Stable groups, List 7

Problem 1. We work in a monster model $\mathfrak{C}$ of an ω-stable theory T. Recall that Problem 1 from List 1' says that if $a \in \mathrm{acl}(A, b)$, then $RM(a/A) \leq RM(b/A)$ (prove it). Deduce that T has $\mathfrak{R}$ property, i.e. for every group G interpretable in $\mathfrak{C}$ and for any $a, b \in G$ and $A \subseteq \mathfrak{C}$, if a is generic in G over A, b, and $a \in \mathrm{acl}(A, b)$, then b is also generic in G over A.

Problem 2. (i) Let K be a field such that $x \mapsto x^n$ is onto for all $n > 0$, and, if $\mathrm{char}(K) = p > 0$, then $x \mapsto x^p - x$ is onto as well. Use Galois theory to show that K is radically closed, i.e. K does not have a proper extension of the form $K(a)$ with $a^n \in K$ for some $n > 0$, or $a^p - a \in K$ if $\mathrm{char}(K) = p > 0$.
(ii) Let K be a field such that in every finite extension L of K the functions $x \mapsto x^n$ are onto for all $n > 0$, and, if $\mathrm{char}(K) = p > 0$, then $x \mapsto x^p - x$ is onto as well. Using Galois theory, prove that K is algebraically closed.

Problem 3. Let K be a filed of finite Morley rank. Prove that there does not exist an infinite, definbale, proper subring of K.
Hint. Use the fact (from one of the lectures) that an integral stable ring is a filed, and apply also Macintyre's theorem.

Problem 4. Let K be a filed of finite Morley rank and of characteristic 0. Assume that in the structure K we have a definable operation $\odot$ such that $(K, +, \odot)$ is a field. Prove that the fields $(K, +, \cdot)$ and $(K, +, \odot)$ are definably isomorphic (in K).

Problem 5. Let K be a field of finite Morley rank and of characteristic 0. Prove that if $D \subseteq K$ is infinite and definable, then, for some n, each element of K is a sum of at most n elements of D.

Problem 6. Let $(K, +, \cdot)$ be a pure algebraically closed field. Prove that each definable group of automorphisms of the additive group of K embeds definably into $K^{\times}$.

Stable groups, List 7'

Recall that in a stable theory, Lascar U-rank satisfies Lascar inequalities:

$$U(a/Ab) + U(b/A) \leq U(ab/A) \leq U(a/Ab) \oplus U(b/A).$$

We say that an element a of a stable group G is generic over A, if for any $b \mathop{\smile\!\!\!\!\!\!|}_A a$ one has $a \cdot b \mathop{\smile\!\!\!\!\!\!|} A, b$. Existence and basic theory of generics in stable groups will be presented later. Here, we focus on the superstable case. So assume that G is a sufficiently saturated, superstable group (say a monster model). The goal is to give an alternative proof of Macintyre's theorem (in the superstable context).

Problem 1. Let $a \in G$, $A \subseteq G$. Prove that $\mathrm{tp}(a/A)$ is generic iff $U(a/A)$ is maximal.

Problem 2. Show that if $\bar{a} := (a_1, \ldots, a_n)$ is an independent sequence of generics of G (everything over a given set A) and $\bar{a} \in \mathrm{acl}(\bar{b}, A)$, where $\bar{b} := (b_1, \ldots, b_n)$, then $\bar{b}$ is also an independent sequence of generics of G (over A).

Let us assume that generics exist in superstable groups (this will be proved later for arbitrary stable groups; note that this is clear for us in the ω-stable context).

From now on, let F be an infinite, superstable field which is a monster model. Denote by K an algebraic closure of F. Then goal is to show that $K = F$.

Problem 3. Let $\bar{a} := (a_0, \ldots, a_{n-1})$ be an independent sequence of generics of F (over $\emptyset$). Prove that then all the solutions of

$$X^n + a_{n-1}X^{n-1} + \cdots + a_1 X + a_0 = 0$$

in K belong to F.

Problem 4. Suppose for a contradiction that there is $\alpha \in K \setminus F$, and choose $P(X) = X^n + a_{n-1}X + \ldots a_0$ — the minimal monic polynomial of α over F. Notice that P has n pairwise distinct roots $\alpha_1, \ldots, \alpha_n$. Define $L := F(\alpha_1, \ldots, \alpha_n)$,. Choose $t_0, \ldots, t_{n-1} \in F$ — generics independent over $a_0, \ldots, a_{n-1}$. Define

$$r_i = t_0 + t_1\alpha_i + \cdots + t_{n-1}\alpha_i^{n-1}$$

for $i = 1, \ldots, n$. Let $c_0, \ldots, c_{n-1}$ be the elementary symmetric functions in $r_0, \ldots, r_n$. Work with all these objects to get a contradiction.