Stable groups, List 5

In Problems 3, 4, and 5, we work in a monster model $\mathfrak{C} = \mathfrak{C}^{eq}$ of a stable theory T.

Problem 1. Let T be an ω-stable theory, and $\varphi(x_1), \dots \varphi(x_n)$ — strongly minimal formulas with parameters.
(i) Let M be a model containing the parameters of the above formulas. Prove that if for any proper extension $N \succ M$, $\varphi_i(M) \subsetneq \varphi_i(N)$ for some i, then each type $p \in S_1(M)$ is non-orthogonal to $\varphi_j(x_j)$ for some j.
(ii) Prove that if for every (equivalently, for some $\aleph_0$-saturated) model M containing the parameters of the above formulas each type $p \in S_1(M)$ is non-orthogonal to $\varphi_i(x_i)$ for some i, then for every models $N_1 \prec N_2$ containing the parameters of the above formulas $\varphi_j(N_1) \subsetneq \varphi_j(N_2)$ for some j.

Problem 2. (i) Show that if an $\aleph_0$-saturated, stable structure M is quasi-strongly minimal, then any model containing parameters of the witnessing formulas is also quasi-strongly minimal which is witnessed by the same formulas.
(ii) Prove that if M is stable, $\aleph_0$-saturated, and quasi-strongly minimal, then it is finite dimensional which is witnessed by types of Morley rank 1.
(iii) Let G be a stable, $\aleph_0$-saturated group for which there is a sequence of definable subgroups $G_1 \lhd G_2 \lhd \cdots \lhd G_n$ such that each quotient G_{i+1}/G_i is quasi-strongly minimal and $[G : G_n] < \omega$ (e.g. $G_n = G^0$ for G of finite Morley rank). Prove that G is finite dimensional which is witnessed by types of Morley rank 1.

Problem 3. Let p be a complete type over A, and q — an arbitrary type over A. Prove that p is q-internal iff there are $B \supseteq A$ and an element $a \perp_A B$ realizing p for which there exists $\bar{b} \subseteq q(\mathfrak{C})$ such that $a \in \mathrm{dcl}_B(\bar{b})$.

Problem 4. Let $a, b \in \mathfrak{C}$ and $A \subseteq B \subset \mathfrak{C}$ be such that $\mathrm{tp}(ab/B)$ does not fork over A, $\mathrm{tp}(ab/A)$ is stationary, and $a \in \mathrm{dcl}(Bb)$. Show that $a \in \mathrm{dcl}(Ab)$.

Problem 5. Let p be a type over A, and $\psi(x)$ — a formula over A. Assume that p is φ-internal. Prove that there exists $\bar{c} \subseteq p(\mathfrak{C})$ and an A-definable function $f(\bar{x}, \bar{y})$ such that for every $a \models p$ there exists $\bar{b} \subseteq \psi(\mathfrak{C})$ with $a = f(\bar{c}, \bar{b})$.

Problem 6. Let L be language consisting only of constants $\{c_n : n \in \omega\}$, and T be a theory in L saying that these constants are pairwise distinct. Let $p = \{x = x\}$, and $q = \{x \neq c_n : n \in \omega\}$. Show that p is q-internal, but there is no definable function f such that for every $a \in p(\mathfrak{C}) = \mathfrak{C}$ there is $\bar{b} \subseteq q(\mathfrak{C})$ with $a = f(\bar{b})$.

Stable groups, List 5'

The problems on this auxiliary list are not meant to be solved during the discussion sessions (unless time permits).

Recall one of the fundamental theorems of local stability: If $\delta(x,y)$ is a stable formula and A is a small subset of a monster model $\mathfrak{C}$ (in fact, it is enough to assume that $\mathfrak{C}$ is $(|A|+|T|)^+$-saturated), then there is $(c_i)_{i<\omega}$ with $c_{i+1} \models p \restriction_{Ac_{\leq i}}$ for all i for which there is a δ-definition of p which is a positive Boolean combination of formulas $\delta(c_i, y)$. A proof can be found in "Geometric stability theory" by A. Pillay (see Lemma 2.2 on page 14).

We work in a stable theory T.

Problem 1. Let $p \in S(A)$ be a stationary type, and $(a_i)_{i<\omega}$ — a Morley sequence in p. Deduce from the above theorem that $\mathrm{Cb}(p) \subseteq \mathrm{dcl}(a_i : i < \omega)$.

Problem 2. Prove that $ab \mathop{\smile\!\!\!\!\!\!|}_D C \iff a \mathop{\smile\!\!\!\!\!\!|}_D C \wedge b \mathop{\smile\!\!\!\!\!\!|}_{Da} Ca$.

Problem 3. Assume T is ω-stable. Show that for every type $p \in S(A)$ there is a finite $B \subseteq A$ such that p does not fork over B and $p|_B$ is stationary.

Problem 4. Assume T is ω-stable. Let $M \prec \mathfrak{C}$ be $\aleph_0$-saturated. Show that for any types p and q over M: $p \perp q \iff (\forall a \models p)(\forall b \models q)(a \mathop{\smile\!\!\!\!\!\!|}_{dom(p)} M \wedge b \mathop{\smile\!\!\!\!\!\!|}_{dom(q)} M \rightarrow a \mathop{\smile\!\!\!\!\!\!|}_M b)$.