}|} \mid \bar{a}' \equiv \bar{a}\}.$$

If instead of C we have a tuple, say $\bar{d}$, the meaning of $\binom{\bar{d}}{\bar{a}}$ is the same, i.e. it is $\{\bar{a}' \in D^{|\bar{a}|} \mid \bar{a}' \equiv \bar{a}\}$, where D is the set of all coordinates of $\bar{d}$.

For $r < \omega$, a *coloring* of the realizations of $\mathrm{tp}(\bar{a})$ in C into r colors is any mapping $c : \binom{C}{\bar{a}} \to r$. A subset $S \subseteq \binom{C}{\bar{a}}$ is *monochromatic* with respect to c if $c[S]$ is a singleton.

Definition 4.1. a) A coloring $c : \binom{C}{\bar{a}} \to 2^n$ is *definable* if there are formulae with parameters $\varphi_0(\bar{x}), \dots, \varphi_{n-1}(\bar{x})$ such that:

$$c(\bar{a}')(i) = \begin{cases} 1, & \models \varphi_i(\bar{a}') \\ 0, & \models \neg\varphi_i(\bar{a}') \end{cases}$$

for any $\bar{a}' \in \binom{C}{\bar{a}}$ and $i < n$.

b) A coloring $c : \binom{C}{\bar{a}} \to 2^n$ is *externally definable* if there are formulae without parameters $\varphi_0(\bar{x}, \bar{y}), \dots, \varphi_{n-1}(\bar{x}, \bar{y})$ and types $p_0(\bar{y}), \dots, p_{n-1}(\bar{y}) \in S_{\bar{y}}(\mathfrak{C})$ such that:

$$c(\bar{a}')(i) = \begin{cases} 1, & \varphi_i(\bar{a}', \bar{y}) \in p_i(\bar{y}) \\ 0, & \neg\varphi_i(\bar{a}', \bar{y}) \in p_i(\bar{y}) \end{cases}$$

for any $\bar{a}' \in \binom{C}{\bar{a}}$ and $i < n$.

c) If Δ is a set of formulae, then an externally definable coloring c is called an *externally definable Δ-coloring* if all the formulae $\varphi_i(\bar{x}, \bar{y})$'s defining c are taken from Δ.

Remark 4.2. A coloring $c : \binom{C}{\bar{a}} \to 2^n$ is definable iff it is externally definable via realized (in $\mathfrak{C}$) types $p_0(\bar{y}), \dots, p_{n-1}(\bar{y}) \in S_{\bar{y}}(\mathfrak{C})$. $\square$

Remark 4.3. An externally definable coloring $c : \binom{C}{\bar{a}} \to 2^n$ given by $\varphi_0(\bar{x}, \bar{y}), \dots, \varphi_{n-1}(\bar{x}, \bar{y})$ and $p_0(\bar{y}), \dots, p_{n-1}(\bar{y}) \in S_{\bar{y}}(\mathfrak{C})$ can be defined by using n formulae $\psi_0(\bar{x}, \bar{z}), \dots, \psi_{n-1}(\bar{x}, \bar{z})$ and only one type $p(\bar{z}) \in S_{\bar{z}}(\mathfrak{C})$. Similarly, in the definition of the definable coloring we can assume that all formulae $\varphi_0(\bar{x}), \dots, \varphi_{n-1}(\bar{x})$ have the same parameters $\bar{d}$ and then the coloring is externally definable witnessed by the single realized type $p(\bar{y}) := \operatorname{tp}(\bar{d}/\mathfrak{C})$.

Proof. Let $\bar{z} = (\bar{y}_0, \dots, \bar{y}_{n-1})$, where each $\bar{y}_i$ is of length $|\bar{y}|$. Let $p(\bar{z})$ be any completion of $\bigcup_{i<n} p_i(\bar{y}_i)$ and let $\psi_i(\bar{x}, \bar{z}) := \varphi_i(\bar{x}, \bar{y}_i)$. Now, note that $\psi_i(\bar{a}', \bar{z}) \in p(\bar{z})$ iff $\varphi_i(\bar{a}', \bar{y}) \in p_i(\bar{y})$. The second part of the remark is obvious by adding dummy parameters. $\square$

The next remark explains that Definition 4.1 coincides with the usual definition of [externally] definable map from a type-definable set to a compact, Hausdorff space (in our case this space is finite). Recall that a function $f : X \to C$, where X is a type-definable subset of a sufficiently saturated model and C is a compact, Hausdorff space, is said to be *[externally] definable* if the preimages of any two disjoint closed subsets of C can be separated by a relatively [externally] definable subset of X. In particular, if C is finite, then this is equivalent to saying that all fibers of f are relatively [externally] definable subsets of X.

Remark 4.4. Let $n, r < \omega$.

(i) Let $c\colon \binom{\mathfrak{C}}{\bar{a}} \to 2^n$ be a coloring. Then, c is [externally] definable in the sense of Definition 4.1 iff it is [externally] definable in the above sense (i.e. the fibers are relatively [externally] definable subsets of the type-definable set $\binom{\mathfrak{C}}{\bar{a}}$).

(ii) Let $c\colon \binom{\mathfrak{C}}{\bar{a}} \to r$ be an [externally] definable coloring in the above sense. Define $c' : \binom{\mathfrak{C}}{\bar{a}} \to 2^r$ by: $c'(\bar{a}')(i) = 1$ if $c(\bar{a}') = i$, and $c'(\bar{a}')(i) = 0$ if $c(\bar{a}') \neq i$. Then c' is [externally] definable in the sense of Definition 4.1. Also, for all $\bar{a}', \bar{a}'' \in \binom{\mathfrak{C}}{\bar{a}}$ we have $c(\bar{a}') = c(\bar{a}'') \iff c'(\bar{a}') = c'(\bar{a}'')$. $\square$

In consequence, in the whole development below, we could work with [externally] definable colorings into $r < \omega$ (not necessarily a power of 2) colors in the above sense. But it is more convenient to work with Definition 4.1.

4.1. **Ramsey properties.** Motivated by the embedding Ramsey property for Fraïssé structures, we introduce the following natural notion.

Definition 4.5. A theory T has *EERP* (*the elementary embedding Ramsey property*) if for any two finite tuples $\bar{a} \subseteq \bar{b} \subseteq \mathfrak{C}$ and any $r < \omega$ there exists a finite subset $C \subseteq \mathfrak{C}$ such that for any coloring $c : \binom{C}{\bar{a}} \to r$ there exists $\bar{b}' \in \binom{C}{\bar{b}}$ such that $\binom{\bar{b}'}{\bar{a}}$ is monochromatic with respect to c.

At first sight, the above definition depends on the choice of the monster model $\mathfrak{C} \models T$, but we show that this actually is not the case.

Proposition 4.6. *If $\mathfrak{C}$ and $\mathfrak{C}^*$ are two monster models of T, then $\mathfrak{C}$ satisfies the property given in Definition 4.5 iff $\mathfrak{C}^*$ does.*

Proof. Assume that $\mathfrak{C}$ satisfies the property from Definition 4.5. It suffices to prove the same property for a monster $\mathfrak{C}^*$ such that $\mathfrak{C}^* \succ \mathfrak{C}$ or $\mathfrak{C}^* \prec \mathfrak{C}$.

Assume first that $\mathfrak{C}^* \succ \mathfrak{C}$, and consider any finite $\bar{a} \subseteq \bar{b} \subseteq \mathfrak{C}^*$ and $r < \omega$. We can find an elementary copy $\bar{a}_0 \subseteq \bar{b}_0 \subseteq \mathfrak{C}$ of $\bar{a} \subseteq \bar{b}$. By assumption, we can find finite $C \subseteq \mathfrak{C}$ such that for any coloring $c : \binom{C}{\bar{a}_0} \to r$ there exists $\bar{b}' \in \binom{C}{\bar{b}_0}$ such that $\binom{\bar{b}'}{\bar{a}_0}$ is monochromatic with respect to c. Note that $\binom{C}{\bar{a}_0} = \binom{C}{\bar{a}}$, $\binom{C}{\bar{b}_0} = \binom{C}{\bar{b}}$ and $\binom{\bar{b}'}{\bar{a}_0} = \binom{\bar{b}'}{\bar{a}}$. Hence $C \subseteq \mathfrak{C}^*$ witnesses that $\mathfrak{C}^*$ satisfies the desired property.

Assume now that $\mathfrak{C}^* \prec \mathfrak{C}$, and fix again any finite $\bar{a} \subseteq \bar{b} \subseteq \mathfrak{C}^*$ and $r < \omega$. By assumption, we can find $C \subseteq \mathfrak{C}$ such that for any coloring $c : \binom{C}{\bar{a}} \to r$ there exists $\bar{b}' \in \binom{C}{\bar{b}}$ such that $\binom{\bar{b}'}{\bar{a}}$ is monochromatic with respect to c. Let $C^* \subseteq \mathfrak{C}^*$ be a copy of C by an automorphism of $\mathfrak{C}$; we claim that C^* witnesses the desired property of $\mathfrak{C}^*$. There exists an elementary mapping $f : C \to C^*$ which induces the obvious correspondences $\binom{C}{\bar{a}} \to \binom{C^*}{\bar{a}}$ and $\binom{C}{\bar{b}} \to \binom{C^*}{\bar{b}}$; we may denote them by f, too. For any coloring $c^* : \binom{C^*}{\bar{a}} \to r$, the map $c := c^* \circ f$ is a coloring $\binom{C}{\bar{a}} \to r$. By assumption, there is $\bar{b}' \in \binom{C}{\bar{b}}$ such that $\binom{\bar{b}'}{\bar{a}}$ is monochromatic with respect to c. Then $f(\bar{b}') \in \binom{C^*}{\bar{b}}$ and $\binom{f(\bar{b}')}{\bar{a}}$ is monochromatic with respect to c^*. □

Remark 4.7. In the previous proof, we did not exactly need that $\mathfrak{C}$ and $\mathfrak{C}^*$ are monster models; it is enough that they are $\aleph_0$-saturated. So the definition of *EERP* could have been given with respect to any $\aleph_0$-saturated model. □

Remark 4.8. Definition 4.5 generalizes the definition of the embedding Ramsey property for Fraïssé structures in the following sense: If K is a Fraïssé structure which is $\aleph_0$-saturated, then K has the embedding Ramsey property iff $\mathrm{Th}(K)$ has EERP.

Proof. Recall that if K is an $\aleph_0$-saturated Fraïssé structure, then $\mathrm{Th}(K)$ has quantifier elimination, so $\binom{C}{\bar{a}}^{\mathrm{qf}} = \binom{C}{\bar{a}}$ for any finite $\bar{a}, C \subseteq K$. Therefore, we conclude using Fact 2.11 and Remark 4.7. □

In the following lemma, we give an equivalent, more model-theoretic condition for T to have *EERP*, whose proof is standard in Ramsey theory.

Lemma 4.9. *A theory T has EERP iff for any two finite tuples $\bar{a} \subseteq \bar{b} \subseteq \mathfrak{C}$, any $r < \omega$, and any coloring $c : \binom{\mathfrak{C}}{\bar{a}} \to r$ there exists $\bar{b}' \in \binom{\mathfrak{C}}{\bar{b}}$ such that $\binom{\bar{b}'}{\bar{a}}$ is monochromatic with respect to c.*

Proof. $(\Rightarrow)$ is clear. For the converse, suppose that $\bar{a} \subseteq \bar{b} \subseteq \mathfrak{C}$ and $r < \omega$ are such that the conclusion of *EERP* fails. Denote by $\mathcal{F}$ the family of all finite subsets of $\mathfrak{C}$; $\mathcal{F}$ is naturally directed by inclusion. For $C \in \mathcal{F}$ let:

$$\mathcal{K}_C = \left\{ c : \binom{C}{\bar{a}} \to r \mid \left(\forall \bar{b}' \in \binom{C}{\bar{b}}\right) \#c\left[\binom{\bar{b}'}{\bar{a}}\right] > 1 \right\}.$$

By the choice of $\bar{a}$ and $\bar{b}$, each $\mathcal{K}_C$ is non-empty. Also, for any $C \subseteq C'$ there is a mapping $\mathcal{K}_{C'} \to \mathcal{K}_C$ given by the restriction of colorings defined on $\binom{C'}{\bar{a}}$ to those defined on $\binom{C}{\bar{a}}$. Let $\mathcal{K} = \varprojlim_{C \in \mathcal{F}} \mathcal{K}_C$; we have that $\mathcal{K} \neq \emptyset$, as all $\mathcal{K}_C$'s are finite and non-empty, so we can choose $\eta \in \mathcal{K}$. The formula $c(\bar{a}') := \eta(C)(\bar{a}')$, where $C \in \mathcal{F}$ is such that $\bar{a}' \subseteq C$, yields a well-defined coloring $c : \binom{\mathfrak{C}}{\bar{a}} \to r$. Take any $\bar{b}' \in \binom{\mathfrak{C}}{\bar{b}}$, and let $C \in \mathcal{F}$ be such that $\bar{b}' \subseteq C$. Since $\eta(C) \in \mathcal{K}_C$ and $\eta(C) = c_{\restriction \binom{C}{\bar{a}}}$, we have that $\#c[\binom{\bar{b}'}{\bar{a}}] > 1$. Since $\bar{b}' \in \binom{\mathfrak{C}}{\bar{b}}$ was arbitrary, this contradicts our assumption. □

In [18], the class of first order structures with *ERP* (*the embedding Ramsey property*) is introduced. A first order structure M has *ERP* if for any finite $\bar{a} \subseteq \bar{b} \subseteq M$, any $r < \omega$ and any coloring $c : \binom{M}{\bar{a}}^{\mathrm{Aut}} \to r$ there is $\bar{b}' \in \binom{M}{\bar{b}}^{\mathrm{Aut}}$ such that $\binom{\bar{b}'}{\bar{a}}^{\mathrm{Aut}}$ is monochromatic with respect to c. Here, for finite $\bar{a} \subseteq M$ and $C \subseteq M$:

$$\binom{C}{\bar{a}}^{\mathrm{Aut}} := \{\bar{a}' \subseteq C \mid \bar{a}' = f(\bar{a}) \text{ for some } f \in \mathrm{Aut}(M)\}.$$

For Fraïssé structures the next fact is one of the main results from [15], which was later generalized to arbitrary locally finite ultrahomogeneous structures in [30]. The formulation below comes from [18], but it can be checked (by passing to canonical ultrahomogeneous expansions and using an argument as in Fact 2.11) that it is equivalent to the one from [30].

Fact 4.10 ([18, Theorem 3.2])**.** *A first order structure M has ERP iff $\mathrm{Aut}(M)$ is extremely amenable as a topological group.* □

Note that if M is strongly $\aleph_0$-homogeneous, then $\binom{C}{\bar{a}}^{\mathrm{Aut}} = \binom{C}{\bar{a}}$, so if we in addition assume that M is $\aleph_0$-saturated, then by Remark 4.7 and Lemma 4.9, M has *ERP* iff $\mathrm{Th}(M)$ has *EERP*. Hence, by Fact 4.10, we obtain the following corollary.

Corollary 4.11. *For a theory T, $\mathrm{Aut}(M)$ is extremely amenable (as a topological group) for some $\aleph_0$-saturated and strongly $\aleph_0$-homogeneous model $M \models T$ iff it is extremely amenable for all $\aleph_0$-saturated and strongly $\aleph_0$-homogeneous models of T.* □

We now introduce and study two new classes of theories by restricting our considerations to definable and externally definable colorings, which makes the whole subject more general and more model-theoretic.

Definition 4.12. A theory T has *DEERP* (*the definable elementary embedding Ramsey property*) iff for any two finite tuples $\bar{a} \subseteq \bar{b} \subseteq \mathfrak{C}$, any $n < \omega$ and any definable coloring $c : \binom{\mathfrak{C}}{\bar{a}} \to 2^n$ there exists $\bar{b}' \in \binom{\mathfrak{C}}{\bar{b}}$ such that $\binom{\bar{b}'}{\bar{a}}$ is monochromatic with respect to c.

A theory T has *EDEERP* (*the externally definable elementary embedding Ramsey property*) if in the definition above we consider externally definable colorings c.

Proposition 4.13. *The previous definitions do not depend on the choice of the monster (or just an $\aleph_0$-saturated) model, i.e. the introduced notions of DEERP and EDEERP are indeed properties of T.*

Proof. We fix the following notation. For any finite $\bar{a} \subseteq \bar{b}$ in a model of T, let $\bar{x}'$ be some variables corresponding to $\bar{b}$ and denote by $V_{\bar{a},\bar{b}}$ the set of all $\bar{x} \subseteq \bar{x}'$ corresponding to the elementary copies of $\bar{a}$ within $\bar{b}$. Note that $V_{\bar{a},\bar{b}}$ depends only on $\mathrm{tp}(\bar{a})$, $\mathrm{tp}(\bar{b})$ and the choice of $\bar{x}'$.

In order to see the *DEERP* case, it is enough to note that (even assuming only that $\mathfrak{C}$ is $\aleph_0$-saturated) the statement defining *DEERP* inside $\mathfrak{C}$ is equivalent to the following condition: for any finite $\bar{a} \subseteq \bar{b}$ (in any model of T), any formula $\varphi(\bar{x}') \in \mathrm{tp}(\bar{b})$, and any formulae $\varphi_0(\bar{x}, \bar{y}), \dots, \varphi_{n-1}(\bar{x}, \bar{y})$ without parameters with $\bar{x}$ corresponding to $\bar{a}$:

$$T \vdash (\forall \bar{y})(\exists \bar{x}') \left(\varphi(\bar{x}') \wedge \bigwedge_{\bar{x}_1, \bar{x}_2 \in V_{\bar{a},\bar{b}}} \bigwedge_{i<n} (\varphi_i(\bar{x}_1, \bar{y}) \leftrightarrow \varphi_i(\bar{x}_2, \bar{y})) \right).$$

We now turn to *EDEERP*. Suppose that $\mathfrak{C}$ is a monster (or $\aleph_0$-saturated) model which satisfies the property given in the externally definable case of Definition 4.12. It suffices to prove that any monster (or $\aleph_0$-saturated) model $\mathfrak{C}^*$ such that $\mathfrak{C}^* \prec \mathfrak{C}$ or $\mathfrak{C}^* \succ \mathfrak{C}$ satisfies it as well.

Let $\mathfrak{C}^* \prec \mathfrak{C}$. Consider any finite $\bar{a} \subseteq \bar{b} \subseteq \mathfrak{C}^*$, $n < \omega$, and an externally definable coloring $c^* : \binom{\mathfrak{C}^*}{\bar{a}} \to 2^n$ given by formulae $\varphi_0(\bar{x}, \bar{y}), \dots, \varphi_{n-1}(\bar{x}, \bar{y})$ and a type $p^*(\bar{y}) \in S_{\bar{y}}(\mathfrak{C}^*)$ (see Remark 4.3). By Fact 2.23, let $p(\bar{y}) \in S_{\bar{y}}(\mathfrak{C})$ be a strong heir extension of $p^*(\bar{y})$, and let $c : \binom{\mathfrak{C}}{\bar{a}} \to 2^n$ be the externally definable extension of c^* given by $\varphi_0(\bar{x}, \bar{y}), \dots, \varphi_{n-1}(\bar{x}, \bar{y})$ and $p(\bar{y})$. For a color $\varepsilon \in 2^n$ consider the formula $\theta_\varepsilon(\bar{x}', \bar{y}) := \bigwedge_{\bar{x} \in V_{\bar{a},\bar{b}}} \bigwedge_{i<n} \varphi_i(\bar{x}, \bar{y})^{\varepsilon(i)}$. Note that for $\bar{b}' \in \binom{\mathfrak{C}}{\bar{b}}$, $c[\binom{\bar{b}'}{\bar{a}}] = \{\varepsilon\}$ iff $\theta_\varepsilon(\bar{b}', \bar{y}) \in p(\bar{y})$. By assumption, there is a color $\varepsilon \in 2^n$ and $\bar{b}' \in \binom{\mathfrak{C}}{\bar{b}}$ such that $c[\binom{\bar{b}'}{\bar{a}}] = \{\varepsilon\}$, hence $\theta_\varepsilon(\bar{b}', \bar{y}) \in p(\bar{y})$. Since $p^*(\bar{y}) \subseteq p(\bar{y})$ is a strong heir extension, there is $\bar{b}'' \subseteq \mathfrak{C}^*$ such that $\bar{b}'' \equiv \bar{b}'$ and $\theta_\varepsilon(\bar{b}'', \bar{y}) \in p^*(\bar{y})$. But this means that $\bar{b}'' \in \binom{\mathfrak{C}^*}{\bar{b}}$ and $c[\binom{\bar{b}''}{\bar{a}}] = \{\varepsilon\}$, so $\binom{\bar{b}''}{\bar{a}}$ is monochromatic with respect to c, and hence with respect to c^*.

Let now $\mathfrak{C}^* \succ \mathfrak{C}$, and consider any finite $\bar{a} \subseteq \bar{b} \subseteq \mathfrak{C}^*$, $n < \omega$, and an externally definable coloring $c^* : \binom{\mathfrak{C}^*}{\bar{a}} \to 2^n$ given by formulae $\varphi_0(\bar{x}, \bar{y}), \dots, \varphi_{n-1}(\bar{x}, \bar{y})$ and a type $p^*(\bar{y}) \in S_{\bar{y}}(\mathfrak{C}^*)$. Let $p(\bar{y})$ be the restriction of $p^*(\bar{y})$ to $\mathfrak{C}$, and $c : \binom{\mathfrak{C}}{\bar{a}} \to 2^n$ the restriction of c^* (given by $\varphi_0(\bar{x}, \bar{y}), \dots, \varphi_{n-1}(\bar{x}, \bar{y})$ and $p(\bar{y})$). By assumption, there is $\bar{b}' \in \binom{\mathfrak{C}}{\bar{b}}$ such that $\binom{\bar{b}'}{\bar{a}}$ is monochromatic with respect to c, so also with respect to c^*, as $c^*{\restriction_{\binom{\bar{b}'}{\bar{a}}}} = c{\restriction_{\binom{\bar{b}'}{\bar{a}}}}$. $\square$

The following remark describes the connections between the introduced notions. Both implications below are strict: the lack of the converse of the first implication is witnessed by the theory of the random graph (see Example 6.7) or by $T := ACF_0$ with named constants from the algebraic closure of $\mathbb{Q}$ (see example 6.2), and the second one by the theory of a certain random hypergraph (see Example 6.8).

Remark 4.14. For every theory T, *EERP* $\Longrightarrow$ *EDEERP* $\Longrightarrow$ *DEERP*.

Proof. The first implication is obvious by Lemma 4.9, and the second one by Remark 4.2. □

We now turn to dynamical characterizations of theories with $DEERP$ and $EDEERP$. We first deal with $EDEERP$, and then with $DEERP$ by specialization to realized types.

Theorem 4.15. *A theory T has $EDEERP$ iff there exists $\eta \in \mathrm{EL}(S_{\bar{c}}(\mathfrak{C}))$ such that* $\mathrm{Im}(\eta) \subseteq \mathrm{Inv}_{\bar{c}}(\mathfrak{C})$.

Proof. ($\Rightarrow$) Assume that T has $EDEERP$. First, we prove the following claim.

Claim. *Fix any finite $\bar{a} \subseteq \bar{b} \subseteq \mathfrak{C}$, formulae $\varphi_0(\bar{a},\bar{y}),\dots,\varphi_{n-1}(\bar{a},\bar{y})$, and types $p_0,\dots,p_{n-1} \in S_{\bar{c}}(\mathfrak{C})$ (here, $\bar{y}$ is reserved for $\bar{c}$). There exists $\sigma \in \mathrm{Aut}(\mathfrak{C})$ such that for all $\bar{a}' \in \binom{\bar{b}}{\bar{a}}$ and all $i < n$:*

$$\varphi_i(\bar{a},\bar{y}) \in \sigma(p_i) \quad \textit{iff} \quad \varphi_i(\bar{a}',\bar{y}) \in \sigma(p_i).$$

Proof of Claim. Consider the externally definable coloring $c : \binom{\mathfrak{C}}{\bar{a}} \to 2^n$ given by:

$$c(\bar{a}')(i) = \begin{cases} 1, & \varphi_i(\bar{a}',\bar{y}) \in p_i \\ 0, & \neg\varphi_i(\bar{a}',\bar{y}) \in p_i \end{cases}.$$

By $EDEERP$, we can find $\bar{b}' \in \binom{\mathfrak{C}}{\bar{b}}$ such that $\binom{\bar{b}'}{\bar{a}}$ is monochromatic with respect to c. Let $\sigma \in \mathrm{Aut}(\mathfrak{C})$ be such that $\sigma(\bar{b}') = \bar{b}$. For any $i < n$ and $\bar{a}' \in \binom{\bar{b}}{\bar{a}}$: $\sigma^{-1}(\bar{a}), \sigma^{-1}(\bar{a}') \in \binom{\bar{b}'}{\bar{a}}$, so, by monochromaticity, $\varphi_i(\sigma^{-1}(\bar{a}),\bar{y}) \in p_i$ iff $\varphi_i(\sigma^{-1}(\bar{a}'),\bar{y}) \in p_i$, i.e. $\varphi_i(\bar{a},\bar{y}) \in \sigma(p_i)$ iff $\varphi_i(\bar{a}',\bar{y}) \in \sigma(p_i)$. □ Claim

For a fixed $\bar{a}$ consider the family $\mathcal{F}_{\bar{a}}$ of pairs $(\bar{b}, \{(\varphi_i(\bar{y}), p_i)\}_{i<n})$, where $\bar{b} \supseteq \bar{a}$ is finite, $n < \omega$, $p_0,\dots,p_{n-1} \in S_{\bar{c}}(\mathfrak{C})$ and $\varphi_0(\bar{y}),\dots,\varphi_{n-1}(\bar{y}) \in L(\bar{a})$. We order $\mathcal{F}_{\bar{a}}$ naturally by:

$$(\bar{b}, \{(\varphi_i(\bar{y}), p_i)\}_{i<n}) \leqslant (\bar{b}', \{(\varphi'_i(\bar{y}), p'_i)\}_{i<n'})$$

iff $\bar{b} \subseteq \bar{b}'$, $n \leqslant n'$ and $\{(\varphi_i(\bar{y}), p_i)\}_{i<n} \subseteq \{(\varphi'_i(\bar{y}), p'_i)\}_{i<n'}$; clearly, $\mathcal{F}_{\bar{a}}$ is directed by $\leqslant$. Consider a net $(\sigma_f)_{f\in\mathcal{F}_{\bar{a}}}$ of automorphisms, where each σ_f is chosen to satisfy the claim for $\bar{a}$ and $f \in \mathcal{F}_{\bar{a}}$. Let $\eta_{\bar{a}} \in EL(S_{\bar{c}}(\mathfrak{C}))$ be an accumulation point of this net. We claim that for every $L(\bar{a})$-formula $\varphi(\bar{a},\bar{y})$, every type $p \in S_{\bar{c}}(\mathfrak{C})$, and every $\bar{a}' \equiv \bar{a}$ we have:

$$\varphi(\bar{a},\bar{y}) \in \eta_{\bar{a}}(p) \text{ iff } \varphi(\bar{a}',\bar{y}) \in \eta_{\bar{a}}(p).$$

Suppose not, i.e. there are $\varphi(\bar{a},\bar{y})$, p, and $\bar{a}' \equiv \bar{a}$ such that $\varphi(\bar{a},\bar{y}) \in \eta_{\bar{a}}(p)$ and $\neg\varphi(\bar{a}',\bar{y}) \in \eta_{\bar{a}}(p)$. Consider $f_0 := (\bar{a}^\frown\bar{a}', \{(\varphi(\bar{a},\bar{y}), p)\}) \in \mathcal{F}_{\bar{a}}$. By the definition of $\eta_{\bar{a}}$, we can find $f = (\bar{b}, \{(\varphi_i(\bar{y}), p_i)\}_{i<n}) \in \mathcal{F}_{\bar{a}}$ such that $f_0 \leqslant f$, $\varphi(\bar{a},\bar{y}) \in \sigma_f(p)$ and $\neg\varphi(\bar{a}',\bar{y}) \in \sigma_f(p)$, which contradicts the choice of σ_f.

Consider now the family $\mathcal{F}$ of all finite tuples $\bar{a}$, naturally directed by inclusion. Let $\eta \in \mathrm{EL}(S_{\bar{c}}(\mathfrak{C}))$ be an accumulation point of the net $(\eta_{\bar{a}})_{\bar{a}\in\mathcal{F}}$. We claim that $\mathrm{Im}(\eta) \subseteq \mathrm{Inv}_{\bar{c}}(\mathfrak{C})$. If not, we can find $p \in S_{\bar{c}}(\mathfrak{C})$, finite $\bar{a}_0 \equiv \bar{a}_1$, and a formula $\varphi(\bar{x},\bar{y})$ such that $\varphi(\bar{a}_0,\bar{y}) \in \eta(p)$ and $\neg\varphi(\bar{a}_1,\bar{y}) \in \eta(p)$. By the definition of η, there exists $\bar{a} \supseteq \bar{a}_0{}^\frown\bar{a}_1$ such that $\varphi(\bar{a}_0,\bar{y}) \in \eta_{\bar{a}}(p)$ and $\neg\varphi(\bar{a}_1,\bar{y}) \in \eta_{\bar{a}}(p)$. Let $\sigma \in \mathrm{Aut}(\mathfrak{C})$ be such that $\sigma(\bar{a}_0) = \bar{a}_1$; set $\bar{a}' = \sigma(\bar{a}) \equiv \bar{a}$. Consider the formula $\psi(\bar{a},\bar{y}) := \varphi(\bar{a}_0,\bar{y})$ by adding dummy parameters; note that $\psi(\bar{a}',\bar{y}) = \varphi(\bar{a}_1,\bar{y})$. Hence, we have $\bar{a}' \equiv \bar{a}$, $\psi(\bar{a},\bar{y}) \in \eta_{\bar{a}}(p)$, and $\neg\psi(\bar{a}',\bar{y}) \in \eta_{\bar{a}}(p)$, which contradicts the previous paragraph. This finishes the proof of ($\Rightarrow$).

($\Leftarrow$) Let $\eta \in \mathrm{EL}(S_{\bar{c}}(\mathfrak{C}))$ be such that $\mathrm{Im}(\eta) \subseteq \mathrm{Inv}_{\bar{c}}(\mathfrak{C})$. For any finite $\bar{a} \subseteq \bar{b} \subseteq \mathfrak{C}$, $n < \omega$, and an externally definable coloring $c : \binom{\mathfrak{C}}{\bar{a}} \to 2^n$ we need to find $\bar{b}' \in \binom{\mathfrak{C}}{\bar{b}}$ such that $\binom{\bar{b}'}{\bar{a}}$ is monochromatic with respect to c. Suppose that c is given via formulae $\varphi_0(\bar{x}, \bar{y}), \dots, \varphi_{n-1}(\bar{x}, \bar{y})$ and types $p_0, \dots, p_{n-1} \in S_{\bar{c}}(\mathfrak{C})$. Since the $\eta(p_i)$'s are invariant, we have:

$$\bigwedge_{\bar{a}' \in \binom{\bar{b}}{\bar{a}}} (\varphi_i(\bar{a}, \bar{y}) \leftrightarrow \varphi_i(\bar{a}', \bar{y})) \in \eta(p_i)$$

for all $i < n$. This is an open condition on η, so there exists $\sigma \in \mathrm{Aut}(\mathfrak{C})$ such that:

$$\bigwedge_{\bar{a}' \in \binom{\bar{b}}{\bar{a}}} (\varphi_i(\bar{a}, \bar{y}) \leftrightarrow \varphi_i(\bar{a}', \bar{y})) \in \sigma(p_i)$$

for all $i < n$. For $\bar{b}' := \sigma^{-1}(\bar{b})$, we get that $\binom{\bar{b}'}{\bar{a}}$ is monochromatic with respect to c. □

Corollary 4.16. *If T has EDEERP, then any minimal left ideal $\mathcal{M} \lhd \mathrm{EL}(S_{\bar{c}}(\mathfrak{C}))$ is trivial, hence $u\mathcal{M}$ (i.e. the Ellis group of T) and $\mathrm{Gal}_{\mathrm{L}}(T) = \mathrm{Gal}_{\mathrm{KP}}(T)$ are trivial as well.*

Proof. Assume that T has *EDEERP*. Let $\mathcal{M} \lhd \mathrm{EL}(S_{\bar{c}}(\mathfrak{C}))$ be a minimal left ideal and let $\eta_0 \in \mathcal{M}$ be arbitrary. By Theorem 4.15, there exists $\eta \in \mathrm{EL}(S_{\bar{c}}(\mathfrak{C}))$ with $\mathrm{Im}(\eta) \subseteq \mathrm{Inv}_{\bar{c}}(\mathfrak{C})$. Set $\eta_1 = \eta\eta_0$; clearly $\eta_1 \in \mathcal{M}$ and $\mathrm{Im}(\eta_1) \subseteq \mathrm{Inv}_{\bar{c}}(\mathfrak{C})$. Then $\mathrm{Aut}(\mathfrak{C})\eta_1 = \{\eta_1\}$, so $\{\eta_1\}$ is a minimal subflow. Therefore, $\mathcal{M} = \{\eta_1\}$, and so $u\mathcal{M}$ is trivial. Triviality of $\mathrm{Gal}_L(T)$ follows from the existence of an epimoprhism $u\mathcal{M} \to \mathrm{Gal}_L(T)$ found in [19]. □

Theorem 4.17. *A theory T has DEERP iff T is extremely amenable (in the sense of [13]).*

Proof. ($\Rightarrow$) Assume that T has *DEERP*. We have to prove that $\mathrm{Inv}_{\bar{c}}(\mathfrak{C})$ is non-empty. Using *DEERP* and Remark 4.2 in place of *EDEERP*, we repeat the proof of Theorem 4.15($\Rightarrow$) but working everywhere with realized types from $S_{\bar{c}}(\mathfrak{C})$ in place of arbitrary types. Then, the constructed $\eta \in \mathrm{EL}(S_{\bar{c}}(\mathfrak{C}))$ maps all realized types in $S_{\bar{c}}(\mathfrak{C})$ to $\mathrm{Inv}_{\bar{c}}(\mathfrak{C})$. In particular, $\mathrm{Inv}_{\bar{c}}(\mathfrak{C})$ is non-empty.

($\Leftarrow$) Assume that T is extremely amenable. Consider any finite $\bar{a} \subseteq \bar{b} \subseteq \mathfrak{C}$, $n < \omega$, and a definable coloring $c : \binom{\mathfrak{C}}{\bar{a}} \to 2^n$ given by $\varphi_0(\bar{x}, \bar{d}), \dots, \varphi_{n-1}(\bar{x}, \bar{d})$. By extreme amenability of T, we can find an $\mathrm{Aut}(\mathfrak{C})$-invariant type $p \in S_{\bar{d}}(\mathfrak{C})$. Let $\bar{d}^* \models p$ in a bigger monster $\mathfrak{C}^* \succ \mathfrak{C}$. By $\mathrm{Aut}(\mathfrak{C})$-invariance of p, we have:

$$\models \bigwedge_{i<n} \bigwedge_{\bar{a}_0 \in \binom{\bar{b}}{\bar{a}}} \varphi_i(\bar{a}, \bar{d}^*) \leftrightarrow \varphi_i(\bar{a}_0, \bar{d}^*).$$

Let $\bar{a}^*, \bar{b}^* \subseteq \mathfrak{C}^*$ be such that $\mathrm{tp}(\bar{a}^*, \bar{b}^*, \bar{d}) = \mathrm{tp}(\bar{a}, \bar{b}, \bar{d}^*)$, and $\bar{a}', \bar{b}' \subseteq \mathfrak{C}$ such that $\mathrm{tp}(\bar{a}', \bar{b}'/\bar{d}) = \mathrm{tp}(\bar{a}^*, \bar{b}^*/\bar{d})$; then, $\bar{a}\bar{b}\bar{d}^* \equiv \bar{a}'\bar{b}'\bar{d}$. Therefore:

$$\models \bigwedge_{i<n} \bigwedge_{\bar{a}_0 \in \binom{\bar{b}'}{\bar{a}'}} \varphi_i(\bar{a}', \bar{d}) \leftrightarrow \varphi_i(\bar{a}_0, \bar{d}).$$

Since $\binom{\bar{b}'}{\bar{a}'} = \binom{\bar{b}'}{\bar{a}}$, this means that $\binom{\bar{b}'}{\bar{a}}$ is monochromatic with respect to c. □

In fact, the proof of Theorem 4.17 yields more information.

Corollary 4.18. *For a theory T the following conditions are equivalent.*

(i) T has DEERP.

(ii) There is an element $\eta \in \mathrm{EL}(S_{\bar{c}}(\mathfrak{C}))$ mapping all realized types in $S_{\bar{c}}(\mathfrak{C})$ to $\mathrm{Inv}_{\bar{c}}(\mathfrak{C})$.

(iii) T is extremely amenable, that is $\mathrm{Inv}_{\bar{c}}(\mathfrak{C}) \neq \emptyset$. □

Recall that Corollary 4.11 (i.e. the fact that extreme amenability of $\mathrm{Aut}(M)$ does not depend on the choice of the $\aleph_0$-saturated and strongly $\aleph_0$-homogeneous model M) was deduced from Proposition 4.6 or rather Remark 4.7 (i.e. absoluteness of *EERP*) and Fact 4.10. Similarly, Proposition 4.13 together with the observation that in the proofs of Theorems 4.15 and 4.17 it is enough to assume only $\aleph_0$-saturation and strong $\aleph_0$-homogeneity of $\mathfrak{C}$ and consider *EDEERP* [resp. *DEERP*] only in the chosen model $\mathfrak{C}$ yield that both the existence of $\eta \in \mathrm{EL}(S_{\bar{c}}(\mathfrak{C}))$ with $\mathrm{Im}(\eta) \subseteq \mathrm{Inv}_{\bar{c}}(\mathfrak{C})$ as well as extreme amenability of T are independent of the choice of the $\aleph_0$-saturated and strongly $\aleph_0$-homogeneous model $\mathfrak{C}$. On the other hand, the fact that extreme amenability of T is absolute was easily observed directly in [15], so, using the above observation on the proof of Theorem 4.17, we get the first part of Proposition 4.13, i.e. absoluteness of *DEERP* (at least for $\aleph_0$-saturated and strongly $\aleph_0$-homogeneous models).

Note that *DEERP* implies that $\mathrm{Gal_L}(T)$ is trivial, because, by Theorem 4.17, T is extremely amenable and so $\mathrm{Gal_L}(T)$ is trivial by [13, Proposition 4.31]. However, in contrast with *EDEERP*, Examples 6.8 and 6.9 show that a theory with *DEERP* need not have trivial or even finite Ellis group.

In Corollaries 2.15 and 2.17, we saw that in an $(m+1)$-ary theory, in order to compute the Ellis group or to test the existence of an element in the Ellis semigroup with image contained in invariant types, we can restrict ourselves to the $\mathrm{Aut}(\mathfrak{C})$-flow $S_m(\mathfrak{C})$. The next proposition shows a similar behavior of *EDEERP*.

Proposition 4.19. *Suppose that T is $(m+1)$-ary and each $\bar{a} \subseteq \mathfrak{C}$ of length $|\bar{a}| = m$ satisfies the property given in the definition of EDEERP, i.e. for every $\bar{b} \supseteq \bar{a}$, $n < \omega$, and externally definable coloring $c : \binom{\mathfrak{C}}{\bar{a}} \to 2^n$ there is $\bar{b}' \in \binom{\mathfrak{C}}{\bar{b}}$ such that $\binom{\bar{b}'}{\bar{a}}$ is monochromatic with respect to c. Then T has EDEERP.*

Proof. First, we proceed as in the proof of Theorem 4.15. Namely, the claim from there holds for each $\bar{a}$ of length m. So for each $\bar{a}$ of length m we obtain $\eta_{\bar{a}} \in EL(S_{\bar{c}}(\mathfrak{C}))$ such that for every $L(\bar{a})$-formula $\varphi(\bar{a}, \bar{y})$, type $p \in S_{\bar{c}}(\mathfrak{C})$, and $\bar{a}' \equiv \bar{a}$:

$$\varphi(\bar{a}, \bar{y}) \in \eta_{\bar{a}}(p) \text{ iff } \varphi(\bar{a}', \bar{y}) \in \eta_{\bar{a}}(p).$$

The rest of the proof differs from the proof of Theorem 4.15. First, for finitely many $\bar{a}_0, \ldots, \bar{a}_{k-1}$ of length m put $\eta_{\bar{a}_0,\ldots,\bar{a}_{k-1}} := \eta_{\bar{a}_{k-1}} \circ \cdots \circ \eta_{\bar{a}_0}$. By induction on k, we prove that for every $i < k$, $L(\bar{a}_i)$-formula $\varphi(\bar{a}_i, \bar{y})$, type $p \in S_{\bar{c}}(\mathfrak{C})$, and $\bar{a}_i' \equiv \bar{a}_i$ we have:

$$\varphi(\bar{a}_i, \bar{y}) \in \eta_{\bar{a}_0,\ldots,\bar{a}_{k-1}}(p) \text{ iff } \varphi(\bar{a}_i', \bar{y}) \in \eta_{\bar{a}_0,\ldots,\bar{a}_{k-1}}(p).$$

This holds for $k = 1$ by the choice of $\eta_{\bar{a}_0}$. Assume that this holds for k and consider $\eta_{\bar{a}_0,\ldots,\bar{a}_{k-1},\bar{a}_k}$. Fix $i \leqslant k$. If $i = k$, then the desired equivalence follows from the choice of $\eta_{\bar{a}_k}$, as $\eta_{\bar{a}_0,\ldots,\bar{a}_{k-1},\bar{a}_k}(p) = \eta_{\bar{a}_k}(\eta_{\bar{a}_0,\ldots,\bar{a}_{k-1}}(p))$. Let $i < k$. Toward a contradiction, assume that for some $L(\bar{a}_i)$-formula $\varphi(\bar{a}_i, \bar{y})$, type $p \in S_{\bar{c}}(\mathfrak{C})$, and $\bar{a}_i' \equiv \bar{a}_i$ we have $\varphi(\bar{a}_i, \bar{x}) \in \eta_{\bar{a}_0,\ldots,\bar{a}_{k-1},\bar{a}_k}(p)$

but $\neg\varphi(\bar{a}'_i, \bar{x}) \in \eta_{\bar{a}_0,\dots,\bar{a}_{k-1},\bar{a}_k}(p)$. Set $q = \eta_{\bar{a}_0,\dots,\bar{a}_{k-1}}(p)$, so $\varphi(\bar{a}_i, \bar{y}) \in \eta_{\bar{a}_k}(q)$ and $\neg\varphi(\bar{a}'_i, \bar{y}) \in \eta_{\bar{a}_k}(q)$. Take an automorphism σ such that $\varphi(\bar{a}_i, \bar{y}) \in \sigma(q)$ and $\neg\varphi(\bar{a}'_i, \bar{y}) \in \sigma(q)$, i.e. $\varphi(\sigma^{-1}(\bar{a}_i), \bar{y}) \in q$ and $\neg\varphi(\sigma^{-1}(\bar{a}'_i), \bar{y}) \in q$. But, by induction hypothesis, $\varphi(\sigma^{-1}(\bar{a}_i), \bar{y}) \in q$ iff $\varphi(\bar{a}_i, \bar{y}) \in q$, and $\neg\varphi(\sigma^{-1}(\bar{a}'_i), \bar{y}) \in q$ iff $\neg\varphi(\bar{a}_i, \bar{y}) \in q$, because $\sigma^{-1}(\bar{a}_i) \equiv \bar{a}_i \equiv \sigma^{-1}(\bar{a}'_i)$. This is a contradiction.

Let $\mathcal{F}$ be the family of all finite subsets of tuples of length m, naturally directed by inclusion. For each $S = \{\bar{a}_0, \dots, \bar{a}_{k-1}\} \in \mathcal{F}$ define η_S to be $\eta_{\bar{a}_0,\dots,\bar{a}_{k-1}}$ (we do not care in which order the composition is taken). Take η to be an accumulation point of $(\eta_S)_{S\in\mathcal{F}}$. We claim that $\mathrm{Im}(\eta) \subseteq \mathrm{Inv}_{\bar{c}}(\mathfrak{C})$, and thus T has *EDEERP* by Theorem 4.15, which will complete the proof.

Let $p \in S_{\bar{c}}(\mathfrak{C})$ be any type, $\varphi(\bar{b}, \bar{y})$ be a formula, and $\bar{b}' \equiv \bar{b}$. By $(m+1)$-arity of T, $\varphi(\bar{x}, \bar{y})$ is a Boolean combination of formulae with at most $m+1$ free variables. By using dummy variables if necessary, each of the members of this Boolean combination is of the form $\psi(\bar{x}_0, \bar{y}_0)$, where $|\bar{x}_0| + |\bar{y}_0| = m+1$, $\bar{x}_0 \subseteq \bar{x}$, and $\bar{y}_0 \subseteq \bar{y}$. If $|\bar{y}_0| = 0$, then clearly $\models \psi(\bar{b}_0)$ iff $\models \psi(\bar{b}'_0)$, where $\bar{b}_0 \subseteq \bar{b}$ and $\bar{b}'_0 \subseteq \bar{b}'$ are subtuples corresponding to $\bar{x}_0 \subseteq \bar{x}$. So let $|\bar{y}_0| \geqslant 1$; then $|\bar{x}_0| \leqslant m$, and by using dummy variables we may write $\psi(\bar{x}_0, \bar{y}_0)$ as $\theta(\bar{x}_1, \bar{y}_0)$, where $\bar{x}_1 \supseteq \bar{x}_0$ and $|\bar{x}_1| = m$.

Assume that $\psi(\bar{b}_0, \bar{y}_0) \in \eta(p)$, where $\bar{b}_0 \subseteq \bar{b}$ corresponds to $\bar{x}_0 \subseteq \bar{x}$. Let $\bar{b}'_0 \subseteq \bar{b}'$ be the subtuple determined by the same correspondence. Let $\bar{a} \supseteq \bar{b}_0$ be of length m, and let $\bar{a}' \supseteq \bar{b}'_0$ be such that $\bar{a}' \equiv \bar{a}$, and both these inclusions correspond to $\bar{x}_1 \supseteq \bar{x}_0$. Then $\theta(\bar{a}, \bar{y}_0) \in \eta(p)$. If $\neg\theta(\bar{a}', \bar{y}_0) \in \eta(p)$, then, by the choice of η, we can find $S \in \mathcal{F}$ containing $\bar{a}$ such that $\theta(\bar{a}, \bar{y}_0) \in \eta_S(p)$ and $\neg\theta(\bar{a}', \bar{y}_0) \in \eta_S(p)$. But, by the discussion above, this is not possible, as $\bar{a} \equiv \bar{a}'$. Thus, $\theta(\bar{a}', \bar{y}_0) \in \eta(p)$, i.e. $\psi(\bar{b}'_0, \bar{y}_0) \in \eta(p)$. Similarly, if $\neg\psi(\bar{b}_0, \bar{y}_0) \in \eta(p)$, then $\neg\psi(\bar{b}'_0, \bar{y}_0) \in \eta(p)$.

Finally, since $\varphi(\bar{b}, \bar{y})$ was a Boolean combination of some formulae of the form $\psi(\bar{b}_0, \bar{y}_0)$, we get that $\varphi(\bar{b}, \bar{y}) \in \eta(p)$ implies $\varphi(\bar{b}', \bar{y}) \in \eta(p)$, and we are done. □

4.2. **Separately finite externally definable elementary embedding Ramsey degree.**

By a direct analogy with the introduced notions of elementary embedding Ramsey properties, one may define the notions of finite elementary embedding Ramsey degrees. We introduce here theories with separately finite *EERdeg* and a weak form of separately finite *EDEERdeg*, as they play an essential role in this paper. In order to state this weak version, we need to use externally definable Δ-colorings (see Definition 4.1).

Definition 4.20. a) A theory T has *separately finite EERdeg* (*separately finite elementary embedding Ramsey degree*) if for any finite tuple $\bar{a}$ there exists $l < \omega$ such that for any finite tuple $\bar{b} \subseteq \mathfrak{C}$ containing $\bar{a}$, $r < \omega$, and coloring $c : \binom{\mathfrak{C}}{\bar{a}} \to r$ there exists $\bar{b}' \in \binom{\mathfrak{C}}{\bar{b}}$ such that $\#c[\binom{\bar{b}'}{\bar{a}}] \leqslant l$.

b) A theory T has *separately finite EDEERdeg* (*separately finite externally definable elementary embedding Ramsey degree*) if for any finite tuple $\bar{a}$ and finite set of formulae Δ there exists $l < \omega$ such that for any finite tuple $\bar{b} \subseteq \mathfrak{C}$ containing $\bar{a}$, $n < \omega$, and externally definable Δ-coloring $c : \binom{\mathfrak{C}}{\bar{a}} \to 2^n$ there exists $\bar{b}' \in \binom{\mathfrak{C}}{\bar{b}}$ such that $\#c[\binom{\bar{b}'}{\bar{a}}] \leqslant l$.

The word "separately" is used here to stress that l depends on $\bar{a}$ (in (b), l also depends on Δ). The least such number will be called the *[externally definable] elementary embedding Ramsey degree* of $\bar{a}$ [with respect to Δ].

By an easy modification of the argument in Lemma 4.9, we see that a theory T has sep. fin. $EERdeg$ iff the following holds: For any finite tuple $\bar{a}$ there exists $l < \omega$ such that for any finite tuple $\bar{b} \subseteq \mathfrak{C}$ and any $r < \omega$ there exists a finite subset $C \subseteq \mathfrak{C}$ such that for any coloring $c : \binom{C}{\bar{a}} \to r$ there exists $\bar{b}' \in \binom{C}{\bar{b}}$ such that $\#c[\binom{\bar{b}'}{\bar{a}}] \leqslant l$. By using this characterization, the same argument as in Proposition 4.6 shows that the property of having sep. fin. $EERdeg$ does not depend on the choice of the monster model, i.e. it is indeed a property of the theory. Moreover, the counterpart of Remark 4.7 holds: the definition of sep. fin. $EERdeg$ may be given with respect to any $\aleph_0$-saturated model (rather than with respect to the monster model). The counterpart of Remark 4.8 also holds, namely

Remark 4.21. Definition 4.20(a) generalizes the definition of sep. fin. embedding Ramsey degree for Fraïssé structures in the following sense: If K is a Fraïssé structure which is $\aleph_0$-saturated, then it has sep. fin. embedding Ramsey degree iff $\mathrm{Th}(K)$ has sep. fin. $EERdeg$. □

The property of having sep. fin. $EDEERdeg$ is absolute, too, i.e. does not depend on the choice of the monster model. To see this, one should follow the lines of the proof of absoluteness of $EDEERP$ in Proposition 4.13 with the following modifications. Fix Δ. For any finite $\bar{a} \subseteq \bar{b}$ consider the same $V_{\bar{a},\bar{b}}$ as in the proof of Proposition 4.13. For any $n < \omega$ and formulae $\varphi_0(\bar{x}, \bar{y}), \dots, \varphi_{n-1}(\bar{x}, \bar{y}) \in \Delta$ consider the formula:

$$\theta(\bar{x}', \bar{y}) := \bigvee_{\substack{V \subseteq V_{\bar{a},\bar{b}} \\ |V|=l}} \bigwedge_{\bar{x}_0 \in V_{\bar{a},\bar{b}}} \bigvee_{\bar{x} \in V} \bigwedge_{i<n} \varphi_i(\bar{x}_0, \bar{y}) \leftrightarrow \varphi_i(\bar{x}, \bar{y}),$$

where l is the $EDEERdeg$ of $\bar{a}$ with respect to Δ, computed in $\mathfrak{C}$. Note that for an externally definable Δ-coloring $c : \binom{\mathfrak{C}}{\bar{a}} \to 2^n$ given by $\varphi_0(\bar{x}, \bar{y}), \dots, \varphi_{n-1}(\bar{x}, \bar{y})$ and $p(\bar{y})$, for $\bar{b}' \in \binom{\mathfrak{C}}{\bar{b}}$ we have $\theta(\bar{b}', \bar{y}) \in p(\bar{y})$ iff $\#c[\binom{\bar{b}'}{\bar{a}}] \leqslant l$. The rest of the proof of Proposition 4.13 goes through.

Note that the diagram in Remark 4.14 expands to:

$$\begin{array}{ccccc} EERP & \Rightarrow & EDEERP & \Rightarrow & DEERP \\ \Downarrow & & \Downarrow & & \\ \text{sep. fin. } EERdeg & \Rightarrow & \text{sep. fin. } EDEERdeg & & \end{array}$$

All the implications written above are strict. The converse of the first vertical implication fails for the random graph; the converse of the second one fails for the hypergraph from Example 6.8; the converse of the lower horizontal implication fails e.g. in ACF_0 (see Example 6.2).

We now prove the counterpart of Theorem 4.15 for sep. finite $EDEERdeg$, which is one of the main results of this paper. For a formula $\varphi(\bar{x}, \bar{y})$ put $\varphi^{\mathrm{opp}}(\bar{y}, \bar{x}) := \varphi(\bar{x}, \bar{y})$. For $\Delta = \{\varphi_i(\bar{x}, \bar{y})\}_{i<k}$ put $\Delta^{\mathrm{opp}} := \{\varphi_i^{\mathrm{opp}}(\bar{y}, \bar{x})\}_{i<k}$.

Theorem 4.22. *A theory T has sep. finite EDEERdeg iff for every $\Delta = \{\varphi_i(\bar{x}, \bar{y})\}_{i<k}$ and $\bar{p} = \{p_j\}_{j<m} \subseteq S_{\bar{y}}(T)$ there exists $\eta \in \mathrm{EL}(S_{\bar{c},\Delta}(\bar{p}))$ such that $\mathrm{Im}(\eta)$ is finite.*

Proof. ($\Rightarrow$) Suppose that T has sep. finite *EDEERdeg*. Consider any $\Delta = \{\varphi_i(\bar{x}, \bar{y})\}_{i<k}$ and $\bar{p} = \{p_j\}_{j<m} \subseteq S_{\bar{y}}(T)$ (where $\bar{x}$ corresponds to $\bar{c}$). Fix $\bar{a}_j \models p_j$ for $j < m$, and put $\bar{a} = (\bar{a}_0, \dots, \bar{a}_{m-1})$. Let $\Delta' := \{\phi_{i,j}(\bar{z}, \bar{x}) \mid i < k, j < m\}$, where $\phi_{i,j}(\bar{z}, \bar{x}) := \varphi_i(\bar{x}, \bar{y}_j)$ and $\bar{z} = (\bar{y}_0, \dots, \bar{y}_{m-1})$ with $\bar{y}_i$ corresponding to $\bar{a}_i$. Let $l < \omega$ be the *EDEERdeg* of $\bar{a}$ with respect to Δ'.

For any $n < \omega$, $q_0, \dots, q_{n-1} \in S_{\bar{c}}(\mathfrak{C})$, and finite $\bar{b} \supseteq \bar{a}$ consider the externally definable Δ'-coloring $c : \binom{\mathfrak{C}}{\bar{a}} \to 2^{kmn}$ given by:

$$c(\bar{a}')(i,j,t) = \begin{cases} 1, & \varphi_i(\bar{x}, \bar{a}'_j) \in q_t \\ 0, & \neg\varphi_i(\bar{x}, \bar{a}'_j) \in q_t \end{cases},$$

where $\bar{a}'_j \subseteq \bar{a}'$ is the subtuple corresponding to $\bar{y}_j$. (Note that c is indeed an externally definable Δ'-coloring with respect to the formulae $\phi_{i,j,t}(\bar{z}, \bar{x}) := \varphi_i(\bar{x}, \bar{y}_j)$ and types $r_{i,j,t}(\bar{x}) := q_t(\bar{x})$ for $i < k$, $j < m$, and $t < n$.) By the choice of l, we can find $\bar{b}' \in \binom{\mathfrak{C}}{\bar{b}}$ such that $\#c[\binom{\bar{b}'}{\bar{a}}] \leqslant l$. Let $\sigma_{\bar{b},\bar{q}} \in \mathrm{Aut}(\mathfrak{C})$ be such that $\sigma_{\bar{b},\bar{q}}(\bar{b}') = \bar{b}$; here $\bar{q}$ denotes $(q_0, \dots, q_{n-1})$. Consider the naturally directed family $\mathcal{F}$ of all pairs $(\bar{b}, \bar{q})$, and let η be an accumulation point of the net $(\sigma_{\bar{b},\bar{q}})_{(\bar{b},\bar{q})\in\mathcal{F}}$ in $\mathrm{EL}(S_{\bar{c},\Delta}(\bar{p}))$.

We claim that $\mathrm{Im}(\eta)$ is finite. Suppose not, i.e. we can find $q_0, q_1, \dots \in S_{\bar{c}}(\mathfrak{C})$ such that the $\eta(\hat{q}_t)$'s are pairwise distinct, where $\hat{q}_t \in S_{\bar{c},\Delta}(\bar{p})$ denotes the restriction of q_t. For each $t \neq t'$ the fact that $\eta(\hat{q}_t) \neq \eta(\hat{q}_{t'})$ is witnessed by one of the formulae $\varphi_0(\bar{x}, \bar{y}), \dots, \varphi_{k-1}(\bar{x}, \bar{y})$ and a realization of one of the types $p_0, \dots, p_{m-1}$. By Ramsey theorem, passing to a subsequence, we can assume that there are $i_0 < k$ and $j_0 < m$ such that for each $t \neq t'$ the fact that $\eta(\hat{q}_t) \neq \eta(\hat{q}_{t'})$ is witnessed by $\varphi_{i_0}(\bar{x}, \bar{y})$ and a realization of p_{j_0}. Let $n > 2^l$, and for $t < t' < n$ denote by $\bar{a}^{t,t'}_{j_0}$ a realization of p_{j_0} such that $\varphi_{i_0}(\bar{x}, \bar{a}^{t,t'}_{j_0}) \in \eta(\hat{q}_t)$ iff $\neg\varphi_{i_0}(\bar{x}, \bar{a}^{t,t'}_{j_0}) \in \eta(\hat{q}_{t'})$. Note that this is an open condition on η, so if we let $\bar{a}^{t,t'}$ to be a copy of $\bar{a}$ extending $\bar{a}^{t,t'}_{j_0}$, then we can find a tuple $(\bar{b}, \bar{q})$ greater than $(\bar{a}^\frown(\bar{a}^{t,t'})_{t<t'<n}, (q_0, \dots, q_{n-1}))$ in $\mathcal{F}$ such that $\varphi_{i_0}(\bar{x}, \bar{a}^{t,t'}_{j_0}) \in \sigma_{\bar{b},\bar{q}}(\hat{q}_t)$ iff $\neg\varphi_{i_0}(\bar{x}, \bar{a}^{t,t'}_{j_0}) \in \sigma_{\bar{b},\bar{q}}(\hat{q}_{t'})$. By the choice of $\sigma_{b,\bar{q}}$, $\bar{b}' = \sigma^{-1}_{\bar{b},\bar{q}}(\bar{b})$ satisfies $\#c[\binom{\bar{b}'}{\bar{a}}] \leqslant l$.

For $\bar{a}' \in \binom{\bar{b}'}{\bar{a}}$ let $S(\bar{a}') := \{t < n \mid \varphi_{i_0}(\bar{x}, \bar{a}'_{j_0}) \in q_t\}$. Since $\#c[\binom{\bar{b}'}{\bar{a}}] \leqslant l$, we have $\#\{S(\bar{a}') \mid \bar{a}' \in \binom{\bar{b}'}{\bar{a}}\} \leqslant l$. Indeed, among $l+1$ copies of $\bar{a}$ in $\bar{b}'$, two of them must have the same c-color, which in particular means that they have the same S-sets. Put $l' = \#\{S(\bar{a}') \mid a' \in \binom{\bar{b}'}{\bar{a}}\}$, so $l' \leqslant l$, and let $\{S(\bar{a}') \mid a' \in \binom{\bar{b}'}{\bar{a}}\} = \{S_0, \dots, S_{l'-1}\}$. Note that the mapping $f : n \to \mathcal{P}(\{S_0, \dots, S_{l'-1}\})$ given by $f(t) = \{S_u \mid u < l', t \in S_u\}$ is injective. Indeed, for $t < t' < n$, by the choice of $\bar{b}$, we have $\bar{a}^{t,t'} \subseteq \bar{b}$, so $\bar{a}' := \sigma^{-1}_{\bar{b},\bar{q}}(\bar{a}^{t,t'}) \subseteq \bar{b}'$. But since $\varphi_{i_0}(\bar{x}, \bar{a}^{t,t'}_{j_0}) \in \sigma_{\bar{b},\bar{q}}(\hat{q}_t)$ iff $\neg\varphi_{i_0}(\bar{x}, \bar{a}^{t,t'}_{j_0}) \in \sigma_{\bar{b},\bar{q}}(\hat{q}_{t'})$, we have $\varphi_{i_0}(\bar{x}, \bar{a}'_{j_0}) \in \hat{q}_t$ iff $\neg\varphi_{i_0}(\bar{x}, \bar{a}'_{j_0}) \in \hat{q}_{t'}$, i.e. $\varphi_{i_0}(\bar{x}, \bar{a}'_{j_0}) \in q_t$ iff $\neg\varphi_{i_0}(\bar{x}, \bar{a}'_{j_0}) \in q_{t'}$, hence $t \in S(\bar{a}')$ iff $t' \notin S(\bar{a}')$. Thus, $S(\bar{a}')$ distinguishes $f(t)$ and $f(t')$, and f is injective. Therefore, $n \leqslant 2^{l'} \leqslant 2^l$, which contradicts the choice of n.

($\Leftarrow$) Suppose now that the right hand side of the theorem holds. Take a finite tuple $\bar{a} \subseteq \mathfrak{C}$ and a finite set of formulae Δ in variables $\bar{x}, \bar{y}$ (where since we will be considering Δ-colorings, we can assume that $\bar{y}$ corresponds to $\bar{c}$); let $\#\Delta = k$ and set $p = \mathrm{tp}(\bar{a})$. By assumption, we may find $\eta \in \mathrm{EL}(S_{\bar{c},\Delta^{\mathrm{opp}}}(p))$ with $\mathrm{Im}(\eta)$ finite, say $\#\mathrm{Im}(\eta) = t$. We claim that $l := 2^{kt}$ satisfies our requirements for $\bar{a}$ and Δ.

Fix a finite $\bar{b}$ containing $\bar{a}$, $n < \omega$, and an externally definable Δ-coloring $c : \binom{\mathfrak{C}}{\bar{a}} \to 2^n$ given by formulae $\varphi_0(\bar{x}, \bar{y}), \ldots, \varphi_{n-1}(\bar{x}, \bar{y}) \in \Delta$ and types $q_0(\bar{y}), \ldots, \bar{q}_{n-1}(\bar{y}) \in S_{\bar{c}}(\mathfrak{C})$. Actually, we may assume that the types $q_i(\bar{y})$'s are from $S_{\bar{c},\Delta^{\mathrm{opp}}}(\mathfrak{C})$ by taking their appropriate restrictions, as the coloring c depends only on these restrictions. For each $\bar{a}' \in \binom{\mathfrak{C}}{\bar{a}}$ find $\varepsilon_{\bar{a}'} \in 2^n$ such that $\varphi_i(\bar{a}', \bar{y})^{\varepsilon_{\bar{a}'}(i)} \in \eta(q_i(\bar{y}))$ for all $i < n$. Note that we have only 2^{kt} possibilities for $\varepsilon_{\bar{a}'}$, as we only have k-many formulae in Δ and t-many types in $\mathrm{Im}(\eta)$ (namely, if $\varphi_i(\bar{x}, \bar{y}) = \varphi_{i'}(\bar{x}, \bar{y})$ and $\eta(q_i(\bar{y})) = \eta(q_{i'}(\bar{y}))$, then, by consistency, we must have $\varepsilon_{\bar{a}'}(i) = \varepsilon_{\bar{a}'}(i')$). Consider the open condition on η given by:

$$\bigwedge_{\bar{a}' \in \binom{\bar{b}}{\bar{a}}} \bigwedge_{i<n} \varphi_i(\bar{a}', \bar{y})^{\varepsilon_{\bar{a}'}(i)} \in \eta(q_i(\bar{y})).$$

Let $\sigma \in \mathrm{Aut}(\mathfrak{C})$ be such that:

$$\bigwedge_{\bar{a}' \in \binom{\bar{b}}{\bar{a}}} \bigwedge_{i<n} \varphi_i(\bar{a}', \bar{y})^{\varepsilon_{\bar{a}'}(i)} \in \sigma(q_i(\bar{y})), \qquad \text{i.e.} \quad \bigwedge_{\bar{a}' \in \binom{\bar{b}}{\bar{a}}} \bigwedge_{i<n} \varphi_i(\sigma^{-1}(\bar{a}'), \bar{y})^{\varepsilon_{\bar{a}'}(i)} \in q_i(\bar{y}).$$

Put $\bar{b}' = \sigma^{-1}(\bar{b})$; we have:

$$\bigwedge_{\bar{a}' \in \binom{\bar{b}'}{\bar{a}}} \bigwedge_{i<n} \varphi_i(\bar{a}', \bar{y})^{\varepsilon_{\bar{a}'}(i)} \in q_i(\bar{y}).$$

But since we only had 2^{kt} possibilities for ε's, this exactly means that $\#c[\binom{\bar{b}'}{\bar{a}}] \leqslant 2^{kt}$. $\square$

In fact, a slight elaboration on our proof yields concrete bounds.

Corollary 4.23. *(1) Let $\Delta = \{\varphi_i(\bar{x}, \bar{y})\}_{i<k}$ and $\bar{p} = \{p_j\}_{j<m} \subseteq S_{\bar{y}}(T)$. Fix $\bar{a}_j \models p_j$ for $j < m$, and put $\bar{a} = (\bar{a}_0, \ldots, \bar{a}_{m-1})$. Let $\Delta' := \{\phi_{i,j}(\bar{z}, \bar{x}) \mid i < k, j < m\}$, where $\phi_{i,j}(\bar{z}, \bar{x}) := \varphi_i(\bar{x}, \bar{y}_j)$ and $\bar{z} = (\bar{y}_0, \ldots, \bar{y}_{m-1})$. Assume $l < \omega$ is the EDEERdeg of $\bar{a}$ with respect to Δ'. Then there exists $\eta \in \mathrm{EL}(S_{\bar{c},\Delta}(\bar{p}))$ such that $\#\mathrm{Im}(\eta) \leqslant 2^{mkl}$.*

(2) Let $\bar{a}$ be a finite tuple and $\Delta = \{\varphi_i(\bar{x}, \bar{y})\}_{i<k}$ (where without loss $\bar{y}$ corresponds to $\bar{c}$). Put $p = \mathrm{tp}(\bar{a})$. Assume that there is $\eta \in \mathrm{EL}(S_{\bar{c},\Delta^{opp}}(p))$ with $\#\mathrm{Im}(\eta) = t < \omega$. Then the EDEERdeg of $\bar{a}$ with respect to Δ is bounded by 2^{kt}.

Proof. The second item was obtained in the proof of ($\Leftarrow$) in Theorem 4.22. To get (1), one needs to elaborate on the proof of ($\Rightarrow$) as follows. We do not use Ramsey theorem. Instead, suppose we have n types $q_0, \ldots q_{n-1} \in S_{\bar{c}}(\mathfrak{C})$ with the $\eta(\hat{q}_t)$'s pairwise distinct. Then for any $t < t' < n$ we can find $i(t,t') < k$, $j(t,t') < m$, and a realization $\bar{a}^{t,t'}_{j(t,t')}$ of $p_{j(t,t')}$ such that $\varphi_{i(t,t')}(\bar{x}, \bar{a}^{t,t'}_{j(t,t')}) \in \eta(\hat{q}_t)$ iff $\neg\varphi_{i(t,t')}(\bar{x}, \bar{a}^{t,t'}_{j(t,t')}) \in \eta(\hat{q}_{t'})$. Then we continue with obvious adjustments until the definition of $S(\bar{a}')$. Next, for any $\bar{a}' \in \binom{\bar{b}'}{\bar{a}}$, $i < k$, and $j < m$ put $S(\bar{a}')^{i,j} := \{t < n \mid \varphi_i(\bar{x}, \bar{a}'_j) \in q_t\}$; define $S(\bar{a}') := (S(\bar{a}')^{i,j})_{i<k,\, j<m}$. Then still $\#\{S(\bar{a}') \mid \bar{a}' \in \binom{\bar{b}'}{\bar{a}}\} \leqslant l$, so $\{S(\bar{a}') \mid a' \in \binom{\bar{b}'}{\bar{a}}\} = \{S_0, \ldots, S_{l'-1}\}$ for some $l' \leqslant l$. Let $f : n \to (2^{k\times m})^{l'}$ be defined by saying that $f(t)$ is the unique function $\delta : l' \to 2^{k\times m}$ such that for all $u < l'$, $i < k$, $j < m$ one has $\delta(u)(i,j) = 1 \iff t \in S^{i,j}_u$, where $S^{i,j}_u$ is the (i,j)-th coordinate of S_u. As before, one shows that f is injective, which clearly implies that $n \leqslant 2^{kml}$. $\square$

The fact that the property of having sep. fin. *EDEERdeg* does not depend on the choice of $\mathfrak{C}$ together with the fact that in the proof of Theroem 4.22 we work in the given $\mathfrak{C}$ imply that the right hand side of Theorem 4.22 does not depend on the choice of $\mathfrak{C}$, too. By examining the above proofs, one can also see that one can assume here only that $\mathfrak{C}$ is $\aleph_0$-saturated and strongly $\aleph_0$-homogeneous.

The main consequence of Theorem 4.22 will be Corollary 5.1 which leads to many examples of theories with profinite Ellis group.

4.3. **Elementary embedding convex Ramsey property.**

Recall that in Corollary 4.11 we saw that extreme amenability of the group of automorphisms (as a topological group) of an $\aleph_0$-saturated and strongly $\aleph_0$-homogeneous model of a theory is a property of the theory, in the sense that some $\aleph_0$-saturated and strongly $\aleph_0$-homogeneous model has extremely amenable automorphism group iff all $\aleph_0$-saturated and strongly $\aleph_0$-homogeneous models do. We aim now to prove the counterpart of this result for amenability of automorphism groups.

In [22], Moore characterized amenability of the automorphism group of a Fraïssé structure via the so-called convex Ramsey property. This was generalized to arbitrary structures in [18], where the notion of *ECRP* (the *embedding convex Ramsey property*) for a first order structure is introduced. A structure M has *ECRP* if for any $\epsilon > 0$, any two finite tuples $\bar{a} \subseteq \bar{b} \subseteq M$, any $n < \omega$, and any coloring $c : \binom{M}{\bar{a}}^{\mathrm{Aut}} \to 2^n$ there exist $k < \omega$, $\lambda_0, \dots, \lambda_{k-1} \in [0, 1]$ with $\lambda_0 + \cdots + \lambda_{k-1} = 1$, and $\sigma_0, \dots, \sigma_{k-1} \in \mathrm{Aut}(M)$ such that for any two tuples $\bar{a}', \bar{a}'' \in \binom{\bar{b}}{\bar{a}}^{\mathrm{Aut}}$ we have:

$$\max_{i<n} \Big| \sum_{j<k} \lambda_j c(\sigma_j(\bar{a}'))(i) - \sum_{j<k} \lambda_j c(\sigma_j(\bar{a}''))(i) \Big| \leqslant \epsilon.$$

M has the *strong ECRP* if the previous holds for $\epsilon = 0$.

Recall that for finite $\bar{a} \subseteq M$ and $C \subseteq M$:

$$\binom{C}{\bar{a}}^{\mathrm{Aut}} := \{\bar{a}' \subseteq M \mid \bar{a}' = f(\bar{a}) \text{ for some } f \in \mathrm{Aut}(M)\}.$$

Also, recall that if M is strongly $\aleph_0$-homogeneous, then $\binom{C}{\bar{a}}^{\mathrm{Aut}} = \binom{C}{\bar{a}}$. The following fact was proved in [18].

Fact 4.24. [18, Theorem 4.3] *Let M be a first-order structure. TFAE:*

(1) $\mathrm{Aut}(M)$ *is amenable as a topological group;*

(2) M has ECRP;

(3) M has strong ECRP. □

By analogy with *EERP*, we have the following definition.

Definition 4.25. A theory T has *EECRP* (the *elementary embedding convex Ramsey property*) if it has a monster model $\mathfrak{C}$ with *ECRP*.

We will prove that we could have equivalently asked in Definition 4.25 that some (equivalently, every) $\aleph_0$-saturated and $\aleph_0$-strongly homogeneous model of T has *ECRP*. For this we will need the following characterization of *ECRP* models, which can be proved by a standard argument (with the inverse limit of "bad" colorings), very similarly to Lemma 4.9. The details of the proof are left to the reader.

Lemma 4.26. *If M is an $\aleph_0$-saturated and strongly $\aleph_0$-homogeneous model, then M has* ECRP *iff for every $\epsilon > 0$, any two finite tuples $\bar{a} \subseteq \bar{b} \subseteq M$, and any $n < \omega$ there exists a finite subset $C \subseteq M$ such that for any coloring $c : \binom{C}{\bar{a}} \to 2^n$ there exist $k < \omega$, $\lambda_0, \dots, \lambda_{k-1} \in [0, 1]$ with $\lambda_0 + \dots + \lambda_{k-1} = 1$, and $\sigma_0, \dots, \sigma_{k-1} \in \mathrm{Aut}(M)$ such that $\sigma_0(\bar{b}), \dots, \sigma_{k-1}(\bar{b}) \subseteq C$ and for any two tuples $\bar{a}', \bar{a}'' \in \binom{\bar{b}}{\bar{a}}$ we have:*

$$\max_{i<n} |\sum_{j<k} \lambda_j c(\sigma_j(\bar{a}'))(i) - \sum_{j<k} \lambda_j c(\sigma_j(\bar{a}''))(i)| \leqslant \epsilon. \qquad \square$$

Lemma 4.27. *If M and M^* are two elementarily equivalent $\aleph_0$-saturated and strongly $\aleph_0$-homogeneous models, then M has* ECRP *iff M^* does.*

Proof. Assume that M has *ERCP*. It is enough to prove that M^* has *ECRP* for $M^* \prec M$ and for $M^* \succ M$.

Let $M^* \prec M$. Fix $\epsilon > 0$, finite $\bar{a} \subseteq \bar{b} \subseteq M^*$, and $n < \omega$. By Lemma 4.26, we can find a finite $C \subseteq M$ such that for any coloring $c : \binom{C}{\bar{a}} \to 2^n$ there exist $k < \omega$, $\lambda_0, \dots, \lambda_{k-1} \in [0, 1]$ with $\lambda_0 + \dots + \lambda_{k-1} = 1$, and $\sigma_0, \dots, \sigma_{k-1} \in \mathrm{Aut}(M)$ such that $\sigma_0(\bar{b}), \dots, \sigma_{k-1}(\bar{b}) \subseteq C$ and for any two tuples $\bar{a}', \bar{a}'' \in \binom{\bar{b}}{\bar{a}}$ we have:

$$\max_{i<n} |\sum_{j<k} \lambda_j c(\sigma_j(\bar{a}'))(i) - \sum_{j<k} \lambda_j c(\sigma_j(\bar{a}''))(i)| \leqslant \epsilon.$$

By $\aleph_0$-saturation of M^*, we can find an elementary copy $C^* \subseteq M^*$ of C; let $f : C \to C^*$ be a partial elementary map. We claim that C^* satisfies the property given in Lemma 4.26. Fix a coloring $c^* : \binom{C^*}{\bar{a}} \to 2^n$. Consider the coloring $c : \binom{C}{\bar{a}} \to 2^n$ given by $c(\bar{a}') = c^*(f(\bar{a}'))$. For this coloring choose $k < \omega$, $\lambda_0, \dots, \lambda_{k-1} \in [0, 1]$ such that $\lambda_0 + \dots + \lambda_{k-1} = 1$, and $\sigma_0, \dots, \sigma_{k-1} \in \mathrm{Aut}(M)$ such that $\sigma_0(\bar{b}), \dots, \sigma_{k-1}(\bar{b}) \subseteq C$ and for any two tuples $\bar{a}', \bar{a}'' \in \binom{\bar{b}}{\bar{a}}$ we have:

$$\max_{i<n} |\sum_{j<k} \lambda_j c(\sigma_j(\bar{a}'))(i) - \sum_{j<k} \lambda_j c(\sigma_j(\bar{a}''))(i)| \leqslant \epsilon.$$

By strong $\aleph_0$-homogeneity of M^*, for every $j < k$ there is $\sigma_j^* \in \mathrm{Aut}(M^*)$ such that $\sigma_j^*(\bar{b}) = f(\sigma_j(\bar{b}))$; note that $\sigma_0^*(\bar{b}), \dots, \sigma_{k-1}^*(\bar{b}) \subseteq C^*$. For any $\bar{a}' \in \binom{\bar{b}}{\bar{a}}$, $c^*(\sigma_j^*(\bar{a}')) = c^*(f(\sigma_j(\bar{a}'))) = c(\sigma_j(\bar{a}'))$, so for any two tuples $\bar{a}', \bar{a}'' \in \binom{\bar{b}}{\bar{a}}$ we have:

$$\max_{i<n} |\sum_{j<k} \lambda_j c^*(\sigma_j^*(\bar{a}'))(i) - \sum_{j<k} \lambda_j c^*(\sigma_j^*(\bar{a}''))(i)| \leqslant \epsilon.$$

This finishes the proof of the first case.

Assume now that $M^* \succ M$. Fix $\epsilon > 0$, finite $\bar{a}^* \subseteq \bar{b}^* \subseteq M^*$, and $n < \omega$. By $\aleph_0$-saturation of M, we can find an elementary copy $\bar{a} \subseteq \bar{b} \subseteq M$ of $\bar{a}^* \subseteq \bar{b}^*$; let f be a partial elementary map such that $f(\bar{b}^*) = \bar{b}$ and $f(\bar{a}^*) = \bar{a}$. By Lemma 4.26, there is a finite $C \subseteq M$ such that for any coloring $c : \binom{C}{\bar{a}} \to 2^n$ there exist $k < \omega$, $\lambda_0, \dots, \lambda_{k-1} \in [0, 1]$ with $\lambda_0 + \dots + \lambda_{k-1} = 1$, and $\sigma_0, \dots, \sigma_{k-1} \in \mathrm{Aut}(M)$ such that $\sigma_0(\bar{b}), \dots, \sigma_{k-1}(\bar{b}) \subseteq C$

and for any two tuples $\bar{a}', \bar{a}'' \in \binom{\bar{b}}{\bar{a}}$ we have:

$$\max_{i<n} |\sum_{j<k} \lambda_j c(\sigma_j(\bar{a}'))(i) - \sum_{j<k} \lambda_j c(\sigma_j(\bar{a}''))(i)| \leqslant \epsilon.$$

We claim that C is also good for M^*. Note that $\binom{C}{\bar{a}^*} = \binom{C}{\bar{a}}$, so for a coloring $c^* : \binom{C}{\bar{a}^*} \to 2^n$ we can find $k < \omega$, $\lambda_0, \dots, \lambda_{k-1} \in [0,1]$ with $\lambda_0 + \dots + \lambda_{k-1} = 1$, and $\sigma_0, \dots, \sigma_{k-1} \in \mathrm{Aut}(M)$ such that $\sigma_0(\bar{b}), \dots, \sigma_{k-1}(\bar{b}) \subseteq C$ and for any two tuples $\bar{a}', \bar{a}'' \in \binom{\bar{b}}{\bar{a}}$ we have:

$$\max_{i<n} |\sum_{j<k} \lambda_j c^*(\sigma_j(\bar{a}'))(i) - \sum_{j<k} \lambda_j c^*(\sigma_j(\bar{a}''))(i)| \leqslant \epsilon.$$

By strong $\aleph_0$-homogeneity of M^*, find $\sigma_0^*, \dots, \sigma_{k-1}^* \in \mathrm{Aut}(M^*)$ such that $\sigma_j^*(\bar{b}^*) = \sigma_j(\bar{b})$ for all $j < k$; then $\sigma_j^* = \sigma_j \circ f$ on $\bar{b}^*$ and $\sigma_0^*(\bar{b}^*), \dots, \sigma_{k-1}^*(\bar{b}^*) \subseteq C$. For any two tuples $\bar{a}'^*, \bar{a}''^* \in \binom{\bar{b}^*}{\bar{a}^*}$ we have $f(\bar{a}'^*), f(\bar{a}''^*) \in \binom{\bar{b}}{\bar{a}}$, hence:

$$\max_{i<n} |\sum_{j<k} \lambda_j c^*(\sigma_j(f(\bar{a}'^*)))(i) - \sum_{j<k} \lambda_j c^*(\sigma_j(f(\bar{a}''^*)))(i)| \leqslant \epsilon.$$

Since $\sigma_j(f(\bar{a}'^*)) = \sigma_j^*(\bar{a}'^*)$ for each $\bar{a}'^* \in \binom{\bar{b}^*}{\bar{a}^*}$ and $j < k$, the desired conclusion follows. □

Directly by Lemma 4.27 and Fact 4.24, we get the main conclusion of Subsection 4.3.

Corollary 4.28. *For a theory T, $\mathrm{Aut}(M)$ is amenable (as a topological group) for some $\aleph_0$-saturated and strongly $\aleph_0$-homogeneous model $M \models T$ iff it is amenable for all $\aleph_0$-saturated and strongly $\aleph_0$-homogeneous models of T.* □

4.4. **Definable elementary embedding convex Ramsey property.**

Recall that Theorem 4.17 says that a theory T is extremely amenably iff it has $DEERP$. In this subsection, we will get an analogous result for amenable theories (in the sense of [13]). The corresponding Ramsey property-like condition which describes amenability of a theory is the *definable elementary embedding convex Ramsey property.*

Definition 4.29. A theory T has $DEECRP$ (the *definable elementary embedding convex Ramsey property*) iff for any $\epsilon > 0$, any two finite tuples $\bar{a} \subseteq \bar{b} \subseteq \mathfrak{C}$, any $n < \omega$, and any definable coloring $c : \binom{\mathfrak{C}}{\bar{a}} \to 2^n$ there exist $k < \omega$, $\lambda_0, \dots, \lambda_{k-1} \in [0,1]$ with $\lambda_0 + \dots + \lambda_{k-1} = 1$, and $\sigma_0, \dots, \sigma_{k-1} \in \mathrm{Aut}(\mathfrak{C})$ such that for any two tuples $\bar{a}', \bar{a}'' \in \binom{\bar{b}}{\bar{a}}$ we have:

$$\max_{i<n} |\sum_{j<k} \lambda_j c(\sigma_j(\bar{a}'))(i) - \sum_{j<k} \lambda_j c(\sigma_j(\bar{a}''))(i)| \leqslant \epsilon.$$

A theory T has the *strong* $DEECRP$ if the previous holds for $\epsilon = 0$.

Proposition 4.30. *The previous definition does not depend on the choice of the monster model.*

Proof. First, we show that if $\mathfrak{C} \prec \mathfrak{C}^*$ are monster models and $\mathfrak{C}$ satisfies the property given in Definition 4.29, then $\mathfrak{C}^*$ satisfies it as well. Consider any $\epsilon > 0$, $\bar{a}^* \subseteq \bar{b}^* \subseteq \mathfrak{C}^*$, $n < \omega$, and definable coloring $c^* : \binom{\mathfrak{C}^*}{\bar{a}^*} \to 2^n$ given by $\varphi_0(\bar{x}, \bar{d}^*), \dots, \varphi_{n-1}(\bar{x}, \bar{d}^*)$ for $\bar{d}^* \subseteq \mathfrak{C}^*$. Choose $\bar{a}, \bar{b}, \bar{d} \subseteq \mathfrak{C}$ such that $\mathrm{tp}(\bar{a}, \bar{b}, \bar{d}) = \mathrm{tp}(\bar{a}^*, \bar{b}^*, \bar{d}^*)$ and $\tau \in \mathrm{Aut}(\mathfrak{C}^*)$ with $\tau(\bar{a}, \bar{b}, \bar{d}) = (\bar{a}^*, \bar{b}^*, \bar{d}^*)$, and let $c : \binom{\mathfrak{C}}{\bar{a}} \to 2^n$ be the coloring defined by the formulae

$\varphi_0(\bar{x},\bar{d}),\dots,\varphi_{n-1}(\bar{x},\bar{d})$. By assumption, find $k<\omega$, $\lambda_0,\dots,\lambda_{k-1}\in[0,1]$ with $\lambda_0+\dots+\lambda_{k-1}=1$, and $\sigma_0,\dots,\sigma_{k-1}\in \mathrm{Aut}(\mathfrak{C})$ such that for any two tuples $\bar{a}',\bar{a}''\in\binom{\bar{b}}{\bar{a}}$ we have:

$$\max_{i<n}|\sum_{j<k}\lambda_j c(\sigma_j(\bar{a}'))(i)-\sum_{j<k}\lambda_j c(\sigma_j(\bar{a}''))(i)|\leqslant\epsilon.$$

Let $\binom{\bar{b}}{\bar{a}}=\{\bar{a}_s\}_{s<t}$ and $\bar{a}^*_s=\tau(\bar{a}_s)$ for $s<t$; then $\binom{\bar{b}^*}{\bar{a}^*}=\{\bar{a}^*_s\}_{s<t}$. Also, for each $j<k$ and $s<t$ put $\bar{a}^{*j}_s=\tau(\sigma_j(\bar{a}_s))$. Since for each $j<k$ we have $\mathrm{tp}((\bar{a}^*_s)_{s<t})=\mathrm{tp}((\bar{a}^{*j}_s)_{s<t})$ we can find $\sigma^*_j\in\mathrm{Aut}(\mathfrak{C}^*)$ such that $\sigma^*_j(\bar{a}^*_s)=\bar{a}^{*j}_s$ for every $s<t$. We now have:

$$c^*(\sigma^*_j(\bar{a}^*_s))(i)=1 \text{ iff } c^*(\bar{a}^{*j}_s)(i)=1 \text{ iff } \models\varphi_i(\bar{a}^{*j}_s,\bar{d}^*) \text{ iff } \models\varphi_i(\sigma_j(\bar{a}_s),\bar{d}) \text{ iff } c(\sigma_j(\bar{a}_s))(i)=1,$$

so we see that the desired property in $\mathfrak{C}^*$ is witnessed by k, $\lambda_0,\dots,\lambda_{k-1}$, and $\sigma^*_0,\dots,\sigma^*_{k-1}$.

Suppose now that $\mathfrak{C}^*\prec\mathfrak{C}$ and that the property from Definition 4.29 holds in $\mathfrak{C}$. Take $\epsilon>0$, $\bar{a}\subseteq\bar{b}\subseteq\mathfrak{C}^*$, $n<\omega$, and a definable coloring $c^*:\binom{\mathfrak{C}^*}{\bar{a}}\to 2^n$ given by $\varphi_0(\bar{x},\bar{d}),\dots,\varphi_{n-1}(\bar{x},\bar{d})$ for $\bar{d}\subseteq\mathfrak{C}^*$. The same formulae give a definable extension $c:\binom{\mathfrak{C}}{\bar{a}}\to 2^n$ of c^*. By assumption, we can find $k<\omega$, $\lambda_0,\dots,\lambda_{k-1}\in[0,1]$ with $\lambda_0+\dots+\lambda_{k-1}=1$, and $\sigma_0,\dots,\sigma_{k-1}\in\mathrm{Aut}(\mathfrak{C})$ such that for any two tuples $\bar{a}',\bar{a}''\in\binom{\bar{b}}{\bar{a}}$ we have:

$$\max_{i<n}|\sum_{j<k}\lambda_j c(\sigma_j(\bar{a}'))(i)-\sum_{j<k}\lambda_j c(\sigma_j(\bar{a}''))(i)|\leqslant\epsilon.$$

Again, let $\binom{\bar{b}}{\bar{a}}=\{\bar{a}_s\}_{s<t}$. For each $j<k$ choose $(\bar{a}^j_s)_{s<t}$ in $\mathfrak{C}^*$ such that $\mathrm{tp}((\bar{a}^j_s)_{s<t}/\bar{d})=\mathrm{tp}((\sigma_j(\bar{a}_s))_{s<t}/\bar{d})$ and then find $\sigma^*_j\in\mathrm{Aut}(\mathfrak{C}^*)$ such that $\sigma^*_j((\bar{a}_s)_{s<t})=(\bar{a}^j_s)_{s<t}$. Then:

$$c^*(\sigma^*_j(\bar{a}_s))(i)=1 \text{ iff } c^*(\bar{a}^j_s)(i)=1 \text{ iff } \models\varphi_i(\bar{a}^j_s,\bar{d}) \text{ iff } \models\varphi_i(\sigma_j(\bar{a}_s),\bar{d}) \text{ iff } c(\sigma_j(\bar{a}_s))(i)=1,$$

so we see that the desired property in $\mathfrak{C}^*$ is witnessed by k, $\lambda_0,\dots,\lambda_{k-1}$, and $\sigma^*_0,\dots,\sigma^*_{k-1}$. □

Theorem 4.31. *For a theory T the following conditions are equivalent:*

(1) T is amenable;
(2) T has DEECRP;
(3) T has strong DEECRP.

Proof. (3)⇒(2) is obvious.

(2)⇒(1) Assume that T has *DEECRP*. We have to prove that for any finite $\bar{d}$ there exists an invariant, finitely additive probability measure on the Boolean algebra of clopens in $S_{\bar{d}}(\mathfrak{C})$.

Claim. *Fix $\epsilon>0$, finite $\bar{a}\subseteq\bar{b}\subseteq\mathfrak{C}$, and formulae $\varphi_0(\bar{x},\bar{y}),\dots,\varphi_{n-1}(\bar{x},\bar{y})$ ($\bar{x}$ is reserved for $\bar{a}$, $\bar{y}$ for $\bar{d}$). There exists $\mu\in\mathfrak{M}_{\bar{d}}:=\mathfrak{M}_{\mathrm{tp}(\bar{d})}$ such that for every $\bar{a}',\bar{a}''\in\binom{\bar{b}}{\bar{a}}$:*

$$\max_{i<n}|\mu([\varphi_i(\bar{a}',\bar{y})])-\mu([\varphi_i(\bar{a}'',\bar{y})])|\leqslant\epsilon.$$

Proof of Claim. Consider the definable coloring $c:\binom{\mathfrak{C}}{\bar{a}}\to 2^n$ given by:

$$c(\bar{a}')(i)=\begin{cases}1, & \models\varphi_i(\bar{a}',\bar{d})\\ 0, & \models\neg\varphi_i(\bar{a}',\bar{d})\end{cases}.$$

By *DEECRP*, there are $k < \omega$, $\lambda_0, \dots, \lambda_{k-1} \in [0,1]$ with $\lambda_0 + \dots, \lambda_{k-1} = 1$, and $\sigma_0, \dots, \sigma_{k-1} \in \mathrm{Aut}(\mathfrak{C})$ such that for any two tuples $\bar{a}', \bar{a}'' \in \binom{\bar{b}}{\bar{a}}$ we have:

$$\max_{i<n} |\sum_{j<k} \lambda_j c(\sigma_j(\bar{a}'))(i) - \sum_{j<k} \lambda_j c(\sigma_j(\bar{a}''))(i)| \leqslant \epsilon.$$

Consider $\mu = \lambda_0 \operatorname{tp}(\sigma_0^{-1}(\bar{d})/\mathfrak{C}) + \dots + \lambda_{k-1} \operatorname{tp}(\sigma_{k-1}^{-1}(\bar{d})/\mathfrak{C})$, which is defined by:

$$\mu([\phi(\bar{y})]) := \sum_{j<k} \{\lambda_j \mid \phi(\bar{y}) \in \operatorname{tp}(\sigma_j^{-1}(\bar{d})/\mathfrak{C})\},$$

for $\phi(\bar{y}) \in L(\mathfrak{C})$; it is easy to see that $\mu \in \mathfrak{M}_{\bar{d}}$. For $i < n$, $j < k$, and $\bar{a}' \in \binom{\bar{b}}{\bar{a}}$ we have $c(\sigma_j(\bar{a}'))(i) = 1$ iff $\models \varphi_i(\sigma_j(\bar{a}'), \bar{d})$ iff $\models \varphi_i(\bar{a}', \sigma_j^{-1}(\bar{d}))$ iff $\varphi_i(\bar{a}', \bar{y}) \in \operatorname{tp}(\sigma_j^{-1}(\bar{d})/\mathfrak{C})$. Thus, $\sum_{j<k} \lambda_j c(\sigma_j(\bar{a}'))(i) = \mu([\varphi_i(\bar{a}', \bar{y})])$, so the conclusion of the claim follows. □ Claim

Fix for a moment $\epsilon > 0$ and $\bar{a}$. Consider the family $\mathcal{F}_{\bar{a}}$ of pairs $(\bar{b}, \{\varphi_i(\bar{x}, \bar{y})\}_{i<n})$, where $\bar{b} \supseteq \bar{a}$ is finite, $n < \omega$, and $\varphi_0(\bar{x}, \bar{y}), \dots, \varphi_{n-1}(\bar{x}, \bar{y}) \in L$, where $\bar{x}$ is reserved for $\bar{a}$ and $\bar{y}$ for $\bar{d}$. We direct $\mathcal{F}_{\bar{a}}$ naturally by:

$$(\bar{b}, \{\varphi_i(\bar{x}, \bar{y})\}_{i<n}) \ \leqslant\ (\bar{b}', \{\varphi_i'(\bar{x}, \bar{y})\}_{i<n'})$$

if $\bar{b} \subseteq \bar{b}'$, $n \leqslant n'$, and $\{\varphi_i(\bar{x}, \bar{y})\}_{i<n} \subseteq \{\varphi_i'(\bar{x}, \bar{y})\}_{i<n'}$. Let $\mu_f \in \mathfrak{M}_{\bar{d}}$ be a measure given by the claim for ϵ, $\bar{a}$, and $f \in \mathcal{F}_{\bar{a}}$. Denote by $\mu_{\epsilon,\bar{a}}$ an accumulation point of the net $(\mu_f)_{f \in \mathcal{F}_{\bar{a}}}$ in $\mathfrak{M}_{\bar{d}}$. We claim that for every formula $\varphi(\bar{x}, \bar{y})$ and every tuples $\bar{a}' \equiv \bar{a}$ we have $|\mu_{\epsilon,\bar{a}}([\varphi(\bar{a}, \bar{y})]) - \mu_{\epsilon,\bar{a}}([\varphi(\bar{a}', \bar{y})])| \leqslant \epsilon$. Suppose that this does not hold for some $\varphi(\bar{x}, \bar{y})$ and some $\bar{a}' \equiv \bar{a}$. Consider $f_0 = (\bar{a}^\frown \bar{a}', \{\varphi(\bar{x}, \bar{y})\}) \in \mathcal{F}_{\bar{a}}$. By the definition of $\mu_{\epsilon,\bar{a}}$, we can find $f \in \mathcal{F}_{\bar{a}}$ such that $f \geqslant f_0$ and $|\mu_f([\varphi(\bar{a}, \bar{y})]) - \mu_f([\varphi(\bar{a}', \bar{y})])| > \epsilon$, which contradicts the choice of μ_f.

Consider now the family $\mathcal{F}$ of all pairs $(\epsilon, \bar{a})$ naturally directed by: $(\epsilon, \bar{a}) \leqslant (\epsilon', \bar{a}')$ iff $\epsilon \geqslant \epsilon'$ and $\bar{a} \subseteq \bar{a}'$. Let $\mu \in \mathfrak{M}_{\bar{d}}$ be an accumulation point of the net $(\mu_{\epsilon,\bar{a}})_{(\epsilon,\bar{a}) \in \mathcal{F}}$. We claim that μ is invariant. If not, we can find $\varphi(\bar{x}, \bar{y})$, $\epsilon_0 > 0$, and $\bar{a}_0 \equiv \bar{a}_1$ such that $|\mu([\varphi(\bar{a}_0, \bar{y})]) - \mu([\varphi(\bar{a}_1, \bar{y})])| > \epsilon_0$. By the definition of μ, we can find $\epsilon \leqslant \epsilon_0$ and $\bar{a} \supseteq \bar{a}_0^\frown \bar{a}_1$ such that $|\mu_{\epsilon,\bar{a}}([\varphi(\bar{a}_0, \bar{y})]) - \mu_{\epsilon,\bar{a}}([\varphi(\bar{a}_1, \bar{y})])| > \epsilon_0 \geqslant \epsilon$. Let $\sigma \in \mathrm{Aut}(\mathfrak{C})$ be such that $\sigma(\bar{a}_0) = \bar{a}_1$ and put $\bar{a}' := \sigma(\bar{a})$. Consider $\psi(\bar{a}, \bar{y}) := \varphi(\bar{a}_0, \bar{y})$; then $\psi(\bar{a}', \bar{y}) = \varphi(\bar{a}_1, \bar{y})$. Hence, $\bar{a} \equiv \bar{a}'$ and $|\mu_{\epsilon,\bar{a}}([\psi(\bar{a}, \bar{y})]) - \mu_{\epsilon,\bar{a}}([\psi(\bar{a}', \bar{y})])| > \epsilon$, which is not possible by the previous paragraph.

(1)⇒(3) Assume that T is amenable. Fix $\bar{a} \subseteq \bar{b} \subseteq \mathfrak{C}$, $n < \omega$, and a definable coloring $c : \binom{\mathfrak{C}}{\bar{a}} \to 2^n$ given by $\varphi_0(\bar{x}, \bar{d}), \dots, \varphi_{n-1}(\bar{x}, \bar{d})$. By amenability, we can find an invariant, finitely additive probability measure μ on the clopens of $S_{\bar{d}}(\mathfrak{C})$. Let $\binom{\bar{b}}{\bar{a}} = \{\bar{a}_s\}_{s<t}$ and let us consider clopens $[\varphi_i(\bar{a}_s, \bar{y})]$ in $S_{\bar{d}}(\mathfrak{C})$ for $i < n$, $s < t$. Let $\mathcal{F}$ be the family of all $\varepsilon \in 2^{n \times t}$ such that $\psi_\varepsilon(\bar{y}) := \bigwedge_{\substack{i<n \\ s<t}} \varphi_i(\bar{a}_s, \bar{y})^{\varepsilon(i,s)}$ is consistent with $\operatorname{tp}(\bar{d})$. (Note that the $[\psi_\varepsilon(\bar{y})]$'s are the atoms of the Boolean algebra generated by the $[\varphi_i(\bar{a}_s, \bar{y})]$'s in $S_{\bar{d}}(\mathfrak{C})$.) Set $k = \#\mathcal{F}$ and $\lambda_\varepsilon = \mu([\psi_\varepsilon(\bar{y})])$ for $\varepsilon \in \mathcal{F}$; clearly $\sum_{\varepsilon \in \mathcal{F}} \lambda_\varepsilon = 1$. Further, for each $\varepsilon \in \mathcal{F}$ choose $\sigma_\varepsilon \in \mathrm{Aut}(\mathfrak{C})$ such that $\sigma_\varepsilon^{-1}(\bar{d}) \models \psi_\varepsilon(\bar{y})$. We claim that the chosen λ_ε's and σ_ε's satisfy our requirements.

First, note the following: for $s < t$ and $\varepsilon \in \mathcal{F}$, $c(\sigma_\varepsilon(\bar{a}_s))(i) = 1$ iff $\models \varphi_i(\sigma_\varepsilon(\bar{a}_s), \bar{d})$ iff $\models \varphi_i(\bar{a}_s, \sigma_\varepsilon^{-1}(\bar{d}))$ iff $\varepsilon(i,s) = 1$. Hence,

$$\sum_{\varepsilon\in\mathcal{F}} \lambda_\varepsilon c(\sigma_\varepsilon(\bar{a}_s))(i) = \sum_{\substack{\varepsilon\in\mathcal{F}\\ \varepsilon(i,s)=1}} \mu([\psi_\varepsilon(\bar{y})]) = \mu(\bigsqcup_{\substack{\varepsilon\in\mathcal{F}\\ \varepsilon(i,s)=1}} [\psi_\varepsilon(\bar{y})]) = \mu([\varphi_i(\bar{a}_s, \bar{y})]),$$

so for any $s < s' < t$ we have:

$$\sum_{\varepsilon\in\mathcal{F}} \lambda_\varepsilon c(\sigma_\varepsilon(\bar{a}_s))(i) - \sum_{\varepsilon\in\mathcal{F}} \lambda_\varepsilon c(\sigma_\varepsilon(\bar{a}_{s'}))(i) = \mu([\varphi_i(\bar{a}_s, \bar{y})]) - \mu([\varphi_i(\bar{a}_{s'}, \bar{y})]) = 0$$

by the invariance of μ. Thus, (3) holds. $\square$

As usual, Proposition 4.30 together with the fact that in the proof of Theorem 4.31 we work in the given $\mathfrak{C}$ imply that amenability of T is absolute (i.e. does not depend on the choice of $\mathfrak{C}$). But this was proved directly in [13], which together with the observation that in the proof of Theorem 4.31 we work in the given $\mathfrak{C}$ implies Proposition 4.30. By examining the above proofs, one can also see that we can assume here only that $\mathfrak{C}$ is $\aleph_0$-saturated and strongly $\aleph_0$-homogeneous.

Observe also that one could easily modify (or rather simplify) the above proof of (2) $\Rightarrow$ (1) in Theorem 4.31 to get an alternative proof of Theorem 4.17($\Rightarrow$). We decided to give a proof of Theorem 4.17 involving the Ellis semigroup in order to have a uniform treatment of Theorems 4.15 and 4.17 as well as to get Corollary 4.18 as an immediate conclusion.

5. Around profiniteness of the Ellis group

In this section, we will prove Theorem 3 as well as several other criteria for [pro]finiteness of the Ellis group of the theory in question.

Proof of Theorem 3. The implications (D) $\Longrightarrow$ (C) $\Longrightarrow$ (B) $\Longrightarrow$ (A) $\Longrightarrow$ (A') $\Longrightarrow$ (A") follow from various facts or observations made so far: the first implication follows by Fact 2.1, the second one is clear by Lemma 2.19, the third one by Fact 2.7, and the last two by Proposition 3.1.

It remains to show (B) $\Longrightarrow$ (C). Denote by $\mathcal{F}$ the set of all pairs $(\Delta, \bar{p})$, where Δ is a finite set of formulae and $\bar{p}$ a finite set of types from $S(T)$ with the same variables as the parametric variables of the formulae in Δ. For any $t = (\Delta, \bar{p}) \in \mathcal{F}$ put $S_t := S_{\bar{c},\Delta}(\bar{p})$.

Let $F : S_{\bar{c}}(\mathfrak{C}) \to \varprojlim_{t\in\mathcal{F}} S_t$ be the flow isomorphism given by restrictions, and let $G : S_{\bar{c}}(\mathfrak{C}) \to \varprojlim_{i\in I} X_i$ be a flow isomorphism. Also, let $f_{t_0} : \varprojlim_{t\in\mathcal{F}} S_t \to S_{t_0}$ and $g_{i_0} : \varprojlim_{i\in I} X_i \to X_{i_0}$ be projections. Consider: $F_t := \{(p,q) \in S_{\bar{c}}(\mathfrak{C})^2 \mid f_t \circ F(p) = f_t \circ F(q)\}$ and $G_i := \{(p,q) \in S_{\bar{c}}(\mathfrak{C})^2 \mid g_i \circ G(p) = g_i \circ G(q)\}$ for $t \in \mathcal{F}$ and $i \in I$. Both the F_t's and the G_i's are obviously equivalence relations on $S_{\bar{c}}(\mathfrak{C})$. Further, they are closed: F_t is the inverse image of the diagonal on S_t under $f_t \circ F$, and similarly for the G_i's. Moreover, they are clearly $\mathrm{Aut}(\mathfrak{C})$-invariant and also directed in the sense that $t \leqslant t' \in \mathcal{F}$ implies $F_{t'} \subseteq F_t$, and $i \leqslant i' \in I$ implies $G_{i'} \subseteq G_i$.

Notice that $\bigcap_{t\in\mathcal{F}} F_t = D$ and $\bigcap_{i\in I} G_i = D$, where $D = \{(p,p) \mid p \in S_{\bar{c}}(\mathfrak{C})\}$ is the diagonal on $S_{\bar{c}}(\mathfrak{C})$.

Claim. *For any $t \in \mathcal{F}$ there is $i \in I$ such that $G_i \subseteq F_t$.*

Proof of Claim. Observe that for any clopen $C \subseteq S_{\bar{c}}(\mathfrak{C})$ we can find $i_0 \in I$ such that C is a union of G_{i_0}-classes. To see this, note that $\bigcap_{i \in I} G_i = D$ implies that $\bigcup_{i \in I} G_i^c \cup (C \times C) \cup (C^c \times C^c) = S_{\bar{c}}(\mathfrak{C})^2$, where each member is open. By compactness, $\bigcup_{i \in I_0} G_i^c \cup (C \times C) \cup (C^c \times C^c) = S_{\bar{c}}(\mathfrak{C})^2$ for some finite I_0, so $\bigcap_{i \in I_0} G_i \subseteq (C \times C) \cup (C^c \times C^c)$. By choosing i_0 to be greater than all elements of I_0, we get $G_{i_0} \subseteq (C \times C) \cup (C^c \cup C^c)$, and the conclusion follows.

Let $t = (\Delta, \bar{p})$, where $\Delta = \{\varphi_l(\bar{x}, \bar{y})\}_{l<k}$ and $\bar{p} = \{p_j(\bar{y})\}_{j<m} \subseteq S_{\bar{y}}(T)$. For $j < m$ choose $\bar{a}_j \models p_j$. Then $[\varphi_l(\bar{x}, \bar{a}_j)]$ is a clopen subset of S_t for $l < k$ and $j < m$, so $(f_t \circ F)^{-1}[[\varphi_l(\bar{x}, \bar{a}_j)]]$ is clopen in $S_{\bar{c}}(\mathfrak{C})$. By the the previous paragraph, we can find $i_{l,j} \in I$ such that $(f_t \circ F)^{-1}[[\varphi_l(\bar{x}, \bar{a}_j)]]$ is a union of $G_{i_{l,j}}$-classes. Since $G_{i_{l,j}}$ is $\mathrm{Aut}(\mathfrak{C})$-invariant and $f_t \circ F$ is a homomorphism of flows, we get that for every $\bar{a} \models p_j$, $(f_t \circ F)^{-1}[[\varphi_l(\bar{x}, \bar{a})]]$ is a union of $G_{i_{l,j}}$-classes. Let $i \in I$ be greater than or equal to all $i_{l,j}$'s, for $l < k$ and $j < m$. It follows that for every clopen C in S_t we have that $(f_t \circ F)^{-1}[C]$ is a union of G_i-classes.

Therefore, for any $q \in S_t$, since

$$(f_t \circ F)^{-1}[\{q\}] = \bigcap_{\substack{C\colon q \in C, \\ C \text{ clopen}}} (f_t \circ F)^{-1}[C],$$

we get that $(f_t \circ F)^{-1}[\{q\}]$ is a union of G_i-classes. This means that any F_t-class is a union of G_i-classes, hence $G_i \subseteq F_t$. □ Claim

By the claim, for any $t \in \mathcal{F}$ we can find $i \in I$ such that the identity map $S_{\bar{c}}(\mathfrak{C}) \to S_{\bar{c}}(\mathfrak{C})$ induces an $\mathrm{Aut}(\mathfrak{C})$-flow epimorphism $S_{\bar{c}}(\mathfrak{C})/G_i \to S_{\bar{c}}(\mathfrak{C})/F_t$. On the other hand, $S_{\bar{c}}(\mathfrak{C})/G_i \cong X_i$ and $S_{\bar{c}}(\mathfrak{C})/F_t \cong S_t$ as $\mathrm{Aut}(\mathfrak{C})$-flows. Therefore, there exists a flow epimorphism $X_i \to S_t$, and we are done by Fact 2.4(i) and Fact 2.2. □

By Theorem 4.22, for a theory with sep. finite *EDEERdeg* Condition (D) holds. Hence, since (D) $\implies$ (A), we obtain:

Corollary 5.1. *If T has sep. fin. EDEERdeg, then the Ellis group of T is profinite.* □

As mentioned in the introduction, we will give examples showing that (A") does not imply (A') (see Example 6.12) and (A') does not imply (B) (see Example 6.11). We do not know however whether Example 6.11 satisfies (A), so we do not know whether it shows that (A') does not imply (A) or that (A) does not imply (B). We have also not found an example showing that (C) does not imply (D).

We do not expect that (A) $\implies$ (C) is true. However, we can easily see that (A) is equivalent to a weaker version of (C):

(C') for every finite set of formulae Δ and types $\bar{p} \subseteq S(T)$, the Ellis group of the flow $(\mathrm{Aut}(\mathfrak{C}), S_{\bar{c},\Delta}(\bar{p}))$ is profinite.

By Lemma 2.19 and Fact 2.7, we see that (C') $\implies$ (B') $\implies$ (A), where (B') is the weaker version of (B) in which we require only profinitenes of the Ellis groups of the flows X_i.

Proposition 5.2. *Conditions (A), (B'), and (C') are equivalent.*

Proof. It remains to prove (A) $\Longrightarrow$ (C'). Consider an $\mathrm{Aut}(\mathfrak{C})$-flow epimorphism $S_{\bar{c}}(\mathfrak{C}) \to S_{\bar{c},\Delta}(\bar{p})$ given by the restriction. By Fact 2.4, it induces an $\mathrm{Aut}(\mathfrak{C})$-flow and semigroup epimorphism $f : \mathrm{EL}(S_{\bar{c}}(\mathfrak{C})) \to \mathrm{EL}(S_{\bar{c},\Delta}(\bar{p}))$. Let $\mathcal{M}$ be a minimal left ideal of $\mathrm{EL}(S_{\bar{c}}(\mathfrak{C}))$ and $u \in \mathcal{J}(\mathcal{M})$. By Fact 2.2, $\mathcal{M}' := f[\mathcal{M}]$ is a minimal left ideal of $\mathrm{EL}(S_{\bar{c},\Delta}(\bar{p}))$, $u' := f(u) \in \mathcal{J}(\mathcal{M}')$, and $f_{\restriction u\mathcal{M}} : u\mathcal{M} \to u'\mathcal{M}'$ is a group epimorphism and quotient map (in the τ-topology). By Remark 3.2(b), it is enough to prove that $u'\mathcal{M}'$ is 0-dimensional. For this, it suffices to prove that $f_{\restriction u\mathcal{M}}[U]$ is clopen for any clopen $U \subseteq u\mathcal{M}$, as $u\mathcal{M}$ is 0-dimensional.

For any $U \subseteq u\mathcal{M}$ we have $f^{-1}_{\restriction u\mathcal{M}}[f_{\restriction u\mathcal{M}}[U]] = \ker(f_{\restriction u\mathcal{M}}) \cdot U$. If U is clopen, then $f^{-1}_{\restriction u\mathcal{M}}[f_{\restriction u\mathcal{M}}[U]] = \bigcup_{a\in\ker(f_{\restriction u\mathcal{M}})} aU$ is open, but also $\ker(f_{\restriction u\mathcal{M}}) = f^{-1}_{\restriction u\mathcal{M}}[\{u'\}]$ is closed as $u'\mathcal{M}'$ is T_1, so $f^{-1}_{\restriction u\mathcal{M}}[f_{\restriction u\mathcal{M}}[U]] = \ker(f_{\restriction u\mathcal{M}}) \cdot U$ is closed (multiplication $u\mathcal{M} \times u\mathcal{M} \to u\mathcal{M}$ is continuous, $u\mathcal{M} \times u\mathcal{M}$ is compact, and $u\mathcal{M}$ is Hausdorff, hence multiplication is closed, so $\ker(f_{\restriction u\mathcal{M}}) \cdot U$ is closed as a product of two closed sets). Therefore, $f^{-1}_{\restriction u\mathcal{M}}[f_{\restriction u\mathcal{M}}[U]] = \ker(f_{\restriction u\mathcal{M}}) \cdot U$ is clopen for clopen U, so $f_{\restriction u\mathcal{M}}[U]$ is clopen, since $f_{\restriction u\mathcal{M}}$ is a quotient map. □

In the next remark, we observe that, alternatively, (A') can be deduced from (B) (or from a stronger version of (D), namely (D+) below) by using the criteria from Proposition 3.8.

(D+) There is $\eta \in \mathrm{EL}(S_{\bar{c}}(\mathfrak{C}))$ which for every finite $\Delta, \bar{p}$ induces (via the restriction map $S_{\bar{c}}(\mathfrak{C}) \to S_{\bar{c},\Delta}(\bar{p})$ and Fact 2.4(i)) an element $\eta_{\Delta,\bar{p}} \in \mathrm{EL}(S_{\bar{c},\Delta}(\bar{p}))$ with finite image.

Remark 5.3.
(i) (B) implies that $u\mathcal{M}$ is closed in $\mathrm{EL}(X)$, i.e. the assumption of Proposition 3.8(i) holds.
(ii) (D+) implies that there is $\eta \in \mathcal{M}$ witnessing (D+). For any such η, $\mathrm{Im}(\eta)$ is closed. In particular, (D+) implies the assumption of Proposition 3.8(iii)

Proof. (i) By Fact 2.7, present $\mathcal{M}$ as the inverse limit $\varprojlim_{i\in I} \mathcal{M}_i$ (where $\mathcal{M}_i$ is a minimal left ideal in $\mathrm{EL}(X_i)$) and $u\mathcal{M}$ as the subset $\varprojlim_{i\in I} u_i\mathcal{M}_i$. Since the $u_i\mathcal{M}_i$'s are finite (so closed), $u\mathcal{M}$ is closed.

(ii) Let η be a witness for (D+). Replacing η by $\eta\eta_0$ for some $\eta_0 \in \mathcal{M}$, we get a witness for (D+) which is in $\mathcal{M}$.

Now, let $\eta \in \mathcal{M}$ witness (D+). Replacing η by an idempotent $u \in \eta\mathcal{M}$, we reduce the situation to the case when η is an idempotent. As usual, we can identify $S_{\bar{c}}(\mathfrak{C})$ with $\varprojlim_{(\Delta,\bar{p})} S_{\bar{c},\Delta}(\bar{p})$. After this identification, we claim that $\mathrm{Im}(\eta) = \varprojlim_{(\Delta,\bar{p})} \mathrm{Im}(\eta_{\Delta,\bar{p}})$, which clearly shows that $\mathrm{Im}(\eta)$ is closed. The inclusion $(\subseteq)$ is obvious. For the opposite inclusion, take $(\xi_{\Delta,\bar{p}})_{(\Delta,\bar{p})} \in \varprojlim_{(\Delta,\bar{p})} \mathrm{Im}(\eta_{\Delta,\bar{p}})$. Then $\xi_{\Delta,\bar{p}} = \eta_{\Delta,\bar{p}}(\xi'_{\Delta,\bar{p}}) = \eta_{\Delta,\bar{p}}(\eta_{\Delta,\bar{p}}(\xi'_{\Delta,\bar{p}})) = \eta_{\Delta,\bar{p}}(\xi_{\Delta,\bar{p}})$ for some $\xi'_{\Delta,\bar{p}}$. Hence, $(\xi_{\Delta,\bar{p}})_{(\Delta,\bar{p})} = \eta((\xi_{\Delta,\bar{p}})_{(\Delta,\bar{p})}) \in \mathrm{Im}(\eta)$. □

We now state two criteria which in some quite common situations guarantee profiniteness (or even finiteness) of the Ellis group.

Proposition 5.4. *Suppose that $L \subseteq L^*$ are finite relational languages, T^* is a complete L^*-theory with quantifier elimination such that $T := T^*_{\restriction L}$ also has quantifier elimination. Let $\mathfrak{C}^*$ be a monster model of T^* such that $\mathfrak{C} := \mathfrak{C}^*_{\restriction L}$ is a monster model of T. If there is $\eta^* \in \mathrm{EL}(S_{\bar{c}}(\mathfrak{C}^*))$ such that $\mathrm{Im}(\eta^*) \subseteq \mathrm{Inv}_{\bar{c}}(\mathfrak{C}^*)$, then for any finite $\bar{z}$ there exists $\eta \in \mathrm{EL}(S_{\bar{z}}(\mathfrak{C}))$ such that $\mathrm{Im}(\eta)$ is finite.*

In particular, the Ellis group of the flow $(\mathrm{Aut}(\mathfrak{C}), S_{\bar{z}}(\mathfrak{C}))$ *is finite.*

We should stress that here we consider two different flows: $(\mathrm{Aut}(\mathfrak{C}^*), S_{\bar{c}}(\mathfrak{C}^*))$ and $(\mathrm{Aut}(\mathfrak{C}), S_{\bar{c}}(\mathfrak{C}))$, defined with respect to two different languages. In order to avoid confusions, we denote by asterisk notions defined with respect to the language L^*, and without asterisk notions defined with respect to the language L.

Proof. By Lemma 2.18(iv), there is $\eta \in \mathrm{EL}(S_{\bar{z}}(\mathfrak{C}))$ with $\mathrm{Im}(\eta) = \mathrm{Inv}^*_{\bar{z}}(\mathfrak{C})$, where $\mathrm{Inv}^*_{\bar{z}}(\mathfrak{C}) \subseteq S_{\bar{z}}(\mathfrak{C})$ is the set of all $\mathrm{Aut}(\mathfrak{C}^*)$-invariant types. Hence, since T has quantifier elimination, each type $p \in \mathrm{Im}(\eta)$ is determined by saying whether $R(\bar{z}, \bar{b}) \in p$ or not for some (every) $\bar{b} \models q$, for all $R(\bar{z}, \bar{y}) \in L$ and $q \in S_{\bar{y}}(T^*)$. On the other hand, $S_{\bar{y}}(T^*)$ is finite by elimination of quantifiers for T^* and finiteness of L^*. Since L is also finite, we see that we have only finitely many possibilities for $p \in \mathrm{Im}(\eta)$, and thus $\mathrm{Im}(\eta)$ is finite. (Implicitly we also used here that the languages are relational.)

The "In particular" part follows by Fact 2.1. □

Corollary 5.5. *Under the assumptions of Proposition 5.4, the Ellis group of the flow* $(\mathrm{Aut}(\mathfrak{C}), S_{\bar{c}}(\mathfrak{C}))$ *is finite.*

Proof. Since L is a finite relational language and T has quantifier elimination, T is m-ary, where m is the maximal arity of the symbols in L. By Corollary 2.15, the Ellis groups of the flows $(\mathrm{Aut}(\mathfrak{C}), S_{\bar{c}}(\mathfrak{C}))$ and $(\mathrm{Aut}(\mathfrak{C}), S_{m-1}(\mathfrak{C}))$ are isomorphic. The latter is finite by Proposition 5.4. □

Proposition 5.6. *Suppose that* $L \subseteq L^*$*,* T^* *is a complete* L^**-theory, and* $T := T^*_{\restriction L}$*. Let* $\mathfrak{C}^*$ *be a monster of* T^* *such that* $\mathfrak{C} := \mathfrak{C}^*_{\restriction L}$ *is a monster of* T*. Assume that for any finitely many variables* $\bar{y}$ *there are only finitely many extensions in* $S_{\bar{y}}(T^*)$ *of each type from* $S_{\bar{y}}(T)$*. If there is* $\eta^* \in \mathrm{EL}(S_{\bar{c}}(\mathfrak{C}^*))$ *such that* $\mathrm{Im}(\eta^*) \subseteq \mathrm{Inv}_{\bar{c}}(\mathfrak{C}^*)$*, then (D+) holds.*

In particular, the Ellis group of T *is profinite.*

Proof. The existence of $\eta^* \in \mathrm{EL}(S_{\bar{c}}(\mathfrak{C}^*))$ with $\mathrm{Im}(\eta^*) \subseteq \mathrm{Inv}_{\bar{c}}(\mathfrak{C}^*)$ implies, by Lemma 2.18(ii), that there is $\eta \in \mathrm{EL}(S_{\bar{c}}(\mathfrak{C}))$ with $\mathrm{Im}(\eta) \subseteq \mathrm{Inv}^*_{\bar{c}}(\mathfrak{C})$, where $\mathrm{Inv}^*_{\bar{c}}(\mathfrak{C}) \subseteq S_{\bar{c}}(\mathfrak{C})$ is the set of all $\mathrm{Aut}(\mathfrak{C}^*)$-invariant types. Then, for each finite $\Delta, \bar{p}$, the induced $\eta_{\Delta,\bar{p}} \in \mathrm{EL}(S_{\bar{c},\Delta}(\bar{p}))$ satisfies $\mathrm{Im}(\eta_{\Delta,\bar{p}}) \subseteq \mathrm{Inv}^*_{\bar{c},\Delta}(\bar{p})$, where $\mathrm{Inv}^*_{\bar{c},\Delta}(\bar{p})$ is the set of all $\mathrm{Aut}(\mathfrak{C}^*)$-invariant types from $S_{\bar{c},\Delta}(\bar{p})$.

Consider any finite Δ and $\bar{p}$. For each type $q_{\Delta,\bar{p}} \in \mathrm{Inv}^*_{\bar{c},\Delta}(\bar{p})$ and formula $\varphi(\bar{x}, \bar{y}) \in \Delta$ we have: $\varphi(\bar{x}, \bar{b}) \in q_{\Delta,\bar{p}}(\bar{x})$, where $\mathrm{tp}(\bar{b}) \in \bar{p}$, iff $\varphi(\bar{x}, \bar{b}') \in q_{\Delta,\bar{p}}(\bar{x})$ for all $\bar{b}'$ realizing the same extension in $S_{\bar{y}}(T^*)$ of $\mathrm{tp}(\bar{b}) \in S_{\bar{y}}(T)$ as $\bar{b}$. Since Δ and $\bar{p}$ are finite and each $p_0 \in \bar{p}$ has only finitely many extensions in $S_{\bar{y}}(T^*)$, $q_{\Delta,\bar{p}}(\bar{x})$ is completely determined by finitely many conditions. Therefore, $\mathrm{Inv}^*_{\bar{c},\Delta}(\bar{p})$ is finite, and consequently $\mathrm{Im}(\eta_{\Delta,\bar{p}})$ is finite.

The "In particular" part follows from (D+), as (D+) $\implies$ (D) $\implies$ (A). □

Remark 5.7. The assumption of Proposition 5.6 that for each finite $\bar{y}$ there are only finitely many extensions in $S_{\bar{y}}(T^*)$ of each type from $S_{\bar{y}}(T)$ is naturally fulfilled in some situations. For example, if L^* is relational, $L^* \setminus L$ is finite, and T^* has quantifier elimination, then this assumption holds. □

Corollary 5.8. *The assumptions of Proposition 5.4 imply the assumptions of Proposition 5.6 which in turn imply that T has sep. fin. $EDEERdeg$.*

Proof. The first part follows from Remark 5.7. To see the rest, note that since Proposition 5.6 yields (D+) and so (D), we can use the right to left implication in Theorem 4.22. □

Thus, the assumptions of Proposition 5.6 yield a criterion for having sep. fin. $EDEERdeg$. In fact, this is a counterpart of the criterion for having sep. fin. Ramsey degree proved by Zucker in [35, Theorem 8.14], which we now explain (but for the definitions the reader is referred to [35] and [27]; see also the paragraph preceding Example 6.6). Zucker shows that a Fraïssé structure has sep. fin. Ramsey degree iff it has a Fraïssé, precompact (relational) expansion whose age has the embedding Ramsey property and the expansion property relative to the age of the original structure. Combining this with [27, Theorem 6] (and noting that the Fraïssé subclass of $\mathrm{Age}(\boldsymbol{F}^*)$ obtained there is a reasonable expansion of $\mathrm{Age}(\boldsymbol{F})$), we get that a Fraïssé structure has sep. fin. Ramsey degree iff it has a Fraïssé, precompact (relational) expansion whose age has the embedding Ramsey property. Now, if both the original and the expanded structure are $\aleph_0$-saturated (so have elimination of quantifiers), then being precompact exactly means that for any finitely many variables $\bar{y}$ there are only finitely many extensions in $S_{\bar{y}}(T^*)$ of each type from $S_{\bar{y}}(T)$. Moreover, by Theorem 4.15, the assumption of Proposition 5.6 saying that there is $\eta^* \in \mathrm{EL}(S_{\bar{c}}(\mathfrak{C}^*))$ such that $\mathrm{Im}(\eta^*) \subseteq \mathrm{Inv}_{\bar{c}}(\mathfrak{C}^*)$ is equivalent to T^* having $EDEERP$ (which is the appropriate counterpart of the embedding Ramsey property).

However, Example 6.3 shows that, in contrast with Zucker's result, in our case the assumptions of Proposition 5.6 are only sufficient to have sep. finite $EDEERdeg$ for T, but they are not necessary. It could be interesting to find an "iff" criterion of this kind; we have not tried to do that.

6. Applications and examples

In this section, we use our analysis to prove [pro]finiteness (or triviality) of the Ellis groups in some classes of theories, as well as give examples illustrating various phenomena and showing the lack of implications between some key conditions considered in this paper.

Example 6.1 (Stable theories)**.** We prove that stable theories have sep. fin. $EDEERdeg$. Denote by $\mathrm{NF}_{\bar{c}}(\mathfrak{C})$ the space of all types from $S_{\bar{c}}(\mathfrak{C})$ which do not fork over $\emptyset$.

Claim. *There exists an idempotent u in a minimal left ideal $\mathcal{M}$ of $\mathrm{EL}(S_{\bar{c}}(\mathfrak{C}))$ with $\mathrm{Im}(u) \subseteq \mathrm{NF}_{\bar{c}}(\mathfrak{C})$.*

Proof of Claim. This is basically the proof of [16, Proposition 7.11]. For a type $q(\bar{x}) \in S_{\bar{c}}(\mathfrak{C})$ and a formula $\varphi(\bar{x}, \bar{a})$ which forks over $\emptyset$ put:

$$X_{q,\varphi} := \{\eta \in \mathrm{EL}(S_{\bar{c}}(\mathfrak{C})) \mid \neg\varphi(\bar{x}, \bar{a}) \in \eta(q)\}.$$

Note that $X_{q,\varphi}$ is a clopen set in $\mathrm{EL}(S_{\bar{c}}(\mathfrak{C}))$. We claim that $\bigcap_{q,\varphi} X_{q,\varphi} \neq \emptyset$. By compactness, it suffices to show that finite intersections are non-empty. So take $q_0(\bar{x}), \ldots, q_{n-1}(\bar{x}) \in S_{\bar{c}}(\mathfrak{C})$ and $\varphi_0(\bar{x}, \bar{a}_0), \ldots, \varphi_{n-1}(\bar{x}, \bar{a}_{n-1})$ which fork over $\emptyset$. Put $\bar{a} = \bar{a}_0 {}^\frown \ldots {}^\frown \bar{a}_{n-1}$ and consider $\psi(\bar{x}, \bar{a}) := \bigvee_{i<n} \varphi_i(\bar{x}, \bar{a}_i)$; it forks, hence divides over $\emptyset$. Let $(\bar{b}_j)_{j<\omega}$ be an $\emptyset$-indiscernible sequence with $\bar{b}_0 = \bar{a}$ such that $\{\psi(\bar{x}, \bar{b}_j)\}_{j<\omega}$ is inconsistent. There exists $m < \omega$ such that $\neg\psi(\bar{x}, \bar{b}_m) \in q_i(\bar{x})$ for all $i < n$, hence $\neg\varphi_i(\bar{x}, \bar{b}^i_m) \in q_i(\bar{x})$ for all $i < n$, where $\bar{b}^i_m$ in $\bar{b}_m$ corresponds to $\bar{a}_i$ in $\bar{a}$. If $\sigma \in \mathrm{Aut}(\mathfrak{C})$ is such that $\sigma(\bar{b}_m) = \bar{a}$, then $\neg\varphi_i(\bar{x}, \bar{a}_i) \in \sigma(q_i(\bar{x}))$ for all $i < n$. Therefore, $\sigma \in X_{q_0,\varphi_0} \cap \dots \cap X_{q_{n-1},\varphi_{n-1}}$.

Choose $\eta \in \bigcap_{q,\varphi} X_{q,\varphi}$. Then $\mathrm{Im}(\eta) \subseteq \mathrm{NF}_{\bar{c}}(\mathfrak{C})$. If $\eta_0 \in \mathcal{M}$, then $\eta\eta_0 \in \mathcal{M}$ and $\mathrm{Im}(\eta\eta_0) \subseteq \mathrm{Im}(\eta) \subseteq \mathrm{NF}_{\bar{c}}(\mathfrak{C})$. If $u \in \mathcal{J}(\mathcal{M})$ is such that $\eta\eta_0 \in u\mathcal{M}$, then $\mathrm{Im}(u) = \mathrm{Im}(\eta\eta_0)$ (as in the proof of Fact 2.1), so $\mathrm{Im}(u) \subseteq \mathrm{NF}_{\bar{c}}(\mathfrak{C})$. $\square$ Claim

(Note that in the proof of the claim we only used one consequence of stability, namely that forking equals dividing over $\emptyset$.)

Choose $u \in \mathcal{J}(\mathcal{M})$ given by the claim. Fix a finite set of L-formulae Δ. Let $u_\Delta := \hat{f}(u) \in \mathrm{EL}(S_{\bar{c},\Delta}(\mathfrak{C}))$, where $\hat{f} : \mathrm{EL}(S_{\bar{c}}(\mathfrak{C})) \to \mathrm{EL}(S_{\bar{c},\Delta}(\mathfrak{C}))$ is induced by the restriction map $f : S_{\bar{c}}(\mathfrak{C}) \to S_{\bar{c},\Delta}(\mathfrak{C})$ via Fact 2.4(i). Denote by $\mathrm{NF}_{\bar{c},\Delta}(\mathfrak{C})$ the restrictions of all types from $\mathrm{NF}_{\bar{c}}(\mathfrak{C})$ to $S_{\bar{c},\Delta}(\mathfrak{C})$. By Corollary 2.5, we conclude that $\mathrm{Im}(u_\Delta) \subseteq \mathrm{NF}_{\bar{c},\Delta}(\mathfrak{C})$. On the other hand, by basic properties of stable theories (e.g. finiteness of the Δ-multiplicity of $\mathrm{tp}(\bar{c})$), we know that $\mathrm{NF}_{\bar{c},\Delta}(\mathfrak{C})$ is finite. Hence, $\mathrm{Im}(u_\Delta)$ is finite.

Take a finite tuple of types $\bar{p}$. Let $u_{\Delta,\bar{p}} := \hat{g}(u_\Delta) \in \mathrm{EL}(S_{\bar{c},\Delta}(\bar{p}))$, where $\hat{g} : \mathrm{EL}(S_{\bar{c},\Delta}(\mathfrak{C})) \to \mathrm{EL}(S_{\bar{c},\Delta}(\bar{p}))$ is induced by the restriction map $g : S_{\bar{c},\Delta}(\mathfrak{C}) \to S_{\bar{c},\Delta}(\bar{p})$ via Fact 2.4(i). By Corollary 2.5 and the last paragraph, we conclude that $\mathrm{Im}(u_{\Delta,\bar{p}})$ is finite.

Thus, Condition (D) (even (D+)) is fulfilled, and hence T has sep. fin. *EDEERdeg* by Theorem 4.22.

If we in addition assume that $\mathrm{tp}(\bar{c})$ is stationary (equivalently, all types in $S(\emptyset)$ are stationary), then T has *EDEERP*. Indeed, under this assumption, $\mathrm{NF}_{\bar{c}}(\mathfrak{C})$ is a singleton consisting of an invariant type, so u given by the claim has image contained in the invariant types, and hence T has *EDEERP* by Theorem 4.15. Conversely, if $\mathrm{tp}(\bar{c})$ is not stationary, then there is no global invariant extension of this type, i.e. T is not extremely amenable, so T does not have *DEERP*.

This implies that, in stable theories, the properties *EDEERP* and *DEERP* are equivalent (which can also be seen immediately from definitions, using definability of types).

The next concrete example shows that a stable theory does not need to have sep. fin. *EERdeg*; in particular, sep. fin. *EDEERdeg* does not imply sep. fin. *EERdeg*.

Example 6.2. Take $T := \mathrm{ACF}_0$. Consider on $\mathfrak{C}^\times$ the relations $\sim_n$ given by $a \sim_n a'$ iff $a' = \xi_n^k a$ for some k, where ξ_n denotes an n-th primitive root of unity. The $\sim_n$'s are equivalence relations on $\mathfrak{C}^\times$, and let X_n be a set of representatives of the $\sim_n$-classes; then $\mathfrak{C}^\times = \bigsqcup_{k<n} \xi_n^k X_n$. For any transcendental a, take $\bar{b} = (a, \xi_n a, \dots, \xi_n^{n-1} a)$ and consider the coloring $c : \binom{\mathfrak{C}}{a} \to n$ given by: $c(a') = k$ iff $a' \in \xi_n^k X_n$. For any $\bar{b}' \in \binom{\mathfrak{C}}{\bar{b}}$ we have that $\bar{b}' = (a', \omega_n a', \dots, \omega_n^{n-1} a')$ for some transcendental a' and primitive n-th root of unity ω_n. Note that $\#c[\binom{\bar{b}'}{a}] = n$. Thus, a transcendental element cannot have finite *EERdeg*, so T does not have sep. fin. *EERdeg*, while T is stable and as such has sep. fin. *EDEERdeg*.

If we modify T by naming all constants from the algebraic closure of $\mathbb{Q}$, then all types in $S(\emptyset)$ are stationary, so the resulting theory has *EDEERP*, but the above argument shows that it does not have sep. fin. *EERdeg*.

The next example shows that the criterion for having sep. fin. $EDEERdeg$ given in Proposition 5.6 (see also Corollary 5.8) is not a necessary condition.

Example 6.3. Let T be a stable theory with a finitary type $p \in S_{\bar{y}}(T)$ of infinite multiplicity (i.e. with infinitely many global non-forking extensions). (For example, one can take $T := (\mathbb{Z}, +)$ and $p(y) := \mathrm{tp}(1)$.) Then it has sep. fin. $EDEERdeg$, but we will show that there is no expansion T^* of T satisfying the assumptions of Proposition 5.6. Suppose for a contradiction that T^* is such an expansion. The assumption that $\mathrm{Im}(\eta^*) \subseteq \mathrm{Inv}_{\bar{c}}(\mathfrak{C}^*)$ implies that T^* is extremely amenable, so $\mathrm{acl}^{eq,*}(\emptyset) = \mathrm{dcl}^{eq,*}(\emptyset)$ (both computed in $\mathfrak{C}^*$). Since p has infinite multiplicity, it has infinitely many extensions to complete types over $\mathrm{acl}^{eq}(\emptyset)$ computed in $\mathfrak{C}$, so also over $\mathrm{acl}^{eq,*}(\emptyset) \supseteq \mathrm{acl}^{eq}(\emptyset)$. Therefore, we conclude that p has infinitely many extensions in $S_{\bar{y}}(T^*)$, a contradiction.

In the following example, we list some Fraïssé classes which are known to have the embedding Ramsey property and whose Fraïssé limits are $\aleph_0$-categorical, and hence $\aleph_0$-saturated. By Remark 4.8, the theories of these Fraïssé limits have $EERP$, so $EDEERP$ as well, and thus their Ellis groups are trivial by Corollary 4.16.

By a hypergraph we mean a structure in a finite relational language such that each basic relation R is irreflexive (i.e. $(R(a_0, \dots, a_{n-1})$ implies that the a_i's are pairwise distinct) and symmetric (i.e. invariant under all permutations of coordinates).

Example 6.4. (a) The class of all finite (linearly) ordered hypergraphs omitting a fixed class of finite irreducible hypergraphs in a finite relational language containing $\leqslant$. See [24, Theorem A]. In particular, the theories of the ordered random graph and the ordered K_n-free random graph have $EERP$.

(b) The class of all finite sets with n linear orderings (for a fixed n) in the language $L = \{\leqslant_1, \dots, \leqslant_n\}$. See [32, Theorem 4].

(c) The class of all finite structures in the language $L = \{\sqsubseteq, \leqslant\}$ for which $\sqsubseteq$ is a partial ordering and $\leqslant$ is a linear ordering extending $\sqsubseteq$. A proof can be found in [31].

(d) The class of all finite structures in the language $L = \{\sqsubseteq, \leqslant, \preccurlyeq\}$ for which $\sqsubseteq$ is a partial ordering, $\leqslant$ is a linear ordering, and $\preccurlyeq$ is a linear ordering extending $\sqsubseteq$. See [32, Theorem 1].

(e) The class of all naturally ordered finite vector spaces over a fixed finite field F in the language $L = \{+, m_a, \leqslant\}_{a \in F}$, where $+$ is the addition, m_a is the unary multiplication by $a \in F$, and $\leqslant$ is the anti-lexicographical linear ordering induced by an ordering of a basis. This is in [15] together with [10, Corollary 2].

(f) The class of all naturally ordered finite Boolean algebras in the language of Boolean algebras expanded by $\leqslant$, where $\leqslant$ is the anti-lexicographical linear order induced by an ordering of atoms. This is [15, Proposition 6.13] together with [11].

(g) The class of all finite linearly ordered structures in the infinite language consisting of relational symbols R_n, $n > 0$, where R_n is n-ary, such that each R_n is irreflexive and symmetric. It is easy to see that this is a Fraïssé class whose limit is $\aleph_0$-categorical. The fact that it has the embedding Ramsey property follows from the fact that the restrictions to the finite sublanguages have it by (a). This holds more generally in a situation when for each n there are only finitely many relational symbols of arity n.

In the examples of Fraïssé limits with the embedding Ramsey property whose Fraïssé limits are not $\aleph_0$-saturated, the situation is not so obvious, as we cannot use Remark 4.8 to deduce $EERP$ and so $EDEERP$. However, one can often use some methods developed in this paper to show directly that for the monster model $\mathfrak{C}$ of the theory of the Fraïssé limit there is $\eta \in \mathrm{EL}(S_{\bar{c}}(\mathfrak{C}))$ with $\mathrm{Im}(\eta) \subseteq \mathrm{Inv}_{\bar{c}}(\mathfrak{C})$, hence we have $EDEERP$ by Theorem 4.15, and so the Ellis group is trivial by Corollary 4.16.

We will discuss one of such examples, leaving the technical details of the proof of quantifier elimination to the reader.

Example 6.5. Consider the Fraïssé class of all (linearly) ordered finite metric spaces with rational distances in the countable language $\{R_q : q \in \mathbb{Q}\} \cup \{\leqslant\}$, where the R_q's are binary relation symbols interpreted in the finite metric spaces via $R_q(x, y) \iff d(x, y) < q$ (where d is the metric). It is well-know that this is a Farïssé class with the embedding Ramsey property (see [23, Theorem 1.2]). Let M be its Fraïssé limit (i.e. the ordered rational Urysohn space) and $\mathfrak{C} \succ M$ a monster model. Here, M is not $\aleph_0$-saturated, as it does not realize a (consistent) type $\{\neg R_q(x, y) \mid q \in \mathbb{Q}\}$.

Define $\delta(x, y)$ (for $x, y \in \mathfrak{C}$) as the supremum of the q's such that $\neg R_q(x, y)$ holds. It is easy to see that δ is a pseudometric with values in $[0, \infty]$, extending the original metric on M. The relation ϵ saying that two elements are at distance 0 and the relation E saying that two elements are at distance less than ∞ are both equivalence relations on $\mathfrak{C}$.

By a standard back-and-forth argument, we can show that $\mathrm{Th}(M)$ has quantifier elimination. Namely, suppose we have two finite tuples $\bar{a}$ and $\bar{b}$ in $\mathfrak{C}$ with the same quantifier-free type, and let α be any element in $\mathfrak{C}$. The goal is to show that there is $\beta \in \mathfrak{C}$ such that the tuples $\bar{a}\alpha$ and $\bar{b}\beta$ have the same qf-type. Let $A_0, \dots, A_{n-1}$ be all E-classes on $\bar{a}$, and $B_0, \dots, B_{n-1}$ the corresponding E-classes on $\bar{b}$. Then one of the following two cases holds:

(i) α is not E-related to any element of $\bar{a}$;
(ii) α is E-related to all elements of exactly one of the classes A_i (say A_i) and is not related to any element of any other class A_j.

In case (i), using universality and ultrahomogeneity of M, by compactness, we easily get $\beta \in \mathfrak{C}$ not E-related to any element of $\bar{b}$ and satisfying the same qf-order type together with $\bar{b}$ as α together with $\bar{a}$. Thus, $\bar{a}\alpha \equiv^{qf} \bar{b}\beta$.

In case (ii), consider any finite $Q \subseteq \mathbb{Q}$. Put $L_Q := \{R_q \mid q \in Q\} \cup \{\leqslant\}$. Let $\pi(\bar{x}, y) := \mathrm{tp}^{\mathrm{qf}}_{L_Q}(\bar{a}, \alpha)$; it is clearly equivalent to a formula. We leave as a non-trivial exercise to check that there is a finite $Q' \supseteq Q$ such that for every $\bar{a}' \models \mathrm{tp}^{\mathrm{qf}}_{L_{Q'}}(\bar{a}) =: \pi'(\bar{x})$ there exists α' such that $(\bar{a}', \alpha') \models \pi(\bar{x}, y)$; in other words, $\models \pi'(\bar{x}) \to (\exists y)(\pi(\bar{x}, y))$. Since $\bar{b} \models \pi'(\bar{x})$, we can find β' such that $(\bar{b}, \beta') \models \pi(\bar{x}, y)$. By compactness, we get the desired β, and the proof of quantifier elimination is finished.

Our goal is to show that there is $\eta \in \mathrm{EL}(S_{\bar{c}}(\mathfrak{C}))$ with $\mathrm{Im}(\eta) \subseteq \mathrm{Inv}_{\bar{c}}(\mathfrak{C})$. By quantifier elimination, our theory is binary, so, by Corollary 2.17, it is enough to show that there is $\eta \in \mathrm{EL}(S_1(\mathfrak{C}))$ with $\mathrm{Im}(\eta) \subseteq \mathrm{Inv}_1(\mathfrak{C})$.

Claim. *For any finite $A \subseteq \mathfrak{C}$, any $b \in \mathfrak{C}$, and any positive $r \in \mathbb{Q}$, there exists $\sigma_{A,b,r} \in \mathrm{Aut}(\mathfrak{C})$ such that for each $a \in A$ we have $\sigma_{A,b,r}(a) > b$ and $\delta(\sigma_{A,b,r}(a), a) = r$.*

Proof of Claim. Consider A, b, r as in the claim; let $n = \#A$ and $A = \{a_0, \dots, a_{n-1}\}$.

Let Q be any finite subset of $\mathbb{Q}$, and as before: $L_Q := \{R_q : q \in Q\} \cup \{\leqslant\}$. By universality and ultrahomogeneity of M, it is easy to see that for every (finite) tuple $\bar{\alpha} = (\alpha_0, \dots, \alpha_{n-1}) \subseteq M$ and $\beta \in M$ there exists $\bar{\alpha}' = (\alpha'_0, \dots, \alpha'_{n-1}) \subseteq M$ with the same qf-type as $\bar{\alpha}$ (i.e. there is an order preserving isometry between these tuples) and such that $\alpha'_j > \beta$ and $\delta(\alpha'_j, \alpha_j) = r$ for all $j < n$. If we replace "qf-type" by "qf L_Q-type" and "$\delta(\alpha'_j, \alpha_j) = r$" by the approximation "$\neg R_r(\alpha'_j, \alpha_j) \wedge R_q(\alpha'_j, \alpha_j)$" for a rational $q > r$, then the property from the previous sentence becomes a sentence in our language, so it holds in $\mathfrak{C}$ as well. Therefore, by compactness, there is $\bar{a}' = (a'_0, \dots, a'_{n-1}) \subseteq \mathfrak{C}$ with the same qf-type as $\bar{a} := (a_0, \dots, a_{n-1})$ and such that $a'_j > b$ and $\delta(a'_j, a_j) = r$ for all $j < n$. Hence, by q. e., we can choose $\sigma_{A,b,r} \in \mathrm{Aut}(\mathfrak{C})$ mapping $\bar{a}$ to $\bar{a}'$, and it is as required. $\square_{\text{Claim}}$

For each finite A, element $b \in \mathfrak{C}$, and positive $r \in \mathbb{Q}$, choose $\sigma_{A,b,r} \in \mathrm{Aut}(\mathfrak{C})$ as in the claim. Consider the net $(\sigma_{A,b,r})_{A,b,r}$, where the A's are ordered by inclusion, b's by the linear order $\leqslant$, and r's by the usual order on rationals. Let $\eta \in \mathrm{EL}(S_1(\mathfrak{C}))$ be an accumulation point of this net.

Let $p^+_\infty(x)$ be the type in $S_1(\mathfrak{C})$ determined by the conditions $x > c$ and $\delta(x, c) = \infty$ for all $c \in \mathfrak{C}$, and $p^-_\infty \in S_1(\mathfrak{C})$ by the conditions $x < c$ and $\delta(x, c) = \infty$ for all $c \in \mathfrak{C}$. The next claim completes the analysis of Example 6.5.

Claim. *Let $p \in S_1(\mathfrak{C})$. If p contains a formula of the form $x > c$, then $\eta(p) = p^+_\infty$; otherwise, $\eta(p) = p^-_\infty$. In particular, $\mathrm{Im}(\eta) \subseteq \mathrm{Inv}_1(\mathfrak{C})$.*

Proof of Claim. First, consider what happens with $\leqslant$. If $(x > c) \in p(x)$ for some $c \in \mathfrak{C}$, then for any $b \in \mathfrak{C}$: whenever $b' > b$ and $c \in A$, we have $(x > b) \in \sigma_{A,b',r}(p)$. Hence, $\eta(p)$ contains all formulae $x > b$ for $b \in \mathfrak{C}$. On the other hand, it is trivial that if p contains no formula $x > c$, then so does $\eta(p)$.

Now, we study what happens with the relations R_q. Let C_i, $i < \lambda$, be all the classes of E. There are two cases: either for every $c \in \mathfrak{C}$, $p(x)$ implies that $\delta(x, c) = \infty$, or there is exactly one class C_i such that $p(x)$ implies that $\delta(x, c) < \infty$ iff $c \in C_i$. In the first case, clearly $\eta(p)$ still implies $\delta(x, c) = \infty$ for all $c \in \mathfrak{C}$. In the second case, suppose for a contradiction that $R_q(x, c) \in \eta(p)$ for some $q \in \mathbb{Q}$ and $c \in \mathfrak{C}$. First, consider the case $c \in C_i$. Then $R_{q'}(x, c) \in p(x)$ for some rational q'. For a sufficiently large index (A, b, r) we have that $\delta(\sigma_{A,b,r}(c), c) = r > q + q'$ and $R_q(x, c) \in \sigma_{A,b,r}(p)$; also, clearly $R_{q'}(x, \sigma_{A,b,r}(c)) \in \sigma_{A,b,r}(p)$. So considering δ on a bigger monster model in which we take $a \models \sigma_{A,b,r}(p)$, we get $q + q' < \delta(c, \sigma_{A,b,r}(c)) \leqslant \delta(c, a) + \delta(a, \sigma_{A,b,r}(c)) \leqslant q + q'$, a contradiction. Finally, consider the case $c \notin C_i$. Then for any $q' \in \mathbb{Q}$ we have $\neg R_{q'}(x, c) \in p(x)$. For a sufficiently large index (A, b, r) we have that $R_q(x, c) \in \sigma_{A,b,r}(p)$ and $\delta(\sigma_{A,b,r}(c), c) = r < \infty$; also, clearly $\neg R_{q'}(x, \sigma_{A,b,r}(c)) \in \sigma_{A,b,r}(p)$ for all rationals q'. Take $a \models \sigma_{A,b,r}(p)$. Then $\infty = \delta(a, \sigma_{A,b,r}(c)) \leqslant \delta(a, c) + \delta(c, \sigma_{A,b,r}(c)) \leqslant q + r < \infty$, a contradiction. $\square_{\text{Claim}}$

The above analysis goes through also without using the ordering $\leqslant$, showing that the theory of the Fraïssé limit of all finite metric spaces with rational distances in the language $\{R_q \mid q \in \mathbb{Q}\}$ (i.e. the rational Urysohn space) has *EDEERP*, and so trivial Ellis group, although the Fraïssé limit does not have the embedding Ramsey property (as 2-element metric spaces are not rigid).

We now list some Fraïssé classes which are known to have sep. fin. embedding Ramsey degree and whose Fraïssé limits are $\aleph_0$-categorical, and hence $\aleph_0$-saturated. By Remark 4.21, the theories of these Fraïssé limits have sep. fin. $EERdeg$, so sep. fin. $EDEERdeg$, hence their Ellis groups are profinite by Corollary 5.1.

As was recalled after Corollary 5.8, by [35] and [27], to see that a Fraïssé class $\mathcal{K}$ has sep. fin. embedding Ramsey degree one needs to find a Fraïssé class which is a reasonable, precompact, relational expansion of $\mathcal{K}$ with the embedding Ramsey property.

Recall that a Fraïssé class $\mathcal{K}^*$ in a language $L^* \supseteq L$ (where $L^* \setminus L$ consists of relational symbols) is an *expansion* of a Fraïssé class $\mathcal{K}$ in L if $\mathcal{K}$ consists of the reducts to L of the members of $\mathcal{K}^*$. An expansion $\mathcal{K}^*$ of $\mathcal{K}$ is called *reasonable* if for any $A, B \in \mathcal{K}$, embedding $f : A \to B$, and expansion $A^* \in \mathcal{K}^*$ of A, there is an expansion $B^* \in \mathcal{K}^*$ of B with $f : A^* \to B^*$ an embedding. This is equivalent to the property that the reduct to L of the Fraïssé limit of $\mathcal{K}^*$ is the Fraïssé limit of $\mathcal{K}$. An expansion $\mathcal{K}^*$ of $\mathcal{K}$ is called *precompact* if each member of $\mathcal{K}$ has only finitely many expansions to the members of $\mathcal{K}^*$.

Example 6.6. (a) The class of all finite hypergraphs omitting a fixed class of finite irreducible hypergraphs in a finite relational language. Example 6.4(a) gives the desired expansion of this class.

(b) The class of all finite vector spaces over a fixed finite field. Example 6.4(e) gives the desired expansion of this class.

(c) The class of all finite Boolean algebras in the language of all Boolean algebras. Example 6.4(f) gives the desired expansion of this class.

(d) The age of any homogeneous directed graph. See [14].

(e) The class of all finite structures in the infinite language consisting of relational symbols R_n, $n > 0$, where R_n is n-ary, such that each R_n is irreflexive and symmetric. Example 6.4(g) gives the desired expansion of this class. This holds more generally in a situation when for each n there are only finitely many relational symbols of arity n.

Sometimes we can say more. For example, in Example 6.6(b) the Fraïssé limit is just an infinite dimensional vector space over a finite field, so it is strongly minimal with the property that $\mathrm{NF}_{\bar{c}}(\mathfrak{C}) = \mathrm{Inv}_{\bar{c}}(\mathfrak{C})$, and hence the Ellis group in this case is trivial by the claim in Example 6.1 and the argument from the proof of Corollary 4.16. The situation in Example 6.6(a) is more interesting. Namely, the Ellis group there is finite. This follows by Corollary 5.5. Indeed, let L be a finite relational language and $L^* = L \cup \{\leqslant\}$. Let K^* be the Fraïssé limit of the class of all linearly ordered finite L-hypergraphs (possibly omitting a fixed class of finite irreducible hypergraphs). Then $K := K^*_{\restriction L}$ is the Fraïssé limit of all finite L-hypergraphs (omitting this fixed class of finite irreducible hypergraphs). We may find a monster model $\mathfrak{C}^*$ of $T^* := \mathrm{Th}(K^*)$ such that $\mathfrak{C} := \mathfrak{C}^*_{\restriction L}$ is a monster model of $T := \mathrm{Th}(K) = T^*_{\restriction L}$. By Example 6.4(a), T^* has $EDEERP$, so we can find $\eta^* \in \mathrm{EL}(S_{\bar{c}}(\mathfrak{C}^*))$ such that $\mathrm{Im}(\eta^*) \subseteq \mathrm{Inv}_{\bar{c}}(\mathfrak{C}^*)$ by Theorem 4.15. Moreover, both theories T and T^* have quantifier elimination. Thus, by Corollary 5.5, the Ellis group of T is finite. The same holds for the homogeneous directed graphs from Example 6.6(d).

Example 6.7. If T is the theory of the random n-hypergraph (so L in Example 6.6(a) contains only one n-ary relational symbol R), then one can say even more: T has $EDEERP$, so the Ellis group is trivial. To see this, consider $L^* := L \cup \{\leqslant\}$. Let T be the theory of the random n-hypergraph (the theory of the Fraïssé limit of the class of all finite L-hypergraphs), and T^* the theory of the ordered random n-hypergraph (the theory of the Fraïssé limit of the class of all ordered finite n-hypergraphs); then $T = T^*_{\restriction L}$. Choose a monster model $\mathfrak{C}^* \models T^*$ such that $\mathfrak{C} := \mathfrak{C}^*_{\restriction L}$ is a monster model of T.

By Example 6.4(a), T^* has $EDEERP$, so, by Theorem 4.15, we can find $\eta_0^* \in \mathrm{EL}(S_{\bar{c}}(\mathfrak{C}^*))$ such that $\mathrm{Im}(\eta_0^*) \subseteq \mathrm{Inv}_{\bar{c}}(\mathfrak{C}^*)$. By Lemma 2.18(ii), there is $\eta_0 \in \mathrm{EL}(S_{\bar{c}}(\mathfrak{C}))$ such that $\mathrm{Im}(\eta_0) \subseteq \mathrm{Inv}^*_{\bar{c}}(\mathfrak{C})$, where $\mathrm{Inv}^*_{\bar{c}}(\mathfrak{C})$ is the set of all $\mathrm{Aut}(\mathfrak{C}^*)$-invariant types in $S_{\bar{c}}(\mathfrak{C})$. We claim that $\mathrm{Inv}^*_{\bar{c}}(\mathfrak{C}) \subseteq \mathrm{Inv}_{\bar{c}}(\mathfrak{C})$, which finishes the proof by Theorem 4.15.

Let $p(\bar{x}) \in S_{\bar{c}}(\mathfrak{C})$ be $\mathrm{Aut}(\mathfrak{C}^*)$-invariant and we prove that it is $\mathrm{Aut}(\mathfrak{C})$-invariant. By quantifier elimination of T, it is enough to prove that whenever $R(\bar{z}, \bar{a}) \in p(\bar{x})$ and $\bar{a} \equiv^L \bar{b}$, then $R(\bar{z}, \bar{b}) \in p(\bar{x})$, where $\bar{z} \subseteq \bar{x}$, $\bar{a} \subseteq \mathfrak{C}$, $|\bar{z}| + |\bar{a}| = n$, and $|\bar{z}| \geqslant 1$ (so $|\bar{a}| \leqslant n-1$). Since T^* has quantifier elimination, $|\bar{a}| \leqslant n-1$, the arity of all symbols in L is n, and all relations from L are irreflexive and symmetric in $\mathfrak{C}$, we see that the L^*-type of $\bar{a}$ is determined completely by the order, so for some permutation $\bar{b}'$ of $\bar{b}$ we have $\bar{a} \equiv^{L^*} \bar{b}'$. By $\mathrm{Aut}(\mathfrak{C}^*)$-invariance of $p(\bar{x})$, we get $R(\bar{z}, \bar{b}') \in p(\bar{x})$, so, by symmetry of R, $R(\bar{z}, \bar{b}) \in p(\bar{x})$.

Note that in the previous example, the same argument is viable if L contains finitely many relational symbols each of which is of one of two consecutive arities n or $n+1$. Only the very last paragraph should be additionally explained. If $R \in L$ is of arity n, $R(\bar{z}, \bar{a}) \in p(\bar{x})$, and $\bar{a} \equiv^L \bar{b}$, then $|\bar{a}| \leqslant n-1$, so the L^*-type of $\bar{a}$ is completely determined by the order, and $R(\bar{z}, \bar{b}) \in p(\bar{x})$ as above. If $R \in L$ is of arity $n+1$, $R(\bar{z}, \bar{a}) \in p(\bar{x})$, and $\bar{a} \equiv^L \bar{b}$, then, if $|\bar{a}| \leqslant n-1$, the same argument shows that $R(\bar{z}, \bar{b}) \in p(\bar{x})$. But now it is possible that $|\bar{a}| = n$, so $\bar{z} = z$ is a single variable. In this case, the L^*-type of $\bar{a}$ is determined by the order and by whether or not $R'(\bar{a})$ holds for the n-ary symbols $R' \in L$. The latter does not depend on permutations of $\bar{a}$, as the R''s are symmetric. Thus, again we can find a permutation $\bar{b}'$ of $\bar{b}$ such that $\bar{a} \equiv^{L^*} \bar{b}'$, so $R(z, \bar{b}') \in p(\bar{x})$ by $\mathrm{Aut}(\mathfrak{C}^*)$-invariance of $p(\bar{x})$, and $R(z, \bar{b}) \in p(\bar{x})$ by symmetry of R.

However, the above conclusion does not hold for all random hypergraphs in finite languages. In the following example, we state that the theory of the random $(2,4)$-hypergraph has a non-trivial Ellis group and describe various consequences of this result. All computations around this example are contained in Appendix A.

Example 6.8. Let $L = \{R_2, R_4\}$, where R_2 is a binary and R_4 is a quaternary relation. The Fraïssé limit K of the class of all finite L-hypergraphs is the random $(2,4)$-hypergraph. We will prove that the Ellis group of the theory $T := \mathrm{Th}(K)$ is $\mathbb{Z}/2\mathbb{Z}$. Hence, T does not have $EDEERP$ by Corollary 4.16, although it does have sep. fin. $EERdeg$ by Example 6.6(a). Furthermore, this theory does have $DEERP$ by Theorem 4.17, as the class above has free amalgamation and so T is extremely amenable (see the discussion after Corollary 4.16 in [13]). So the theory T shows that in general sep. fin. $EDEERdeg$ (or even $EERdeg$) does not imply $EDEERP$, and that $DEERP$ does not imply $EDEERP$.

Since T is extremely amenable, $\mathrm{Gal}_L(T) = \mathrm{Gal}_{KP}(T)$ is trivial by [13, Proposition 4.31], whereas the Ellis group of T is non-trivial. On the other hand, [13, Theorem 0.7] says that a certain natural epimorphism from the Ellis group of an amenable theory with NIP to its KP-Galois group is always an isomorphism (this is a variant of Newelski's conjecture for groups of automorphisms). Thus, our theory T shows that in this theorem the assumption that the theory has NIP cannot be dropped (even assuming extreme amenability of the theory in question). Such an example was not known so far.

Now, we will describe a variant of the above example which yields a Fraïssé structure whose theory T satisfies the assumptions of Remark 5.7 and Proposition 5.6, and so has sep. fin. $EDEERdeg$, and the Ellis group of T is infinite (and clearly profinite). On the other hand, T is extremely amenable, so $\mathrm{Gal}_{KP}(T)$ is trivial. It will be a many-sorted example. But this is not a problem, because, as we said before, the whole theory developed in this paper works in a many-sorted context after minor adjustments. In the case of Proposition 5.6, it is easy to see that the proof goes through (only in Corollary 2.14, which is involved in this proof, one has to consider the tuple $\bar{z}$ in which there are infinitely many variables associated with each sort).

Example 6.9. Consider the language L consisting of infinitely many sorts S_n, $n < \omega$, and relational symbols R_2^n and R_4^n, $n < \omega$, where each R_2^n is binary, each R_4^n quaternary, and they are both associated with the sort S_n. Let $L^* := L \cup \{\leqslant_n : n < \omega\}$, where each $\leqslant_n$ is a binary symbol associated with S_n. Let K be the L^*-structure which is the disjoint union of copies of the ordered random $(2,4)$-hypergraph; then $K := K^*_{\restriction L}$ is the disjoint union of copies of the random $(2,4)$-hypergraph. Put $T := \mathrm{Th}(K)$ and $T^* := \mathrm{Th}(K^*)$. Choose a monster model $\mathfrak{C}^*$ of T^* such that $\mathfrak{C} := \mathfrak{C}^*_{\restriction L}$ is a monster model of T.

It is clear that L^* is relational, T^* has quantifier elimination (as each sort has it and there are no interactions between distinct sorts), and $L^* \setminus L$ is finite among relations involving variables from any given finitely many sorts. So the assumptions of the obvious many-sorted version of Remark 5.7 are satisfied. Now, we check the remaining assumption of Proposition 5.6 that there is $\eta^* \in \mathrm{EL}(S_{\bar{c}}(\mathfrak{C}^*))$ such that $\mathrm{Im}(\eta^*) \subseteq \mathrm{Inv}_{\bar{c}}(\mathfrak{C}^*)$. Let $\mathfrak{C}^*_n := S_n(\mathfrak{C}^*)$ be the $\{R_2^n, R_4^n, \leqslant_n\}$-structure induced from $\mathfrak{C}^*$, for $n < \omega$. Then $\mathrm{Aut}(\mathfrak{C}^*) = \prod_{n \leqslant \omega} \mathrm{Aut}(\mathfrak{C}^*_n)$ and $S_{\bar{c}}(\mathfrak{C}^*) = \prod_{n \leqslant \omega} S_{\bar{c}_n}(\mathfrak{C}^*_n)$ after the clear identifications, where $\bar{c}_n \subseteq \bar{c}$ is the enumeration of $\mathfrak{C}^*_n$. By [29, Lemma 6.44], $\mathrm{EL}(S_{\bar{c}}(\mathfrak{C}^*))$ is naturally isomorphic (as a semigroup and as an $\mathrm{Aut}(\mathfrak{C}^*)$-flow) with $\prod_{n \leqslant \omega} \mathrm{EL}(S_{\bar{c}_n}(\mathfrak{C}^*_n))$. But in each $\mathrm{EL}(S_{\bar{c}_n}(\mathfrak{C}^*_n))$ we have η^*_n whose image is contained in the $\mathrm{Aut}(\mathfrak{C}^*_n)$-invariant types (by Example 6.4(a) and Theorem 4.15), so the corresponding $\eta^* \in \mathrm{EL}(S_{\bar{c}}(\mathfrak{C}^*))$ has image contained in $\mathrm{Inv}_{\bar{c}}(\mathfrak{C}^*)$.

Now, we will compute the Ellis group of T. Let $\mathfrak{C}_n := S_n(\mathfrak{C})$ be the $\{R_2^n, R_4^n\}$-structure induced from $\mathfrak{C}$, for $n < \omega$. Then $\mathrm{Aut}(\mathfrak{C}) = \prod_{n \leqslant \omega} \mathrm{Aut}(\mathfrak{C}_n)$ and $S_{\bar{c}}(\mathfrak{C}) = \prod_{n \leqslant \omega} S_{\bar{c}_n}(\mathfrak{C}_n)$ after the clear identifications (where $\bar{c}_n \subseteq \bar{c}$ is the enumeration of $\mathfrak{C}_n$). By [29, Lemma 6.44], $u\mathcal{M} \cong \prod_{n \leqslant \omega} u_n\mathcal{M}_n$ (also in the τ-typologies), where $\mathcal{M}$ and $\mathcal{M}_n$ are minimal left ideals of $\mathrm{EL}(S_{\bar{c}}(\mathfrak{C}))$ and $\mathrm{EL}(S_{\bar{c}_n}(\mathfrak{C}_n))$, respectively, and $u \in \mathcal{M}$ and $u_n \in \mathcal{M}_n$ are idempotents. Since each $u_n\mathcal{M}_n \cong \mathbb{Z}/2\mathbb{Z}$ by Example 6.8, we conclude that $u\mathcal{M} \cong (\mathbb{Z}/2\mathbb{Z})^\omega$ as a topological group; in particular, $u\mathcal{M}$ is infinite.

Extreme amenability of T can be seen directly: each finitary type $p(\bar{x})$ has a global invariant extension determined by the formulae: $y \neq a$, $\neg R^n_2(y, b)$, and $\neg R^n_4(\bar{y}, \bar{b}')$, for all $n < \omega$, and $a, b, \bar{b}' \subseteq \mathfrak{C}$ (with $|\bar{b}'| \geqslant 1$), $y \in \bar{x}$, $\bar{y}' \subseteq \bar{x}$ from the sorts for which these formulae make sense.

We now give an example of a theory with sep. fin. $EDEERdeg$ (even $EDEERP$) such that for some finite set of formulae Δ we do not have $\eta \in \mathrm{EL}(S_{\bar{c},\Delta}(\mathfrak{C}))$ with finite image. This shows that we really need to consider spaces $S_{\bar{c},\Delta}(\bar{p})$, rather than the classical spaces of Δ-types $S_{\bar{c},\Delta}(\mathfrak{C})$.

Example 6.10. Consider $L = \{R_2, P_n\}_{n<\omega}$, where the P_n's are unary and R_2 is binary. Put $P_\omega := \bigwedge_{n<\omega} \neg P_n$. Consider the class of all finite L-structures A such that:

- P^A_n are mutually disjoint for $n < \omega$, and
- R^A_2 is irreflexive and symmetric.

This class is Fraïssé. Its Fraïssé limit K consists of infinitely many disjoint parts $P_n(K)$, for $n \leqslant \omega$, each of which is isomorphic to the random graph, but also there is the random interaction between them. Also, $T := \mathrm{Th}(K)$ has quantifier elimination, hence it is binary, K is $\aleph_0$-saturated, although it is not $\aleph_0$-categorical.

To see that T has $EDEERP$, by Proposition 4.19, we need to prove that for each singleton a, finite $\bar{b}$ containing a, $n < \omega$, and externally definable coloring $c : \binom{\mathfrak{C}}{a} \to 2^n$ there is $\bar{b}' \in \binom{\mathfrak{C}}{\bar{b}}$ such that $\binom{\bar{b}'}{a}$ is monochromatic with respect to c. We will prove more, namely we will not restrict ourselves to externally definable colorings.

For each $n \leqslant \omega$, $P_n(\mathfrak{C})$ is the set of realizations of a complete 1-type over $\emptyset$. Fix a singleton a; then $a \in P_n(\mathfrak{C})$ for some $n \leqslant \omega$. Take a finite $\bar{b} \ni a$ and write $\bar{b} = \bar{b}_0\bar{b}_1$, where $\bar{b}_0 \subseteq P_n(\mathfrak{C})$ and $\bar{b}_1 \cap P_n(\mathfrak{C}) = \emptyset$. Take $r < \omega$ and a coloring $c : \binom{\mathfrak{C}}{a} \to r$ (so $c : P_n(\mathfrak{C}) \to r$). By saturation, we can find a copy $G \subseteq P_n(\mathfrak{C})$ of the random graph. The restriction of c to G corresponds to a finite partition of G, so we can find a monochromatic isomorphic copy G' of G (we use here the following combinatorial fact: For any finite partition of the random graph G, at least one of the parts is isomorphic to G; see [3, Proposition 3.3]). Let $\bar{b}'_0 \subseteq G'$ be the copy of $\bar{b}_0$; by quantifier elimination, we have $\bar{b}'_0 \equiv \bar{b}_0$. Take $\sigma \in \mathrm{Aut}(\mathfrak{C})$ such that $\sigma(\bar{b}_0) = \bar{b}'_0$ and put $\bar{b}' := \sigma(\bar{b}) = \bar{b}'_0\sigma(\bar{b}_1)$. Then $\bar{b}' \in \binom{\mathfrak{C}}{\bar{b}}$ and $\binom{\bar{b}'}{a} = \binom{\bar{b}'_0}{a} = \bar{b}'_0 \subseteq G'$, so $\binom{\bar{b}'}{a}$ is monochromatic with respect to c.

Consider now $\Delta := \{R\}$. We check that $\mathrm{EL}(S_{\bar{c},\Delta}(\mathfrak{C}))$ does not contain an element with finite image. Let $\bar{x}$ correspond to $\bar{c}$ and let $x_0 \in \bar{x}$ be any fixed single variable. For every $S \subseteq \omega + 1$ consider:

$$\pi_S(x_0) := \{R(x_0, a) \mid a \in P_n(\mathfrak{C}) \text{ for some } n \in S\} \cup \{\neg R(x_0, a) \mid a \in P_n(\mathfrak{C}) \text{ for some } n \notin S\}.$$

By randomness, $\pi_S(x_0)$ is consistent with $\mathrm{tp}(\bar{c})$, so we can find $p_S(\bar{x}) \in S_{\bar{c},\Delta}(\mathfrak{C})$ extending it. Note that each $\pi_S(x_0)$ is $\mathrm{Aut}(\mathfrak{C})$-invariant, as the $P_n(\mathfrak{C})$'s are sets of realizations of complete 1-types over $\emptyset$. Thus, for each $\eta \in \mathrm{EL}(S_{\bar{c},\Delta}(\mathfrak{C}))$ we have $\pi_S(x_0) \subseteq \eta(p_S)$, so $\eta(p_S)$ are pairwise distinct for $S \subseteq \omega + 1$. Therefore, $\mathrm{Im}(\eta)$ is infinite.

Next, we give an example of a theory T showing that (A') does not imply (B) in Theorem 3. Hence, (D) fails for T and so T does not have sep. fin. $EDEERdeg$ by Theorem 4.22. Also, T turns out to be amenable and so has $DEECRP$ by Theorem 4.31.

Example 6.11. Consider the language $L = \{R, E_n\}_{n<\omega}$, where each symbol is a binary relational symbol, and consider an L-structure M such that:

- R is irreflexive and symmetric;
- each E_n is an equivalence relation with exactly two classes;
- equivalences $\{E_n\}_{n<\omega}$ are independent in the sense that for each choice of E_n-class C_n, the family $\{C_n\}_{n<\omega}$ has the finite intersection property;
- for distinct $a_0, \dots, a_{l-1}, b_0, \dots, b_{m-1} \in M$ and any $n < \omega$, the set $\bigcap_{i<l} R(M, a_i) \cap \bigcap_{j<m} \neg R(M, b_j)$ intersects each $\bigcap_{k\leqslant n} E_k$-class.

Note that in particular we have that (M, R) is a model of the theory of the random graph. Let $T := \mathrm{Th}(M)$. By standard arguments, it is straightforward to see that T has quantifier elimination. Consequently, there is only one type in $S_1(T)$. Let $\mathfrak{C} \models T$ be a monster model.

We will first show that Condition (C) (equivalently (B)) fails for T. More precisely, we will show that the Ellis group of the $\mathrm{Aut}(\mathfrak{C})$-flow $S_{\bar{c},\Delta}(p)$ is infinite, where $\Delta := \{R(x_0, y)\}$ and $p(y) \in S_y(T)$ is the unique 1-type over $\emptyset$ (so, $S_{\bar{c},\Delta}(p) = S_{\bar{c},\Delta}(\mathfrak{C})$). Here, $\bar{x}$ is reserved for $\bar{c}$ and $x_0 \in \bar{x}$ is any fixed variable. Consider the following global partial types:

$$\pi_n(\bar{x}) := \{(R(x_0, a) \leftrightarrow R(x_0, b)) \leftrightarrow E_n(a, b) \mid a, b \in \mathfrak{C}\}.$$

In fact, by randomness, each $\pi_n(\bar{x}) \cup \mathrm{tp}(\bar{c})$ is consistent. So $X_n := [\pi_n(\bar{x})]$ is a non-empty, closed subset of $S_{\bar{c}}(\mathfrak{C})$, which is moreover an $\mathrm{Aut}(\mathfrak{C})$-subflow, as the $\pi_n(\bar{x})$'s are clearly $\mathrm{Aut}(\mathfrak{C})$-invariant. By independence of the relations E_n, the X_n's are pairwise disjoint. Let $\Phi : S_{\bar{c}}(\mathfrak{C}) \to S_{\bar{c},\Delta}(\mathfrak{C})$ be the $\mathrm{Aut}(\mathfrak{C})$-flow epimorphism given by restriction to Δ-types. Note that $\Phi^{-1}[\Phi[X_n]] = X_n$: If $q(\bar{x}) \notin X_n$, then for some $a, b \in \mathfrak{C}$ we have either $E_n(a, b)$ and $R(x_0, a) \wedge \neg R(x_0, b) \in q(\bar{x})$, or $\neg E_n(a, b)$ and $R(x_0, a) \wedge R(x_0, b) \in q(\bar{x})$, or $\neg E_n(a, b)$ and $\neg R(x_0, a) \wedge \neg R(x_0, b) \in q(\bar{x})$. In all three cases, the same conjunction belongs to $\Phi(q(\bar{x}))$, but it does not belong to $\Phi(q'(\bar{x}))$ for any $q'(\bar{x}) \in X_n$. Thus, $q(\bar{x}) \notin \Phi^{-1}[\Phi[X_n]]$. Therefore, $Y_n := \Phi[X_n]$ are pairwise disjoint $\mathrm{Aut}(\mathfrak{C})$-subflows of $S_{\bar{c},\Delta}(\mathfrak{C})$. Since for each $\sigma \in \mathrm{Aut}(\mathfrak{C})$ we have $\sigma[Y_n] \subseteq Y_n$, the same holds for any $\eta \in \mathrm{EL}(S_{\bar{c},\Delta}(\mathfrak{C}))$. Consequently, $\mathrm{Im}(\eta)$ is infinite; hence, Condition (D) does not hold, but we want to show that (C) fails.

Let $u \in \mathrm{EL}(S_{\bar{c},\Delta}(\mathfrak{C}))$ be an idempotent in a minimal left ideal $\mathcal{M}$, and let $q_n(\bar{x}) \in Y_n$ be in $\mathrm{Im}(u)$; so $u(q_n) = q_n$. Consider any $a \in \mathfrak{C}$ and its E_n-class $C_n := E_n(a, \mathfrak{C})$ for $n < \omega$. For each $n < \omega$ find an automorphism $\sigma_n \in \mathrm{Aut}(\mathfrak{C})$ such that σ_n fixes each E_k-class for $k < n$ and swaps the E_n-classes. To see that σ_n exists, just take $b \in \bigcap_{k<n} C_k \cap C_n^c$; since $a \equiv b$, we have the desired σ_n. Note that $u\sigma_n u \in u\mathcal{M}$; we claim that they are all distinct. For each $n < \omega$ let $\epsilon_n \in 2$ be such that $\varphi_n(x_0, a) := R^{\epsilon_n}(x_0, a) \in q_n$. Since $q_n \in Y_n$, this formula completely determines whether $R(x_0, b) \in q_n$ or not, for each $b \in \mathfrak{C}$. For $k < n$ we check that $\varphi_k(x_0, a) \in u\sigma_n(q_k) = u\sigma_n u(q_k)$. If this is not the case, then $\varphi_k(x_0, a) \in q_k = u(q_k)$ and $\neg\varphi_k(x_0, a) \in u\sigma_n(q_k)$, which is an open condition on u. So there is an automorphism σ such that $\varphi_k(x_0, a) \in \sigma(q_k)$ and $\neg\varphi_k(x_0, a) \in \sigma\sigma_n(q_k)$, i.e. $\varphi_k(x_0, \sigma^{-1}(a)) \in q_k$ and $\neg\varphi_k(x_0, \sigma_n^{-1}(\sigma^{-1}(a))) \in q_k$. But this is not possible, since σ_n fixes E_k-classes, as $k < n$. Similar argument shows that we have $\neg\varphi_n(x_0, a) \in u\sigma_n(q_n) = u\sigma_n u(q_n)$, as σ_n swaps the

E_n-class. So, for $k < n$, $u\sigma_k u(q_k) \neq u\sigma_n u(q_k)$, hence $u\sigma_k u \neq u\sigma_n u$, and $u\mathcal{M}$ is infinite. This finishes the proof that Condition (C) does not hold.

We now prove amenability of T. We have to construct an invariant, finitely additive probability measure on clopens in $S_p(\mathfrak{C})$ for every $p(\bar{y}) \in S_{\bar{y}}(T)$, where $\bar{y}$ is finite. Let $\bar{y} = (y_0, \dots, y_m)$. First, let $\Delta := \{E_n(y_0, z)\}_{n<\omega}$ and $S_{y_0,\Delta}(\mathfrak{C})$ be the space of all Δ-types in variable y_0. If we fix an E_n-class C_n for each n, then, by independence of the E_n's, the elements of $S_{y_0,\Delta}(\mathfrak{C})$ are the sets of formulae $\{y_0 \in C_n^{\epsilon(n)} \mid n < \omega\}$ for all $\epsilon \in 2^\omega$. So the clopens in $S_{y_0,\Delta}(\mathfrak{C})$ correspond to the (finite) Boolean combinations of the C_n's, and hence there is an invariant, finitely additive probability measure ν on the algebra of these clopens, which is determined by saying that the clopen corresponding to any intersection of k-many $C_n^{\epsilon(n)}$'s has measure $1/2^k$.

Consider $\Phi : S_p(\mathfrak{C}) \to S_{y_0,\Delta}(\mathfrak{C})$ given by restriction. This is clearly an $\mathrm{Aut}(\mathfrak{C})$-flow epimorphism. Let $q_R(\bar{y}) := \{R(y_i, a) \mid i < m, a \in \mathfrak{C}\}$; by randomness and q.e., $q_R(\bar{y})$ is a a partial type consistent with $p(\bar{y})$. It is $\mathrm{Aut}(\mathfrak{C})$-invariant, so $X := [q_R(\bar{y})]$ is an $\mathrm{Aut}(\mathfrak{C})$-subflow of $S_p(\mathfrak{C})$. Let us consider the $\mathrm{Aut}(\mathfrak{C})$-homomorphism $\Phi_{\restriction X} : X \to S_{y_0,\Delta}(\mathfrak{C})$. Note that for each $r(y_0) \in S_{y_0,\Delta}(\mathfrak{C})$ the type $p(\bar{y}) \cup q_R(\bar{y}) \cup r(y_0)$ extends to a unique element of X. Indeed, by randomness and q.e., it is consistent, so it extends to at least one element of X. On the other hand, by q.e., it is determined by the formulae and their negations of the form $R(y_i, y_j)$, $E_n(y_i, y_j)$, and $y_i = y_j$ (which is already given by $p(\bar{y})$), $R(y_i, a)$ (which is given by $q_R(\bar{y})$) and $E_n(y_i, a)$. The formulae $E_n(y_0, a)$ are determined by $r(y_0)$, and $E_n(y_i, a)$ for $i > 0$ are determined by $E_n(y_0, a)$ and $E_n(y_0, y_i)$ (given by $p(\bar{y})$), as each E_n is an equivalence relation with two classes. (One should note that $q_R(\bar{y})$ implies $y_i \neq a$ for all $i < \omega$ and $a \in \mathfrak{C}$, so we do not have to worry about formulae of this type.) Therefore, $\Phi_{\restriction X}$ is an $\mathrm{Aut}(\mathfrak{C})$-flow isomorphism; in particular, a homeomorphism. Now, one can define μ on clopens of $S_p(\mathfrak{C})$ by $\mu(U) := \nu(\Phi[X \cap U])$. It is easy to check that μ is an invariant, finitely additive probability measure on clopens of $S_p(\mathfrak{C})$.

The rest of the analysis of Example 6.11 is devoted to the proof that Condition (A') holds for T, i.e. our goal is to show that the canonical Hausdorff quotient of the Ellis group of T is profinite. Since the theory is binary, by Corollary 2.15, we can work with $(\mathrm{Aut}(\mathfrak{C}), S_1(\mathfrak{C}))$ in place of $(\mathrm{Aut}(\mathfrak{C}), S_{\bar{c}}(\mathfrak{C}))$.

Put $E_\infty := \bigwedge_{i\in\omega} E_i$ and $H := 2^\omega$. For each $i \in \omega$, fix an enumeration $C_{i,0}, C_{i,1}$ of the E_i-classes, and an enumeration $(C_\epsilon)_{\epsilon\in H}$ of the E_∞-classes. Let $X := S_1(\mathfrak{C})$ and put

$$X' := \bigcap_{a,b\in\mathfrak{C}, E_\infty(a,b)} [R(x,a) \leftrightarrow R(x,b)].$$

It is clear that X' is an $\mathrm{Aut}(\mathfrak{C})$-subflow of X. By q.e., each $p \in X'$ is implied by the union of the following partial types for unique $\epsilon \in H$ and $\delta \in 2^H$,

- $p_\epsilon(x) := \{x \in C_{i,\epsilon(i)} \mid i \in \omega\}$; and
- $q_\delta(x) := \{R(x,a) \mid a \in C_{\epsilon'}, \delta(\epsilon') = 1\} \cup \{\neg R(x,a) \mid a \in C_{\epsilon'}, \delta(\epsilon') = 0\}$.

Conversely, for each $\epsilon \in H$ and $\delta \in 2^H$ the union of the above partial types implies a type in X'. So X' is topologically identified with the space $H \times 2^H$. Let $p_{\epsilon,\delta} \in S_1(\mathfrak{C})$ be the type determined by p_ϵ and q_δ for $(\epsilon, \delta) \in H \times 2^H$. Further, for $\mathrm{Stab}(X') := \{\sigma \in \mathrm{Aut}(\mathfrak{C}) \mid \sigma_{\restriction X'} = \mathrm{id}_{X'}\}$, we see that $\mathrm{Aut}(\mathfrak{C})/\mathrm{Stab}(X') \cong H$ and the flow $(\mathrm{Aut}(\mathfrak{C})/\mathrm{Stab}(X'), X')$ can be identified with the flow $(H, H \times 2^H)$ equipped with the following action: For $\sigma \in H$ and $(\epsilon, \delta) \in H \times 2^H$,

$$\sigma(\epsilon, \delta) := (\sigma + \epsilon, \sigma\delta),$$

where $\sigma\delta(\epsilon') := \delta(\epsilon' - \sigma) = \delta(\epsilon' + \sigma)$ for $\epsilon' \in H$.

Claim. *There is $\eta \in \mathrm{EL}(X)$ whose image is contained in X'.*

Proof of Claim. For each $\epsilon \in H$, put

$$\Delta_\epsilon(x) := \{R(x, a) \leftrightarrow R(x, b) : a, b \in C_\epsilon\}.$$

First, we will show that for every $\epsilon \in H$ there is $\eta_\epsilon \in \mathrm{EL}(X)$ such that:

- η_ϵ is the limit of a net of $\sigma \in \mathrm{Aut}(\mathfrak{C})$ fixing each $C_{\epsilon'}$, $\epsilon' \in H$, setwise (equivalently, this is a net of Shelah strong automorphism);
- $\mathrm{Im}(\eta_\epsilon) \subseteq [\Delta_\epsilon]$.

For this, choose any representatives $a_{\epsilon'}$ of the classes $C_{\epsilon'}$, $\epsilon' \in H$, and consider any $p_0, \dots, p_{k-1} \in X$ and $a_0, \dots, a_{n-1} \in C_\epsilon$. Since $(C_\epsilon, R_{\restriction C_\epsilon})$ is a (monster) model of the theory of the random graph, there are $a'_0, \dots, a'_n \in C_\epsilon$ such that $\bar{a} := a_0 \dots a_{n-1} \equiv a'_0 \dots a'_{n-1} =: \bar{a}'$ and for each $l < k$:

$$p_l \models \bigwedge_{j,j'<n} R(x, a'_j) \leftrightarrow R(x, a'_{j'}).$$

By randomness, q.e., and saturation, we can find $a'_{\epsilon'} \in C_{\epsilon'}$ for all $\epsilon' \in H$ so that

$$(a_j \mid j < n)^\frown(a_{\epsilon'} \mid \epsilon' \in H) \equiv (a'_j \mid j < n)^\frown(a'_{\epsilon'} \mid \epsilon' \in H).$$

So, there is $\sigma_{\bar{p},\bar{a}} \in \mathrm{Aut}(\mathfrak{C})$ mapping the latter sequence above to the former one. Now, it is enough to take η_ϵ to be an accumulation point of the net $(\sigma_{\bar{p},\bar{a}})_{\bar{p},\bar{a}}$.

Since the η_ϵ's are approximated by Shelah strong automorphisms, for any $\epsilon \in H$:

- $\eta_\epsilon[[\Delta_{\epsilon'}]] \subseteq [\Delta_{\epsilon'}]$ for each $\epsilon' \in H$;
- $\mathrm{Im}(\eta_\epsilon) \subseteq [\Delta_\epsilon]$.

For $\bar{\epsilon} = \{\epsilon_i\}_{i<n} \subseteq H$ put $\eta_{\bar{\epsilon}} := \eta_{\epsilon_{n-1}} \circ \dots \circ \eta_{\epsilon_0}$, where $\epsilon_0 < \dots < \epsilon_{n-1}$ (say with respect to the lexicographic order). Then, an accumulation point of the net $(\eta_{\bar{\epsilon}})_{\bar{\epsilon}}$ fulfills our requirements, namely $\mathrm{Im}(\eta) \subseteq \bigcap[\Delta_\epsilon] = X'$. (In fact, $\mathrm{Im}(\eta) = X'$, because η acts trivially on X', as it is approximated by Shelah strong automorphisms.) $\square_{\text{Claim}}$

By the claim and Corollary 2.6, the Ellis groups of the flows $(\mathrm{Aut}(\mathfrak{C}), X)$ and $(\mathrm{Aut}(\mathfrak{C}), X')$ are topologically isomorphic. By the discussion before the claim, the latter Ellis group is topologically isomorphic with the Ellis group of the flow $(H, H \times 2^H)$ which in turn is topologically isomorphic with the Ellis group of the flow $(H, \beta H)$ by Proposition 2.8(ii). As a conclusion, we get that the canonical Hausdorff quotient of the Ellis group of T is topologically isomorphic with the canonical Hausdorff quotient of the Ellis group of βH, i.e. with the generalized Bohr compactification of H. Since H is abelian (and so strongly amenable), Fact 2.9 implies that this generalized Bohr compactification coincides with the Bohr compactification of H, which is profinite by Fact 2.10, as H is abelian of finite exponent. So Condition (A') has been proved.

We do not know whether Condition (A) holds for T. This is equivalent to the question whether the Ellis group of T is Hausdorff.

The next example is a modification of the previous one. It shows that (A") does not imply (A') in Theorem 3. Moreover, the obtained theory is supersimple of SU-rank 1, so we see that even for supersimple theories the Ellis group need not be profinite, while Gal_{KP} group must be profinite by [2]. Our theory is also amenable.

Example 6.12. The idea is to modify the previous example by considering independent equivalence relations with growing finite number of classes in order to get at the end that the canonical Hausdorff quotient of the Ellis group of the resulting theory is topologically isomorphic with the Bohr compactification of the product $\prod_{n\geqslant 2}\mathbb{Z}_n$ which is not profinite by Fact 2.10, as $\prod_{n\geqslant 2}\mathbb{Z}_n$ is not of finite exponent.

Consider the language $L' = \{R, E_n\}_{2\leqslant n<\omega}$, where each symbol is a binary relational symbol, and consider an L'-structure M' such that:

- R is irreflexive and symmetric;
- each E_n is an equivalence relation with exactly n classes;
- equivalences $\{E_n\}_{n\geqslant 2}$ are independent in the sense that for each choice of E_n-class C_n (for $n \geqslant 2$), the family $\{C_n\}_{n\geqslant 2}$ has the finite intersection property;
- for distinct $a_0,\dots,a_{l-1},b_0,\dots,b_{m-1} \in M'$ and any $2 \leqslant n < \omega$, the set $\bigcap_{i<l} R(M',a_i)\cap \bigcap_{j<m} \neg R(M',b_i)$ intersects each $\bigcap_{k\leqslant n} E_k$-class.

Expand now M' to the structure M in the language $L := L' \cup \{O_{n,k}\}_{\substack{2\leqslant n<\omega\\ 1\leqslant k<n}}$, where each $O_{n,k}$ is a binary relational symbol interpreted as follows:

- for each $2 \leqslant n < \omega$ enumerate the E_n-classes as $C_{n,0},\dots,C_{n,n-1}$ and take the cyclic permutation $\rho_n := (0,1,\dots,n-1) \in \mathrm{Sym}(n)$; set $O_{n,k} := \bigcup_{i<n} C_{n,i}\times C_{n,\rho_n^k(i)}$.

Note that $O_{n,k}(a,b)$ means that in our cyclic ordering of the E_n-classes, b belongs to the k-th consecutive E_n-class of the E_n-class of a. Let $T := \mathrm{Th}(M)$. A standard back-and-forth argument shows that T has quantifier elimination; in particular, there is only one type in $S_1(T)$. Let $\mathfrak{C} \models T$ be a monster model.

For $n \geqslant 2$ put $\bar{C}_n := (C_{n,0},\dots,C_{n,n-1})$. Define $O_n(x_0,\dots,x_n)$ as $\bigwedge_{1\leqslant k<n} O_{n,k}(x_0,x_k)$. Observe that $O_n(a_0,\dots,a_{n-1})$ holds iff $(a_0/E_n,\dots,a_{n-1}/E_n)$ belongs to the orbit of $\bar{C}_n$ under the action of $\mathbb{Z}_n$ on $(M/E_n)^n$ defined as follows: For $(b_0,\dots,b_{n-1}) \in (M/E_n)^n$ and $l \in \mathbb{Z}_n$:

$$l(b_0,\dots,b_{n-1}) := (b_{\rho_n^l(0)},\dots,b_{\rho_n^l(n-1)}).$$

(Originally, we wanted to work in the language $L' \cup \{O_n \mid n \geqslant 2\}$ instead of $L' \cup \{O_{n,k} \mid 1 \leqslant k < n < \omega\}$, but it turned out that we do not have q.e. in this language.)

By q.e. and randomness, it is easy to see that for any a and A such that $a \notin A$, $\mathrm{tp}(a/A)$ does not fork over $\emptyset$. Hence, T is supersimple of SU-rank 1. (The same is true in Example 6.11, but supersimplicity is more important here.) Therefore, by [2], $\mathrm{Gal}_{KP}(T)$ is profinite, i.e. (A") holds. Amenability of T can be proved in a similar fashion as in Example 6.11.

Finally, we will prove that (A') fails, following the lines of the argument in Example 6.11. Put $E_\infty := \bigwedge_{n\geqslant 2} E_n$ and $H := \prod_{n\geqslant 2}\mathbb{Z}_n$. Fix an enumeration $(C_\epsilon)_{\epsilon\in H}$ of the E_∞-classes.

Let $X := S_1(\mathfrak{C})$ and put:

$$X' := \bigcap_{a,b\in\mathfrak{C}, E_\infty(a,b)} [R(x,a) \leftrightarrow R(x,b)].$$

It is clear that X' is an $\mathrm{Aut}(\mathfrak{C})$-subflow of X. By q.e., each $p \in X'$ is implied by the union of the following partial types for unique $\epsilon \in H$ and $\delta \in 2^H$,

- $p_\epsilon(x) := \{x \in C_{n,\epsilon(n)} \mid n \geqslant 2\}$; and
- $q_\delta(x) := \{R(x,a) \mid a \in C_{\epsilon'}, \delta(\epsilon') = 1\} \cup \{\neg R(x,a) \mid a \in C_{\epsilon'}, \delta(\epsilon') = 0\}$.

Conversely, for each $\epsilon \in H$ and $\delta \in 2^H$ the union of the above partial types implies a type in X'. So X' is topologically identified with the space $H \times 2^H$.

Now, using the fact that the O_n's are L-formulae, it is easy to see that for $\mathrm{Stab}(X') := \{\sigma \in \mathrm{Aut}(\mathfrak{C}) \mid \sigma_{\restriction X'} = \mathrm{id}_{X'}\}$, $\mathrm{Aut}(\mathfrak{C})/\mathrm{Stab}(X') \cong H$ and the flow $(\mathrm{Aut}(\mathfrak{C})/\mathrm{Stab}(X'), X')$ can be identified with the flow $(H, H \times 2^H)$ equipped with the following action: For $\sigma \in H$ and $(\epsilon, \delta) \in H \times 2^H$,

$$\sigma(\epsilon, \delta) := (\sigma + \epsilon, \sigma\delta),$$

where $\sigma\delta(\epsilon') := \delta(\epsilon' - \sigma)$ for $\epsilon' \in H$.

Next, the proof of the claim from the analysis of Example 6.11 goes through to conclude that there exists $\eta \in \mathrm{EL}(X)$ whose image is contained in X'. Note also that T is binary by quantifier elimination. So the same argument as in the paragraph after this claim shows that the canonical Hausdorff quotient of the Ellis group of T is topologically isomoprhic with the Bohr compactification of H, which is not profinite by Fact 2.10, because $\prod_{n\geqslant 2} \mathbb{Z}_n$ does not have finite exponent. Thus, Condition (A') fails.

By [13, Proposition 4.31], we know that if a theory T is extremely amenable, then $\mathrm{Gal}_L(T) = \mathrm{Gal}_{KP}(T)$ is trivial. The next example (whose details are left to the reader) shows that amenability of T does not even imply that $\mathrm{Gal}_{KP}(T)$ is profinite.

Example 6.13. Let $N = (M, X, \cdot)$ be the two-sorted structure, where:

- M is a real closed field in the language $L_{or}(\mathbb{R})$ of ordered rings with constant symbols for all $r \in \mathbb{R}$;
- $\cdot : S^1 \times X \to X$ is a strictly 1-transitive action of the circle group S^1 on X.

N is clearly interpretable in M, hence $T := \mathrm{Th}(N)$ has NIP. By [7], we easily get that $\mathrm{Gal}_{KP}(T) \cong S^1$, so $\mathrm{Gal}_{KP}(T)$ is not profinite. By [13, Corollary 4.19], we know that a NIP theory is amenable iff $\emptyset$ is an extension base for forking. We leave as a non-difficult exercise to check that in T every set is an extension base. It is convenient to use here the fact that it is enough to test this property only for 1-types; and, in our case, one has to consider two kinds of 1-types, depending on with which of the two sorts the variable of the type in question is associated. Both cases are easy.

We have determined the relationships (implications or lack of implications) between most of the introduced properties. However, there are still a few questions around this. Let us list some of them.

Question 6.14. Is there an example for which (C) holds but (D) does not?

Question 6.15. (i) Is there an example for which (A) holds but (C) does not?
(ii) Is there an example for which (A') holds but (A) does not?

Example 6.11 shows that (A') does not imply (B). So this example either witnesses that (A') does not imply (A), or that (A) does not imply (B); but we do not know which of these two lack of implications is witnessed by Example 6.11. By Remark 3.2(a), it witnesses the lack of the implication (A') $\implies$ (A) if and only if the Ellis group of the theory in this example is not Hausdorff.

By Example 6.1, we know that having sep. fin. *EDEERdeg* does not imply *DEERP*; this is witnessed by any stable theory with a non-stationary type in $S_1(\emptyset)$. Is the converse true, i.e. does *DEERP* (equiv. extreme amenablity) imply sep. fin. *EDEERdeg*? Probably not. Example 6.11 shows that *DEECRP* (equiv. amenability) does not imply sep. fin. *EDEERdeg*. In fact, Example 6.13 shows that amenability does not even imply that Gal_{KP} is profinite (i.e. (A")), whereas extreme amenability implis that Gal_{KP} is trivial by [13, Proposition 4.31].

By Example 6.6(e), we know that the Ellis group there is profinite. Is it infinite? Can one compute it precisely, as we did for the theory from Example 6.8? The same problem for all random hypergraphs, although here we know that the Ellis groups are finite by the paragraph after Example 6.6.

Appendix A. Example 6.8

We now calculate the Ellis group from Example 6.8. Recall that we consider the language $L := \{R_2, R_4\}$, where R_2 is a binary and R_4 is a quaternary relational symbol. We consider the class of all finite $(2,4)$-hypergraphs, i.e. the class of all finite structures in L for which R_2 and R_4 are irreflexive and symmetric (for R_4 this means that $R_4(a_0, a_1, a_2, a_3)$ implies $\bigwedge_{i<j<4} a_i \neq a_j$ and $R(a_{\sigma(0)}, a_{\sigma(1)}, a_{\sigma(2)}, a_{\sigma(3)})$ for every $\sigma \in \mathrm{Sym}(4)$). This is a Fraïssé class, and its Fraïssé limit K is called the random $(2,4)$-hypergraph. The theory $T := \mathrm{Th}(K)$ is $\aleph_0$-categorical, and consequently $\aleph_0$-saturated with quantifier elimination.

We also consider the following expansion of T. Let $L^* = L \cup \{<\}$. Consider the class of all finite linearly ordered $(2,4)$-hypergraphs. This is a Fraïssé class, and its Fraïssé limit K^* is called the ordered random $(2,4)$-hypergraph. The theory $T^* := \mathrm{Th}(K^*)$ is $\aleph_0$-categorical, so $\aleph_0$-saturated with quantifier elimination. Then $K^*_{\restriction L} \cong K$, so $T^*_{\restriction L} = T$, and we may assume that $K^*_{\restriction L} = K$.

Lemma A.1. *There exists $\rho \in \mathrm{Aut}(K)$ such that ρ reverses the order on K^*: $\rho[<] = >$.*

Proof. This is a straightforward back-and-forth construction. Let $K = \{a_n\}_{n<\omega}$. We build an increasing sequence of finite partial L-elementary mappings $\rho_0 \subseteq \rho_1 \subseteq \dots$ such that a_n belongs to both the domain and the range of ρ_n, and for any $a < b$ in the domain of ρ_n we have $\rho_n(a) > \rho_n(b)$. Then $\rho := \bigcup_{n<\omega} \rho_n$ will be the desired automorphism.

For ρ_0 we may set $\rho_0(a_0) = a_0$. If we have ρ_{n-1} defined, we define ρ_n in two steps as follows: If $a_n \in \mathrm{dom}(\rho_{n-1})$, then put $\rho'_n = \rho_{n-1}$ and proceed to the second step. Otherwise, let $\mathrm{dom}(\rho_{n-1}) = \{b_i\}_{i<m}$ be such that $b_0 < b_1 < \dots < b_{m-1}$. Suppose that $b_i < a_n < b_{i+1}$ for $i < m-1$ (the cases $a_n < b_0$ and $b_{m-1} < a_n$ are similar). Consider

$\mathrm{tp}^L_{x,y_0,\dots,y_{m-1}}(a_n, b_0, \dots, b_{m-1}) \cup \{y_0 > \dots > y_i > x > y_{i+1} > \dots y_{m-1}\}$. By randomness, this is a type in T^*, so take $(a'_n, b'_0, \dots, b'_{m-1}) \subseteq K^*$ realizing it. Since by quantifier elimination we have $(b'_0, \dots, b'_{m-1}) \equiv^{L^*} (\rho_{n-1}(b_0), \dots, \rho_{n-1}(b_{m-1}))$, by ultrahomogeneity we can find $\sigma \in \mathrm{Aut}(K^*)$ such that $\sigma(b'_i) = \rho_{n-1}(b_i)$. Now, extend ρ_{n-1} to ρ'_n by setting $\rho'_n(a_n) = \sigma(a'_n)$. The second step, i.e. extending ρ'_n to ρ_n such that a_n belongs to the range of ρ_n is analogous. □

Take the automorphism ρ given by the previous lemma. Take $L^*_\rho := L^* \cup \{\rho\}$, and look at the obvious expansion K^*_ρ of K^*. Take a monster $\mathfrak{C}^*_\rho$ of $\mathrm{Th}(K^*_\rho)$ such that $\mathfrak{C}^* := \mathfrak{C}^*_{\rho\restriction L^*}$ and $\mathfrak{C} := \mathfrak{C}^*_{\rho\restriction L}$ are monster models of T^* and T, respectively. The interpretation of ρ in $\mathfrak{C}^*_\rho$, which will be also denoted by ρ, is an automorphism of $\mathfrak{C}$ reversing $<$. Further on, we fix $\mathfrak{C}^*$, $\mathfrak{C}$, and ρ.

Since T^* has *EDEERP* (even *EERP* by Example 6.4(a)), by Theorem 4.15, we can find $u^* \in \mathrm{EL}(S_{\bar{c}}(\mathfrak{C}^*))$ with $\mathrm{Im}(u^*) \subseteq \mathrm{Inv}_{\bar{c}}(\mathfrak{C}^*)$. By Lemma 2.18(iv), for a single variable z, we can find $u \in \mathrm{EL}(S_z(\mathfrak{C}))$ such that $\mathrm{Im}(u) \subseteq \mathrm{Inv}^*_z(\mathfrak{C})$, where $\mathrm{Inv}^*_z(\mathfrak{C})$ is the set of all $\mathrm{Aut}(\mathfrak{C}^*)$-invariant types in $S_z(\mathfrak{C})$. Moreover, we may assume that u is an idempotent in a minimal left ideal $\mathcal{M}$ of $\mathrm{EL}(S_z(\mathfrak{C}))$.

The elements of $\mathrm{Inv}^*_z(\mathfrak{C})$ are not hard to describe. Let $p(z) \in \mathrm{Inv}^*_z(\mathfrak{C})$. Since there is only one type in $S_y(T^*)$, $p(z)$ either contains $R_2(z,a)$ for all $a \in \mathfrak{C}$ or $\neg R_2(z,a)$ for all $a \in \mathfrak{C}$. Note that $S_{\pi(\bar{y})}(T^*)$, where $\bar{y} = (y_0, y_1, y_2)$ and $\pi(\bar{y}) = \{y_0 \neq y_1 \neq y_2 \neq y_0\}$, is completely determined by restriction to $\{R_2, <\}$. Let us write $[\mathfrak{C}]^3 = O_0 \sqcup O_1 \sqcup O_2 \sqcup O_3$, where O_i is the set of all $\{a,b,c\} \in [\mathfrak{C}]^3$ with exactly i-many R_2-edges on the set $\{a,b,c\}$. Note that by symmetry of R_4, either $R_4(z,a,b,c) \in p(z)$ for all $\{a,b,c\} \in O_0$, or $\neg R_4(z,a,b,c) \in p(z)$ for all $\{a,b,c\} \in O_0$. The same holds for O_3. The sets O_1 and O_2 are more interesting. Write $O_1 = O_1^- \sqcup O_1^\circ \sqcup O_1^+$, where for $\{a,b,c\} \in O_1$ with b being R_2-unconnected with a and c we put:

$$\{a,b,c\} \in \begin{cases} O_1^- & \text{if } b \text{ is minimal among } \{a,b,c\} \\ O_1^\circ & \text{if } b \text{ is the middle one among } \{a,b,c\} \\ O_1^+ & \text{if } b \text{ is maximal among } \{a,b,c\} \end{cases}.$$

Similarly, we write $O_2 = O_2^- \sqcup O_2^\circ \sqcup O_2^+$, where the division is determined by the element R_2-connected to both other elements. By symmetry of R_4, we have either $R_4(z,a,b,c) \in p(z)$ for all $\{a,b,c\} \in O_1^-$, or $\neg R_4(z,a,b,c) \in p(z)$ for all $\{a,b,c\} \in O_1^-$. The same holds for $O_1^\circ, O_1^+, O_2^-, O_2^\circ$ and O_2^+. By quantifier elimination, $p(z)$ is completely determined by the previous information. Moreover, by randomness, each described possibility occurs. Thus, we see that $\mathrm{Inv}^*_z(\mathfrak{C})$ has 2^9 elements.

Lemma A.2. *$u(p) = p$ for all $p \in \mathrm{Inv}^*_z(\mathfrak{C})$. In particular,* $\mathrm{Im}(u) = \mathrm{Inv}^*_z(\mathfrak{C})$.

Proof. It is enough to prove the first part. If $R_2(z,a)^\epsilon \in p$ for $\epsilon \in 2$ and $a \in \mathfrak{C}$, then $R_2(z,a)^\epsilon \in \sigma(p)$ for every $\sigma \in \mathrm{Aut}(\mathfrak{C})$, so $R_2(z,a)^\epsilon \in u(p)$ holds as well. Similarly, if $R_4(z,a,b,c)^\epsilon \in p$ for $\epsilon \in 2$ and $\{a,b,c\} \in O_0$, then $R_4(z,a,b,c)^\epsilon \in \sigma(p)$ for every $\sigma \in \mathrm{Aut}(\mathfrak{C})$, so $R_4(z,a,b,c)^\epsilon \in u(p)$. The same holds for O_3.

Note that if we put $q = u(p)$, then, by idempotency, $u(q) = u^2(p) = u(p) = q$.

Let us focus on O_1. We say that $p \in \mathrm{Inv}^*_z(\mathfrak{C})$ is of type $(\epsilon_-, \epsilon_\circ, \epsilon_+)$, where $\epsilon_-, \epsilon_\circ, \epsilon_+ \in 2$, if $R_4^{\epsilon_\star}(z, a, b, c) \in p$ for all $\{a, b, c\} \in O_1^\star$, for each $\star \in \{-, \circ, +\}$.

Claim. *p and $q = u(p)$ have the same type.*

Proof of Claim. Note that if p is of type $(0, 0, 0)$ or $(1, 1, 1)$, then q is of the same type, as all automorphisms in these cases preserve the type of p, and hence u preserves it, too. Take a, b, c, d such that $a < b < c < d$, $R_2(a, d)$, $R_2(b, c)$, and there are no other R_2-edges between a, b, c, d. Let q be of type $(\epsilon_-, \epsilon_\circ, \epsilon_+)$. Then the formula $\phi(z, a, b, c, d) :=$

$$R_4^{\epsilon_-}(z, a, b, c) \wedge R_4^{\epsilon_\circ}(z, b, a, d) \wedge R_4^{\epsilon_\circ}(z, c, a, d) \wedge R_4^{\epsilon_+}(z, d, b, c) \ \in\ q = u(p) = u(q),$$

which is an open condition on u, so we can find an automorphism $\sigma \in \mathrm{Aut}(\mathfrak{C})$ such that $\phi(z, a, b, c, d) \in \sigma(p), \sigma(q)$, i.e. $\phi(z, a', b', c', d') \in p, q$, where $\sigma(a', b', c', d') = (a, b, c, d)$. We have the following cases.

Case 1. q is of type $(1, 0, 0)$. Then $R_4(z, a', b', c') \in q$ implies $a' < b', c'$, so $R_4(z, a', b', c') \in p$ implies that p is of one of the types $(1, 0, 0)$, $(1, 1, 0)$ or $(1, 0, 1)$.
- 1.1. p is of type $(1, 1, 0)$. Then $\neg R_4(z, b', a', d') \wedge \neg R_4(z, d', b', c') \in p$ implies $b' > a', d'$ and $d' > b', c'$ which is not possible.
- 1.2. p is of type $(1, 0, 1)$. Then $\neg R_4(z, b', a', d') \wedge \neg R_4(z, c', a', d') \wedge \neg R_4(z, d', b', c') \in p$ implies that b' and c' are between a' and d', and d' is between b' and c'. This is not possible, too.

Thus p is of type $(1, 0, 0)$.

Case 2. q is of type $(0, 0, 1)$. This is completely dual by interchanging a' and d' as well as $<$ and $>$.

Case 3. q is of type $(0, 1, 0)$. Then $R_4(z, b', a', d') \in q$ implies that b' is between a' and d', so $R_4(z, b', a', d') \in p$ implies that the type of p is either $(0, 1, 0)$, $(0, 1, 1)$ or $(1, 1, 0)$.
- 3.1. p is of type $(0, 1, 1)$. Then $\neg R_4(z, a', b', c') \wedge \neg R_4(z, d', b', c') \in p$ implies $a' < b', c'$ and $d' < b', c'$. This is not possible, as b' is between a' and d'.
- 3.2. p is of type $(1, 1, 0)$. Then $\neg R_4(z, a', b', c') \wedge \neg R_4(z, d', b', c') \in p$ implies $a' > b', c'$ and $d' > b', c'$. This is again impossible.

Thus p is of type $(0, 1, 0)$.

Case 4. q is of type $(0, 0, 0)$. Then $\neg R_4(z, a', b', c') \wedge \neg R_4(z, b', a', d') \wedge \neg R_4(z, c', a', d') \wedge \neg R_4(z, d', b', c') \in p$.
- 4.1 p is of type $(1, -, -)$. Then $b' < a'$ or $c' < a'$, $a' < b'$ or $d' < b'$, $a' < c'$ or $d' < c'$, and $b' < d'$ or $c' < d'$, and this is not possible. E.g. if $b' < a'$, then $d' < b'$, hence $c' < d'$, so $a' < c'$, and we get $b' < b'$.
- 4.2 p is of type $(-, -, 1)$. This is dual to the previous by interchanging $<$ and $>$.
- 4.3 p is of type $(0, 1, 0)$. This subcase requires a different trick. Choose five elements a_0, a_1, a_2, a_3, a_4 such that $R_2(a_i, a_{i+1})$ for all $i < 5$ (here $+$ is modulo 5), and there are no other R_2-edges between them. Then $\bigwedge_{i<5} \neg R_4(z, a_i, a_{i+2}, a_{i+3}) \in q = u(p) = u(q)$, as q is of type $(0, 0, 0)$. As above, we can approximate u by σ and find a copy $(a'_0, a'_1, a'_2, a'_3, a'_4)$ of $(a_0, a_1, a_2, a_3, a_4)$ such that both p and q contain $\bigwedge_{i<5} \neg R_4(z, a'_i, a'_{i+2}, a'_{i+3})$. Since p is of type $(0, 1, 0)$, we get that

$a'_i < a'_{i+2}, a'_{i+3}$ or $a'_i > a'_{i+2}, a'_{i+3}$, for all $i < 5$. But it is easy to see that this is impossible (just looking at $<$).

Thus p is of type $(0, 0, 0)$.

The remaining cases are completely dual by interchanging 0 and 1, and R_4 and $\neg R_4$ in the previous cases. $\square$ Claim

It remains to discuss O_2. This is analogous to the discussion of O_1, and one can adapt the previous analysis by interchanging all R_2-edges and R_2-non-edges. We leave this to the reader. The lemma is proved. $\square$

We can now easily see that $u\mathcal{M}$ is not trivial. Take $p, q \in \mathrm{Inv}^*_z(\mathfrak{C})$ such that $p(z)$ implies that z is not R_2-connected to anything and only R_4-connected to O_1^-, and $q(z)$ implies that z is not R_2-connected to anything and only R_4-connected to O_1^+. Note that $u\rho u \in u\mathcal{M}$ and that $\rho[O_1^-] = O_1^+$ and vice versa. Thus, by Lemma A.2, $u\rho u(p) = u\rho(p) = u(q) = q$, so $u\rho u \neq u$ as $u(p) = p$. We will see that $u\mathcal{M} = \{u, u\rho u\}$, but this will require more work, involving applications of contents.

Lemma A.3. *$u\mathcal{M} = \{u, u\rho u\}$, so $u\mathcal{M} \cong \mathbb{Z}/2\mathbb{Z}$.*

Proof. Since $\mathrm{Im}(u) \subseteq \mathrm{Inv}^*_z(\mathfrak{C})$, it is enough to prove that for any $\eta \in u\mathcal{M}$: either $\eta(p) = u(p)$ for all $p \in \mathrm{Inv}^*_z(\mathfrak{C})$, or $\eta(p) = u\rho u(p)$ for all $p \in \mathrm{Inv}^*_z(\mathfrak{C})$. We define the notion of O_1-type and O_2-type of $p \in \mathrm{Inv}^*_z(\mathfrak{C})$ as in the proof of Lemma A.2. Fix $\eta \in u\mathcal{M}$; p will always range over $\mathrm{Inv}^*_z(\mathfrak{C})$.

Claim. *If the O_1-type of p is $(\epsilon_-, \epsilon_\circ, \epsilon_+)$, then the O_1-type of $\eta(p)$ is $(\epsilon_-, \epsilon_\circ, \epsilon_+)$ or $(\epsilon_+, \epsilon_\circ, \epsilon_-)$.*

Proof of Claim. If $\epsilon_- = \epsilon_\circ = \epsilon_+$, then each automorphism preserves the O_1-type of p, hence η preserves it, too, and we are done.

Take elements a_0, a_1, a_2, a_3 such that $R_2(a_0, a_2)$ and $R_2(a_1, a_3)$, and there are no other R_2-edges between them. Put $q(y_0, y_1, y_2, y_3) := \mathrm{tp}^L(a_0, a_1, a_2, a_3)$. For a realization $\bar{b}$ of q we will say that it is of type:

A: if $\min(\bar{b})$ is R_2-connected to $\max(\bar{b})$ (min and max are taken in $\mathfrak{C}^*$);

B: if $\min(\bar{b})$ and $\max(\bar{b})$ are not R_2-connected and $\min(\bar{b})^\star < \max(\bar{b})^\star$, where $b_i^\star = b_{i+2}$, so it is the element to which b_i is R_2-connected;

C: if $\min(\bar{b})$ and $\max(\bar{b})$ are not R_2-connected and $\max(\bar{b})^\star < \min(\bar{b})^\star$.

For a type p, let $\delta_0, \delta_1, \delta_2, \delta_3 \in \{0, 1\}$ be the unique numbers such that the formula $\bigwedge_{i<4} R_4^{\delta_i}(z, b_i, b_{i+1}, b_{i+2})$ belongs to p. Note that they depend on p and $\bar{b} \models q$. Denote this formula by $\phi_{p,\bar{b}}(z, \bar{b})$. In the following table, we calculate $\sum_{i<4} \delta_i$ depending on the O_1-type of p and the type of ordering on $\bar{b}$:

	A	B	C
(0,0,0)	0	0	0
(1,0,0)	1	2	1
(0,1,0)	2	0	2
(0,0,1)	1	2	1
(1,1,0)	3	2	3
(1,0,1)	2	4	2
(0,1,1)	3	2	3
(1,1,1)	4	4	4

Recall that by Fact 2.21, $\mathrm{ct}(\eta(p)) \subseteq \mathrm{ct}(p)$. By our choice, $(\phi_{p,\bar{b}}(z,\bar{y}), q(\bar{y})) \in \mathrm{ct}(p)$ for every $\bar{b} \models q$. Consider the following cases.

Case 1. p is of O_1-type $(1,0,0)$ or $(0,0,1)$. If $\eta(p)$ is of type $(0,0,0)$, $(0,1,0)$, $(1,0,1)$ or $(1,1,1)$, then choose $\bar{b} \models q$ of type B. Since $(\phi_{\eta(p),\bar{b}}(z,\bar{y}), q(\bar{y})) \in \mathrm{ct}(\eta(p))$, this pair belongs to $\mathrm{ct}(p)$ as well. But, by the table, this is not possible, since $\phi_{\eta(p),\bar{b}}$ has either 0 or 4 positive occurrences of R_4, whereas this does not happen in $\varphi_{p,\bar{b}'}$ for any $\bar{b}' \models q$ if p is of type $(1,0,0)$ or $(0,0,1)$. Similarly, if $\eta(p)$ is of type $(1,1,0)$ or $(0,1,1)$, by choosing $\bar{b} \models q$ of type A, we have $(\phi_{\eta(p),\bar{b}}(z,\bar{y}), q(\bar{y})) \in \mathrm{ct}(\eta(p)) \subseteq \mathrm{ct}(p)$, but since $\phi_{\eta(p),\bar{b}}$ has 3 positive occurrences of R_4, we again cannot find $\bar{b}' \models q$ such that $\phi_{\eta(p),\bar{b}}(z,\bar{b}') \in p$.

So, $\eta(p)$ is either of O_1-type $(1,0,0)$ or $(0,0,1)$.

Case 2. p is of O_1-type $(0,1,0)$. By similar considerations as in Case 1, we can eliminate the possibilities that $\eta(p)$ is of any O_1-type different from $(0,0,0)$ and $(0,1,0)$. The case when $\eta(p)$ is of O_1-type $(0,0,0)$ requires a different trick, but this can be done in the same way as Case 4.3 in the proof of Lemma A.2. So $\eta(p)$ is of O_1-type $(0,1,0)$.

The remaining cases are dual. $\square$ Claim

Similarly, interchanging R_2 edges and R_2-non-edges, we obtain:

Claim. *If the O_2-type of p is $(\epsilon_-, \epsilon_\circ, \epsilon_+)$, then the O_2-type of $\eta(p)$ is $(\epsilon_-, \epsilon_\circ, \epsilon_+)$ or $(\epsilon_+, \epsilon_\circ, \epsilon_-)$.*

Note that if p is of O_1-type $(\epsilon_-, \epsilon_\circ, \epsilon_+)$, then $\rho(p)$ is of O_1-type $(\epsilon_+, \epsilon_\circ, \epsilon_-)$, and similarly for O_2-types. Therefore, the previous two claims say that $\eta(p)$ has the same O_1-type [O_2-type] as p or as $\rho(p)$.

Claim. *Either for every p the O_1-types of p and $\eta(p)$ are equal, or for every p the O_1-types of $\rho(p)$ and $\eta(p)$ are equal.*

Proof of Claim. Suppose not. Then we have types p, p' with O_1-types $(\epsilon_-, \epsilon_\circ, \epsilon_+)$ and $(\epsilon'_-, \epsilon'_\circ, \epsilon'_+)$ such that $\eta(p)$ and $\eta(p')$ are of O_1-types $(\epsilon_-, \epsilon_\circ, \epsilon_+)$ and $(\epsilon'_+, \epsilon'_\circ, \epsilon'_-)$, respectively, where $\epsilon_- \neq \epsilon_+$ and $\epsilon'_- \neq \epsilon'_+$. We have two cases.

Case 1. $(\epsilon_-, \epsilon_+) = (\epsilon'_-, \epsilon'_+)$. Consider a_0, a_1, a_2, a_3 such that $R_2(a_0, a_2)$, $R_2(a_1, a_3)$, and there are no other R_2-edges between them, and $a_0, a_2 < a_1, a_3$. Let $q(\bar{y}) =$

$\mathrm{tp}^L(a_0, a_1, a_2, a_3)$. Choose $\delta_0, \delta_1, \delta_2, \delta_3$ such that $\bigwedge_{i<4} R_4^{\delta_i}(z, a_i, a_{i+1}, a_{i+2}) \in \eta(p)$. Then we have $\bigwedge_{i<4} R_4^{1-\delta_i}(z, a_i, a_{i+1}, a_{i+2}) \in \eta(p')$, so:

$$\left(\bigwedge_{i<4} R_4^{\delta_i}(z, y_i, y_{i+1}, y_{i+2}), \bigwedge_{i<4} R_4^{1-\delta_i}(z, y_i, y_{i+1}, y_{i+2}), q(\bar{y})\right) \in \mathrm{ct}(\eta(p), \eta(p')).$$

By Fact 2.21, this triple belongs to $\mathrm{ct}(p, p')$, so we can find $\bar{b} \models q$ such that $\bigwedge_{i<4} R_4^{\delta_i}(z, b_i, b_{i+1}, b_{i+2}) \in p$ and $\bigwedge_{i<4} R_4^{1-\delta_i}(z, b_i, b_{i+1}, b_{i+2}) \in p'$. Choose i such that $b_{i+1} = \min(\bar{b})$. Then $R_4^{\delta_i}(z, b_i, b_{i+1}, b_{i+2}) \in p$ implies that $\epsilon_- = \delta_i$, and $R_4^{1-\delta_i}(z, b_i, b_{i+1}, b_{i+2}) \in p'$ implies that $\epsilon'_- = 1 - \delta_i$. Therefore, $\epsilon_- \neq \epsilon'_-$; a contradiction.

Case 2. $(\epsilon_-, \epsilon_+) \neq (\epsilon'_-, \epsilon'_+)$. Then $(\epsilon_-, \epsilon_+) = (\epsilon'_+, \epsilon'_-)$, so we reduce this to Case 1 by considering $\eta(p)$ and $\eta(p')$ instead of p and p', and η^{-1} (computed in $u\mathcal{M}$) instead of η (note that $\eta^{-1}(\eta(p)) = u(p) = p$ and $\eta^{-1}(\eta(p')) = u(p') = p'$ by Lemma A.2).

The proof of the claim is finished. $\square$ Claim

Similarly, we obtain:

Claim. *Either for every p the O_2-types of p and $\eta(p)$ are equal, or for every p the O_2-types of $\rho(p)$ and $\eta(p)$ are equal.*

We finally prove:

Claim. *Either for every p the O_1-types of p and $\eta(p)$ are equal and the O_2-types of p and $\eta(p)$ are equal, or for every p the O_1-types of $\rho(p)$ and $\eta(p)$ are equal and the O_2-types of $\rho(p)$ and $\eta(p)$ are equal.*

Proof of Claim. Let p have both the O_1-type and the O_2-type equal to $(1, 0, 0)$. If the claim fails, then, by the previous two claims, the O_1-types of p and $\eta(p)$ are equal and the O_2-types of $\rho(p)$ and $\eta(p)$ are equal, or the O_1-types of $\rho(p)$ and $\eta(p)$ are equal and the O_2-types of p and $\eta(p)$ are equal.

So, assume first that $\eta(p)$ has O_1-type $(1, 0, 0)$ but O_2-type $(0, 0, 1)$. Consider a_0, a_1, a_2, a_3 such that $R_2(a_0, a_1)$, $R_2(a_0, a_3)$ and there are no other R_2-edges between them, and $a_0 > a_1 > a_2 > a_3$; set $q(\bar{y}) = \mathrm{tp}^L(a_0, a_1, a_2, a_3)$. Then $R_4(z, a_2, a_0, a_1) \wedge \neg R_4(z, a_2, a_0, a_3) \wedge R_4(z, a_0, a_1, a_3) \in \eta(p)$. But, by Fact 2.21, $\mathrm{ct}(\eta(p)) \subseteq \mathrm{ct}(p)$. Hence, we can find $\bar{b} \models q$ such that $R_4(z, b_2, b_0, b_1) \wedge \neg R_4(z, b_2, b_0, b_3) \wedge R_4(z, b_0, b_1, b_3) \in p$. Since the O_1-type and the O_2-type of p are both $(1, 0, 0)$, this implies $b_2 < b_0, b_1$, $b_0 < b_1, b_3$, but b_2 is not less than both b_0 and b_3. Clearly, this is not possible.

If $\eta(p)$ has O_1-type $(0, 0, 1)$ but O_2-type $(1, 0, 0)$, the proof is dual by reversing the order on $\{a_0, a_1, a_2, a_3\}$. $\square$ Claim

We are ready to finish the proof of the lemma. If p contains $R_2^\epsilon(z, a)$ for some $\epsilon \in 2$ and all $a \in \mathfrak{C}$, then $\sigma(p)$ contains it, too, and so does $\eta(p)$. Similarly, if p contains $R_4^\epsilon(z, a, b, c)$ for some $\epsilon \in 2$ and all $\{a, b, c\} \in O_0$ [resp. $\in O_3$], then $\eta(p)$ contains it, too. Thus, the restrictions of $\eta(p)$, p, and $\rho(p)$ to these formulae coincide for every $p \in \mathrm{Inv}^*_z(\mathfrak{C})$. By the previous claim, either for every $p \in \mathrm{Inv}^*_z(\mathfrak{C})$ the restrictions of $\eta(p)$ and p to the formulae $R_4^\epsilon(z, a, b, c)$ for $\epsilon \in 2$ and $\{a, b, c\} \in O_1 \cup O_2$ coincide, or for every $p \in \mathrm{Inv}^*_z(\mathfrak{C})$ the restrictions of $\eta(p)$ and $\rho(p)$ to these formulae coincide. Therefore, either for every $p \in \mathrm{Inv}^*_z(\mathfrak{C})$ we have $\eta(p) = p = u(p)$, or for every $p \in \mathrm{Inv}^*_z(\mathfrak{C})$ we have $\eta(p) = \rho(p) = u\rho u(p)$. But this means that either $\eta = u$, or $\eta = u\rho u$. □

Proposition A.4. *The Ellis group of* $(\mathrm{Aut}(\mathfrak{C}), S_{\bar{c}}(\mathfrak{C}))$ *is* $\mathbb{Z}/2\mathbb{Z}$.

Proof. Take $u^* \in \mathrm{EL}(S_{\bar{c}}(\mathfrak{C}^*))$ such that $\mathrm{Im}(u^*) \subseteq \mathrm{Inv}_{\bar{c}}(\mathfrak{C}^*)$, as was described before Lemma A.2. By Corollary 2.18(ii), there is $u' \in \mathrm{EL}(S_{\bar{c}}(\mathfrak{C}))$ with $\mathrm{Im}(u') \subseteq \mathrm{Inv}^*_{\bar{c}}(\mathfrak{C})$. Furthermore, we may assume that u' is an idempotent in a minimal left ideal $\mathcal{M}'$ of $\mathrm{EL}(S_{\bar{c}}(\mathfrak{C}))$. By Lemma 2.12 (having in mind the natural identification of $S_{\bar{d}}(\mathfrak{C})$ with $S_{\bar{c}}(\mathfrak{C})$), we have the flow and semigroup epimorphism $\Phi : \mathrm{EL}(S_{\bar{c}}(\mathfrak{C})) \to \mathrm{EL}(S_z(\mathfrak{C}))$ (where z is a single variable) given by:

$$\Phi(\eta)(p(z)) = \hat{\eta}(p(z)) := \eta(q(\bar{x}))_{\restriction x'}[x'/z],$$

where $p(z) \in S_z(\mathfrak{C})$, $x' \in \bar{x}$, and $q(\bar{x}) \in S_{\bar{c}}(\mathfrak{C})$ are such that $q(\bar{x})_{\restriction x'}[x'/z] = p(z)$. By Fact 2.2, $u := \Phi(u')$ is an idempotent in the minimal left ideal $\mathcal{M} := \Phi[\mathcal{M}']$ of $\mathrm{EL}(S_z(\mathfrak{C}))$, and $\Phi_{\restriction u'\mathcal{M}'} : u'\mathcal{M}' \to u\mathcal{M}$ is a group epimorphism. By the formula above, $\mathrm{Im}(u) \subseteq \mathrm{Inv}^*_z(\mathfrak{C})$, so by Lemma A.2 and Lemma A.3, we have that $\mathrm{Im}(u) = \mathrm{Inv}^*_z(\mathfrak{C})$ and $u\mathcal{M} = \{u, u\rho u\}$ has two elements. So it remains to show that $\ker(\Phi_{\restriction u'\mathcal{M}'})$ is trivial.

Let $\eta \in u'\mathcal{M}'$ be such that $\hat{\eta} = u$. It is enough to prove that $\eta(q) = q$ for all $q \in \mathrm{Im}(\eta) = \mathrm{Im}(u')$ (recall that all such q's are $\mathrm{Aut}(\mathfrak{C}^*)$-invariant).

If $R_2^\epsilon(x_i, a) \in \eta(q)$, then $R_2^\epsilon(x_i, \sigma(a)) \in q$ for some $\sigma \in \mathrm{Aut}(\mathfrak{C})$, but then $R_2^\epsilon(x_i, a) \in q$ by $\mathrm{Aut}(\mathfrak{C}^*)$-invariance of q (as there is only one type in $S_1(T^*)$). Similarly, if $R_4^\epsilon(x_i, x_j, x_k, a) \in \eta(q)$, then $R_4^\epsilon(x_i, x_j, x_k, a) \in q$ by invariance. If $R_4^\epsilon(x_i, x_j, a, b) \in \eta(q)$, by symmetry of R_4 and invariance of q, the conclusion is the same: $R_4^\epsilon(x_i, x_j, a, b) \in q$.

Let us consider $R_4^\epsilon(x_i, a, b, c) \in \eta(q)$. Let $p(z) = q(\bar{x})_{\restriction x_i}[x_i/z]$; note that $p(z) \in \mathrm{Inv}^*_z(\mathfrak{C})$, so $p(z) \in \mathrm{Im}(u)$. Then $R_4^\epsilon(z, a, b, c) \in \hat{\eta}(p) = u(p) = p = q_{\restriction x_i}[x_i/z]$. Thus $R_4^\epsilon(x_i, a, b, c) \in q$.

By quantifier elimination, $\eta(q) \subseteq q$, so $\eta(q) = q$, and we are done. □

Acknowledgments

The first author would like to thank Pierre Simon for inspiring discussions on random hypergraphs. The authors are also grateful to Tomasz Rzepecki for some useful suggestions which helped us to complete our analysis of Example 6.8 and for suggesting Example 6.9.

References

[1] Auslander, J.: Minimal flows and their extensions. Mathematics Studies 153, North-Holland, The Netherlands (1988)

[2] Buechler, S., Pillay, A., Wagner, W.: Supersimple theories. J. Amer. Math. Soc. (1) **14**, 109-124 (2001)

[3] Cameron, P. J.: The random graph. In: The Mathematics of Paul Erdős II, Springer, New York, 353-378 (2013)

[4] Casanovas, E., Lascar, D., Pillay, A., Ziegler, M.: Galois groups of first order theories. J. Math. Log. **1**, 305-319 (2001)

[5] Ellis, R., Nerurkar, M.: Weakly almost periodic flows. Trans. Amer. Math. Soc. (1) **313**, 103-119 (1989)

[6] Gismatullin, J., Jagiella, G., Krupiński, K.: Bohr compactifications of groups and rings, in preparation (2019)

[7] Gismatullin, J., Newelski, L.: G-compactness and groups. Arch. Math. Logic **47**, 479-501 (2008)

[8] Glasner, S.: Proximal Flows. LNM 517, Springer, Germany (1976)

[9] Glasner, E.: Ergodic Theory via Joinings. Math. Surveys and Monographs 101, AMS, Providence (2003)

[10] Graham, R. L., Leeb, K., Rothschild, B. L.: Ramsey's theorem for a class of categories. Proc. Natl. Acad. Sci. USA (1) **69**, 119-120 (1972)

[11] Graham, R. L., Rothschild, B. L.: Ramsey's theorem for n-parameter sets. Trans. Amer. Math. Soc. **159**, 257-292 (1971)

[12] Hrushovski, E.: Definability patterns and their symmetries. arXiv: 1911.01129 (2019)

[13] Hrushovski, E., Krupiński, K., Pillay, A.: Amenability and definability. arXiv: 1901.02859 (2019)

[14] Jasiński, J., Laflamme, C., Nguyen Van Thé, L., Woodrow, R.: Ramsey precompact expansions of homogeneous directed graphs. arXiv: 1310.6466 (2013)

[15] Kechris, A. S., Pestov, V. G., Todorčević, S.: Fraïssé limits, Ramsey theory, and topological dynamics of automorphism groups. Geom. Funct. Anal. (1) **15**, 106-189 (2005)

[16] Krupiński, K., Newelski, L., Simon, P.: Boundedness and absoluteness of some dynamical invariants in model theory. J. Math. Log. (2) **19**, 1950012, 55 pages (2019)

[17] Krupiński, K., Pillay, A.: Generalized Bohr compactification and model-theoretic connected components. Math. Proc. Cambridge Philos. Soc. **163**, 219-249 (2017)

[18] Krupiński, K., Pillay, A.: On the topological dynamics of automorphism groups; a model-theoretic perspective. arXiv: 1909.13285 (2019)

[19] Krupiński, K., Pillay, A., Rzepecki, T.: Topological dynamics and the complexity of strong types. Israel J. Math. (2) **228**, 863-932 (2018)

[20] Krupiński, K., Rzepecki, T.: Smoothness of bounded invariant equivalence relations. J. Symbolic Logic (1) **81**, 326-356 (2016)

[21] Krupiński, K., Rzepecki, T.: Galois groups as quotients of Polish groups. J. Math. Log., to appear. arXiv: 1804.09247 (2018)

[22] Moore, J.: Amenability and Ramsey theory. Fund. Math. **220**, 263-280 (2013)

[23] Nešetřil, J.: Metric spaces are Ramsey. European J. Combin. (1) **28**, 457-468 (2007)

[24] Nešetřil, J., Rödl, V.: Ramsey classes of set systems. J. Combin. Theory Ser. A (2) **34**, 183-201 (1983)

[25] Newelski, L.: Topological dynamics of definable groups actions. J. Symbolic Logic **74**, 50-72 (2009)

[26] Nguyen Van Thé, L.: Structural Ramsey theory of metric spaces and topological dynamics of isometry groups. arXiv:0804.1593 (2008)

[27] Nguyen Van Thé, L.: More on the Kechris-Pestov-Todorčević correspondence: precompact expansions. Fund. Math. **222**, 19-47 (2013)

[28] Nguyen Van Thé, L.: Fixed points in compactifications and combinatorial counterparts. Ann. Henri Lebesgue **2**, 149-185 (2019)

[29] Rzepecki, T.: Bounded Invariant Equivalence Relations. PhD Thesis, Wrocław, arXiv: 1810.05113 (2018)

[30] Salles, T. S.: Extreme Amenability of Topological Groups. Master's Thesis, Bonn (2011)

[31] Sokić, M.: Ramsey properties of finite posets. Order (1) **29**, 1-30 (2012)

[32] Sokić, M.: Ramsey property, ultrametric spaces, finite posets, and universal minimal flows. Israel J. Math. (2) **194**, 609-640 (2013)

[33] Turek, S.: Minimal actions on Cantor cubes. Bull. Pol. Acad. Sci. Math. (2) **51**, 129-138 (2003)

[34] Ziegler, M.: Introduction to the Lascar group. In: Tits buildings and the model theory of groups (Würzburg, 2000), London Math. Soc. Lecture Note Ser. 291, Cambridge University Press, Cambridge, 279-298 (2002)

[35] Zucker, A.: Topological dynamics of automorphism groups, ultrafilter combinatorics, and the Generic Point Problem. Trans. Amer. Math. Soc. (9) **368**, 6715-6740 (2016)

(K. Krupiński) Uniwersytet Wrocławski, Instytut Matematyczny, Plac Grunwaldzki 2/4, 50-384 Wrocław, Poland

Email address: kkrup@math.uni.wroc.pl

(J. Lee) Uniwersytet Wrocławski, Instytut Matematyczny, Plac Grunwaldzki 2/4, 50-384 Wrocław, Poland

Email address: jlee@math.uni.wroc.pl

(S. Moconja) University of Belgrade, Faculty of Mathematics, Studentski trg 16, 11000 Belgrade, Serbia

Email address: slavko@matf.bg.ac.rs