Topological dynamics in model theory. List 4.

Starting from Problem 3, G is a group and $\mathcal{A}$ is Boolean G-algebra of subsets of G.

Problem 1. Let X be a definable set in a model M, and let C be a compact (Hausdorff) space. Let $N \succ M$ be an $|M|^+$-saturated small elementary substructure of the monster model $\mathfrak{C}$. Prove the following statements.

(i) If $f\colon X \to C$ is externally definable, then it extends uniquely to an externally definable function $f^*\colon S_{X,M}(N)(\mathfrak{C}) \to C$. Moreover, f^* is given by $\{f^*(a)\} = \bigcap_{\varphi \in \mathrm{tp}(a/N)} \mathrm{cl}(f[\varphi(M)])$.

(ii) Conversely, if $f^*\colon S_{X,M}(N)(\mathfrak{C}) \to C$ is an externally definable function, then $f^*|_X\colon X \to C$ is externally definable.

(iii) A function $f^*\colon S_{X,M}(N)(\mathfrak{C}) \to C$ is externally definable if and only if there is a continuous map $h\colon S_{X,M}(N) \to C$ such that $f^* = h \circ r$, where $r\colon S_{X,M}(N)(\mathfrak{C}) \to S_{X,M}(N)$ is given by $r(a) := \mathrm{tp}(a/N)$.

Problem 2. Let G be a group definable in M. Assume that all types in $S_G(M)$ are definable. Using the model-theoretic description of the semigroup operation on $S_{G,ext}(M)$ (in terms of realizations of types from $S_{G,M}(N)$), deduce that the semigroup operation on $S_G(M)$ is given by $p * q = \mathrm{tp}(ab/M)$ for some (equiv. any) $b \models q$ and a satisfying a unique coheir extension of p to a complete type over M, b.

Problem 3.

(i) Check that $(G, S(\mathcal{A}), p_e)$ is a G-ambit.

(ii) Assume that $\mathcal{A}$ is d-closed. Prove that for every $p \in S(\mathcal{A})$, $d_p \in \mathrm{End}(\mathcal{A})$.

(iii) Assume that $\mathcal{A}$ is d-closed. Prove that the map $d\colon S(\mathcal{A}) \to \mathrm{End}(\mathcal{A})$ given by $p \mapsto d_p$ is a bijection.

Problem 4.

(i) Prove that for a group G definable in M the G-algebra $\mathrm{Def}_{G,ext}(M)$ is d-closed.

(ii) Give an example of a group G definable in a structure M such that the G-algebra $\mathrm{Def}(G)$ is not d-closed.

Problem 5. Suppose there is a semigroup operation $*$ on $S(\mathcal{A})$ such that the map $l\colon S(\mathcal{A}) \to E(S(\mathcal{A}))$ given by $l \mapsto l_p$ (where $l_p(q) := p * q$) is a continuous epimorphism of semigroups with $l_{p_g} = \pi_g$ for all $g \in G$ (where $\pi_g(q) := gq$). Prove that $\mathcal{A}$ is d-closed.

Problem 6. Let H be a subgroup of $\mathrm{End}(\mathcal{A})$. Show that all elements of H have the same kernel (denoted by K_H) and the same image (denoted by $\mathcal{A}_H$). Moreover, prove that $f \mapsto f|_{\mathcal{A}_H}$ defines a group embedding of H into $\mathrm{Aut}(\mathcal{A}_H)$.

Problem 7. Assume $\mathcal{A}$ is d-closed. Let I be a minimal left ideal in $S(\mathcal{A})$. Let $\mathcal{B} \in \mathcal{R}$ (i.e. $\mathcal{B} = \mathrm{Im}(d_p)$ for some $p \in I$). Prove that:

(i) $I = \bigcap\{[U^c] : U \in K_I\}$,

(ii) for every $U \in \mathcal{A}$, for every $u \in J(I)$, $[U] \cap I = [d_u(U)] \cap I$,

(iii) for every $U \in \mathcal{A}$ there is a unique $V \in \mathcal{B}$ such that $[U] \cap I = [V] \cap I$ (this means that the sets $[V] \cap I$, $V \in \mathcal{B}$, are pairwise distinct and they are all the (relatively) clopen subsets of I),

(iv) for every $q \in S(\mathcal{B})$ there is a unique $p_q \in I$ such that $q = p_q \cap \mathcal{B}$; moreover, this unique p_q is generated as a filter by $q \cup \{U^c : U \in K_I\}$,

(v) the function mapping q to p_q is a homeomorphism from $S(\mathcal{B})$ to I.

Something to be checked.

Let us call a map f from G to a compact Hausdorff space C *$\mathcal{A}$-definable* if the preimages under f of any two disjoint closed subsets of C can be separated by a set from $\mathcal{A}$. If C is 0-dimensional, this just means that the preimage under f of any clopen set belongs to $\mathcal{A}$. Observe that the map $\pi\colon G \to S(A)$ given by $\pi(g) := p_g$ is $\mathcal{A}$-definable. Let us call a flow (G, X) *$\mathcal{A}$-definable* if for every $x \in X$ the map $g \mapsto gx$ is $\mathcal{A}$-definable.

Proposition -1.1 *If the flow $(G, S(\mathcal{A}))$ is $\mathcal{A}$-definable, then the ambit $(G, S(\mathcal{A}), p_e)$ is universal in the category of $\mathcal{A}$-definable G-ambits.*

Proposition -1.2 *The existence of a left continuous semigroup operation $*$ on $S(\mathcal{A})$ extending the action of G (i.e. $p_g * q = gq$) is equivalent to the flow $(G, S(\mathcal{A}))$ being $\mathcal{A}$-definable.*

This together with the corollary on page 29 of the lecture notes yields

Corollary -1.3 *The following conditions are equivalent.*

(i) There exists a left continuous semigroup operation $*$ on $S(\mathcal{A})$ extending the action of G.

(ii) $\mathcal{A}$ is d-closed.

(iii) The flow $(G, S(\mathcal{A}))$ is $\mathcal{A}$-definable (equivalently, $(G, S(\mathcal{A}), p_e)$ is universal in the category of $\mathcal{A}$-definable G-ambits).

(iv) There is a semigroup operation $*$ on $S(\mathcal{A})$ such that the map $l\colon S(\mathcal{A}) \to E(S(\mathcal{A}))$ given by $l \mapsto l_p$ (where $l_p(q) := p * q$) is a continuous epimorphism of semigroups with $l_{p_g} = \pi_g$ for all $g \in G$ (where $\pi_g(q) := gq$).