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THE COVERING NUMBER AND THE TRANSITIVE COVERING NUMBER MAY BE TOTALLY DIFFERENT

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Abstract. We construct a translation invariant $\sigma-$ideal $T(\kappa)$ (where κ is an infinite cardinal number) such that $\mathrm{cov}_t(T(\kappa)) = 2^\kappa$ while $\mathrm{cov}(T(\kappa)) = \mathrm{cof}(T(\kappa)) = \omega_1$. The constructions can be carried out in $\mathbb{R}$ as well.

0. Introduction

In 1938 Rothberger in [2] proved that there is a family of meagre subsets of the real line of size less or equal than the least cardinality of a Lebesgue nonmeasurable set such that its sum is the whole real line (and the same when we replace meagre sets by Lebesgue null sets and a Lebesgue nonmeasurable set by a set without the Baire property). In other words, he showed that $\mathrm{cov}(\mathrm{Meagre}) \leq \mathrm{non}(\mathrm{Null})$ and $\mathrm{cov}(\mathrm{Null}) \leq \mathrm{non}(\mathrm{Meagre})$, where cov and non stand for a covering number and a uniformity of a given ideal. As a matter of fact, Rothberger proved more.

Theorem 0.1. *Let $\mathcal{J}$ and $\mathcal{I}$ be translation invariant ideals of subsets of a group G, orthogonal to each other (that is there exist $A \in \mathcal{J}$ and $B \in \mathcal{I}$ such that $A \cup B = G$). Then*

$$\mathrm{cov}_t(\mathcal{J}) \leq \mathrm{non}(\mathcal{I}).$$

where $\mathrm{non}(\mathcal{I})$ is the minimal cardinality of the subset of G that does not belong to $\mathcal{I}$. □

In this theorem $\mathrm{cov}_t(\mathcal{J})$ denotes a transitive covering number of an ideal $\mathcal{J}$. The natural question to ask is what in general is a possible difference between a covering number and a transitive covering number of a given ideal. In this paper we show that these two cardinal invariants may be totally different.

1. Definitions and basic properties

We use standard set-theoretical notation and terminology from [1]. In particular, the cardinality of the set of all real numbers is denoted by $\mathfrak{c}$. The cardinality of a set X is denoted by $|X|$. A power set of a set X is denoted by $\mathcal{P}(X)$. If κ is a cardinal number then $\mathrm{cf}(\kappa)$ denotes its cofinality.

Let $(G, +)$ be an infinite abelian group. We consider a $\sigma-$ideal $\mathcal{J}$ of subsets of G which is proper and contains all singletons. Moreover, we assume that $\mathcal{J}$ is

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translation invariant (i.e. $(\forall A \in \mathcal{J})(\forall g \in G)\, A + g = \{a + g : a \in A\} \in \mathcal{J})$ and symmetric (i.e. $(\forall A \in \mathcal{J})\, -A = \{-a : a \in A\} \in \mathcal{J})$.

We say that a family $\mathcal{B} \subseteq \mathcal{J}$ is *cofinal* with $\mathcal{J}$ if for each $A \in \mathcal{J}$ there exists such $B \in \mathcal{B}$ that $A \subseteq B$. We also call such a family $\mathcal{B}$ a *base* of $\mathcal{J}$.

For an ideal $\mathcal{J}$ we consider the following cardinal numbers

$$\begin{aligned}
\mathrm{cov}(\mathcal{J}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{J} \ \ \&\ \ \bigcup \mathcal{A} = G\},\\
\mathrm{cov}_t(\mathcal{J}) &= \min\{|T| : T \subseteq G \ \ \&\ \ (\exists A \in \mathcal{J})\, A + T = G\},\\
\mathrm{cof}(\mathcal{J}) &= \min\{|\mathcal{B}| : \mathcal{B} \subseteq \mathcal{J} \ \ \&\ \ \mathcal{B} \textit{ is a base of } \mathcal{J}\}.
\end{aligned}$$

They are called the *covering number*, the *transitive covering number* and the *cofinality* of $\mathcal{J}$, respectively. Note that the following relations hold:

$$\mathrm{cov}(\mathcal{J}) \le \mathrm{cov}_t(\mathcal{J}) \quad \text{and} \quad \mathrm{cov}(\mathcal{J}) \le \mathrm{cof}(\mathcal{J}).$$

For more information about cardinal invariants of ideals on abelian groups and relations between them – see [1].

A set $\mathcal{H} \subseteq \mathbb{R}$ is called *a Hamel basis* if it is a basis of $(\mathbb{R}, +)$ treated as a linear space over a field $\mathbb{Q}$ of rational numbers.

2. Cofinality versus transitive covering

In this section we show that transitive covering of an ideal may be totally different from its cofinality.

Theorem 2.1. *Let λ be a cardinal number of uncountable cofinality and let $\langle G_\alpha : \alpha < \lambda\rangle$ be a strictly increasing sequence of subgroups of a group G such that $G = \bigcup_{\alpha<\lambda} G_\alpha$. If $\mathcal{J}$ is a $\sigma-$ideal of subsets of G generated by the family $\{G_\alpha : \alpha < \lambda\}$ then* $\mathrm{cof}(\mathcal{J}) = \mathrm{cf}(\lambda)$ *and*

$$\mathrm{cov}_t(\mathcal{J}) = \inf\{|{}^{G}/{}_{G_\alpha}| : \alpha < \lambda\}.$$

Proof. Straight from the fact, that the sequence $\langle G_\alpha : \alpha < \lambda\rangle$ is increasing and $\mathrm{cf}(\lambda)$ is uncountable we can deduce that

$$\mathcal{J} = \{A \subseteq G : (\exists \xi < \lambda)\, A \subseteq G_\xi\}.$$

It is a simple observation that $\mathcal{J}$ is a translation invariant, symmetric $\sigma-$ideal containing singletons. It is also proper because of strict monotonicity of the sequence $\langle G_\alpha : \alpha < \lambda\rangle$.

Let us fix a given sequence of ordinal numbers $\langle \xi_\alpha : \alpha < \mathrm{cf}(\lambda)\rangle$, cofinal in λ. Then the family $\{G_{\xi_\alpha} : \alpha < \mathrm{cf}(\lambda)\}$ is a base for $\mathcal{J}$. Moreover, no family of elements of $\mathcal{J}$ of cardinality strictly less than $\mathrm{cf}(\lambda)$ can be a base of $\mathcal{J}$ as all elements of such a family are contained in G_ξ for some $\xi < \lambda$. Hence $\mathrm{cof}(\mathcal{J}) = \mathrm{cf}(\lambda)$.

Let us observe that a sequence of cardinal numbers $\langle |{}^{G}/{}_{G_\alpha}| : \alpha < \lambda\rangle$ is decreasing so there exists $\zeta < \lambda$ such that $|{}^{G}/{}_{G_\alpha}| = |{}^{G}/{}_{G_\zeta}|$ for $\alpha \ge \zeta$. Let us consider now a set $T \subseteq G$ such that $|T| = \mathrm{cov}_t(\mathcal{J})$ and there exists $A \in \mathcal{J}$ such that $A + T = G$.

Without loss of generality we may assume that $A = G_\xi$ for some $\zeta \le \xi < \lambda$. Then we may get $T' \subseteq T$ such that $(\forall t \in T')\, T' \cap (G_\xi + t) = \{t\}$ and $G_\xi + T' = G$, that is, T' is a selector of the cosets. Thus $\mathrm{cov}_t(\mathcal{J}) = |T'| = |\,{}^{G}/_{G_\xi}|$ and, consequently,

$$\mathrm{cov}_t(\mathcal{J}) = |\,{}^{G}/_{G_\xi}| = |\,{}^{G}/_{G_\zeta}| = \inf\{|\,{}^{G}/_{G_\alpha}| : \alpha < \lambda\},$$

which ends the proof. □

As an application of Theorem 2.1 we construct a $\sigma-$ideal, the transitive covering of which is in general radically bigger than its cofinality and, consequently, its covering number as well. First, we introduce some necessary notation.

From now on let us fix a Hamel basis $\mathcal{H}$ and its enumeration $\mathcal{H} = \{h_\alpha : \alpha < \mathfrak{c}\}$. Then every real number x has the unique representation in this basis, i.e.

$$(\forall x \in \mathbb{R})(\exists!\, r_x \in \mathbb{Q}^{\mathfrak{c}})(|\operatorname{supp}(r_x)| < \omega \ \ \&\ \ x = \sum_{\alpha<\mathfrak{c}} r_x(\alpha) h_\alpha),$$

where $\operatorname{supp}(r_x) = \{\alpha : r_x(\alpha) \neq 0\}$. In order to simplify the notation we replace $\operatorname{supp}(r_x)$ by $\operatorname{supp}(x)$.

Definition. Let $\{P_\xi : \xi < \omega_1\}$ be a fixed partition of $\mathfrak{c}$ into parts of cardinality $\mathfrak{c}$. Let A be any set. We say that a function $f \in \mathbb{R}^A$ is *Hamel-bounded* if

$$(\exists \xi < \omega_1)(\forall a \in A)(\operatorname{supp}(f(a)) \subseteq \bigcup_{\beta<\xi} P_\beta).$$

Then we put $HB(A) = \{f \in \mathbb{R}^A : f\ is\ Hamel-bounded\}$. One can check that $HB(A)$ is a subgroup of $\mathbb{R}^A$ with the standard addition of functions.
For any function $f \in HB(A)$ its Hamel-bound $hb\,(f)$ is defined as follows:

$$hb\,(f) = \min\{\xi < \omega_1 : (\forall a \in A)(\operatorname{supp}(f(a)) \subseteq \bigcup_{\beta<\xi} P_\beta)\}.$$

Let κ be an infinite cardinal number. Let $B_\xi = \{f \in HB(\kappa) : hb\,(f) \le \xi\}$. Of course, $\langle B_\xi : \xi < \omega_1\rangle$ is a strictly increasing sequence of subgroups of the group $HB(\kappa)$ and $HB(\kappa) = \bigcup_{\xi<\omega_1} B_\xi$. We define $T(\kappa)$ as a $\sigma-$ideal generated by the family $\{B_\xi : \xi < \omega_1\}$.

Lemma 2.2. $|\,{}^{HB(\kappa)}/_{B_\xi}| = 2^\kappa$ *for every* $\xi < \omega_1$.

Proof. Let us fix B_ξ for some $\xi < \omega_1$. We consider a set $T \subseteq HB(\kappa)$ such that $(\forall t \in T)\, T \cap (B_\xi + t) = \{t\}$ and $B_\xi + T = HB(\kappa)$.

Let us fix $P \subseteq \kappa$ and a real number x such that $x \in \mathcal{H} \setminus \{h_\alpha : \alpha \in \bigcup_{\beta<\xi} P_\beta\}$. We define a function $f_P \in HB(\kappa)$ as follows:

$$f_P(\alpha) = \chi_P(\alpha) \cdot x$$

where χ_P denotes the characteristic function of a set P. Then there exists $t_P \in T$ and $g \in B_\xi$ such that $f_P = g + t_P$. In particular, for each $\alpha \in P$ we have

$$x = f_P(\alpha) = g(\alpha) + t_P(\alpha).$$

But we know from the assumption that $\mathrm{supp}(x) \not\subseteq \bigcup_{\beta<\xi} P_\beta$, so we have $\mathrm{supp}(x) \subseteq \mathrm{supp}(t_P(\alpha))$ for each $\alpha \in P$. On the other hand, if $\alpha \notin P$ then $f_P(\alpha) = 0$ and, consequently, $\mathrm{supp}(t_P(\alpha)) = \mathrm{supp}(g(\alpha)) \subseteq \bigcup_{\beta<\xi} P_\beta$ for such α's.

Let P_1 and P_2 be two different subsets of κ and $\alpha \in P_1 \triangle P_2$. Suppose that $t_{P_1} = t_{P_2} = t$. Then

$$\mathrm{supp}(x) \subseteq \mathrm{supp}(t(\alpha)) \subseteq \bigcup_{\beta<\xi} P_\beta,$$

which is a contradiction. Hence $t_{P_1} \neq t_{P_2}$ and, consequently,

$$\left| {}^{HB(\kappa)}/_{B_\xi} \right| = |T| \geq |\mathcal{P}(\kappa)| = 2^\kappa,$$

which ends the proof, as $|HB(\kappa)| = 2^\kappa$. □

Corollary 2.3. *For every infinite cardinal number κ we have $\mathrm{cof}(T(\kappa)) = \omega_1$ and $\mathrm{cov}_t(T(\kappa)) = 2^\kappa$.*

Proof. It is enough to apply Theorem 2.1 for $\lambda = \omega_1$, $G = HB(\kappa)$, $G_\xi = B_\xi$ and $\mathcal{J} = T(\kappa)$. Thanks to Lemma 2.2 we get the result. □

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