Stable groups, List 2

We work in a monster model $\mathfrak{C} = \mathfrak{C}^{eq}$ of a stable theory T. G denotes a group $\emptyset$-definable in $\mathfrak{C}$.

Problem 1. Assume T is ω-stable.
(i) Let $p, q \in S_G(A)$ be stationary and of the same Morley rank. Prove that $g \cdot p = q$ iff $g \cdot \tilde{p} = \tilde{q}$.
(ii) Let H be a $\emptyset$-type-definable subgroup of G. Let $p_1, \dots, p_d \in S(\mathrm{acl}(\emptyset))$ be all the generics of H. Let $H_1 := \{g \in G : g \cdot \{p_1, \dots, p_d\} = \{p_1, \dots, p_d\}\}$. Deduce from (i) that $H_1 = \{g \in G : g \cdot \{\tilde{p}_1, \dots, \tilde{p}_d\} = \{\tilde{p}_1, \dots, \tilde{p}_d\}\}$ is a subgroup of G.
(iii) Prove that H_1 defined in (ii) is $\mathrm{acl}(\emptyset)$-definable.

Problem 2. Let E be a definable subset of $\mathfrak{C}$ on which G acts definably. Let $D \subseteq E$ be definable and $g \in G$. Prove that if $gD \subseteq D$, then $gD = D$.

Problem 3. Prove that each superstable group satisfies ωdcc^0.

Problem 4. We have proved the theorem saying that in a stable group, for any given formula $\varphi(x, y)$ there is no infinite strictly decreasing sequence of subgroups which are Boolean combinations of φ-definable subgroups. Deduce from this the Baldwin-Saxl chain condition. (Notice that, in fact, the proof of the aforementioned theorem yields its generalization to the version with subgroups which are Boolean combinations of φ-definable sets.)

Problem 5. Drop the assumption that G is stable.
(i) Prove that if G is centralizer-connected and $N \lhd G$ is finite, then $N \leq Z(G)$.
(ii) Assume G is stable. Prove that if in G there are only finitely many commutators, then G is central-by-finite.
(iii) Prove that if G is centralizer-connected and Z(G) is finite, then $Z(G) = Z_2(G)$.
(iv) Prove that if G is infinite, centralizer-connected and nilpotent, then $Z(G)$ is infinite.

Problem 6. Let H be an arbitrary nilpotent group, and $\{e\} \neq N \lhd H$. Prove that $N \cap Z(H)$ is non-trivial.

Problem 7. Assume T is ω-stable. Assume that G is nilpotent. Prove that for every definable subgroup H of infinite index in G the index $[N(H) : H]$ is infinite.

Stable groups, List 2'

The problems on this auxiliary list are not meant to be solved during the discussion sessions. Try to solve at least the first two problems (the second one is very easy).

Problem 1. Recall that a formula $\varphi(x,y)$ has IP if there are sequences $(a_i)_{i\in\omega}$ and $(b_I)_{I\subseteq\omega}$ such that

$$\varphi(a_i, b_I) \iff i \in I.$$

Let $\varphi^{opp}(y,x) := \varphi(x,y)$. Prove that $\varphi(x,y)$ has IP iff $\varphi^{opp}(y,x)$ has IP.

Problem 2. Note that if $\varphi(x,y)$ has IP, then it has OP (in other words, it is unstable).

Problem 3.* Prove that if $\varphi(x,y)$ has OP, then $\varphi(x,y)$ has IP or some other formula $\psi(x,z)$ has SOP. (This implies that each unstable theory has IP or SOP; recall that a theory has one of these properties if there is a formula with this property).

Problem 4. Prove that $\varphi(x,y)$ has IP if and only if there is an indiscernible sequence $(a_i)_{i\in\omega}$ and a tuple b such that

$$\models \varphi(a_i, b) \iff i \text{ is even}.$$

Deduce that a Boolean combination of NIP formulas (in given variables (x,y)) is also a NIP formula. (This observation, together with the theorem saying that a theory has IP iff there is an IP formula $\varphi(x,y)$ with x of length 1, is very useful to check that some particular theories have NIP. For example, show that every o-minimal theory has NIP.)