On invariant CCC $\sigma-$ideals.

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Abstract

We re-read Recław's proof from [6] on invariant CCC σ–ideals of subsets of reals and obtain a reasonably stronger corollary for such ideals on the Cantor space.

1. Preliminaries. In 1998 Recław in [6] investigated cardinal invariants of CCC σ–ideals of subsets of reals. In particular, he showed that if such a σ–ideal $\mathcal{J}$ is invariant, then $\mathfrak{p} \leq \mathrm{non}(\mathcal{J})$, where $\mathfrak{p}$ is a pseudointersection number (cf. [8] for more details). In this paper we analyze his proof and get an apparently stronger result for $\sigma-$ideals of subsets of the Cantor space 2^ω.

We use standard set-theoretical notation and terminology derived from [1]. Let us remind that the cardinality of the set of all real numbers is denoted by $\mathfrak{c}$. The cardinality of a set X is denoted by $|X|$. By $[\omega]^\omega$ we denote the family of all infinite subsets of ω. If $\varphi : X \to Y$ is a function then $\mathrm{rng}(\varphi)$ denotes the range of φ.

Let $(G, +)$ be an abelian Polish (i.e. separable, completely metrizable, without isolated points) group and let $\mathcal{J}$ be a $\sigma-$ideal of subsets of G (we assume from now on that $\mathcal{J}$ is proper and contains all singletons). We will consider that $\mathcal{J}$ is invariant, that is for every $A \subseteq G$ and $g \in G$ we have $A + g = \{a + g : a \in A\} \in \mathcal{J}$ and $-A = \{-a : a \in A\} \in \mathcal{J}$). Moreover, we will assume that the $\sigma-$ideal $\mathcal{J}$ has a Borel basis i.e. every set from $\mathcal{J}$ is contained in a certain Borel set from the ideal.

We say that $\mathcal{J}$ is CCC (countable chain condition) if the quotient Boolean algebra $\mathcal{B}(G)/\mathcal{J}$ is CCC, where $\mathcal{B}(G)$ is the $\sigma-$algebra of all Borel subsets of G.

We define the following cardinal invariants of $\mathcal{J}$.

$$\begin{aligned}\mathrm{non}(\mathcal{J}) &= \min\{|B| : B \subseteq G \wedge B \notin \mathcal{J}\},\\ \mathrm{cov}_t(\mathcal{J}) &= \min\{|T| : T \subseteq G \wedge (\exists A \in \mathcal{J})\, A + T = G\},\end{aligned}$$

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We define also an operation on the $\sigma-$ideal $\mathcal{J}$ (it was introduced by Seredyński in [7], who denoted it by $\mathcal{J}^*$)

$$s(\mathcal{J}) = \{A \subseteq G : (\forall B \in \mathcal{J})(\exists g \in G)\,(A+g) \cap B = \emptyset\}.$$

If we apply these operations to the $\sigma-$ideals of meagre sets $\mathcal{M}$ and of null sets $\mathcal{N}$ we obtain strongly null sets $s(\mathcal{M})$ and strongly meager sets $s(\mathcal{N})$. The following is well-known

$$\mathrm{non}(s(\mathcal{J})) = \mathrm{cov}_t(\mathcal{J}).$$

We define

$$Pif = \{f : f\ is\ a\ function \wedge \mathrm{dom}(f) \in [\omega]^\omega \wedge \mathrm{rng}(f) \subseteq 2\}.$$

If $f \in Pif$ then we put

$$[f] = \{x \in 2^\omega : f \subseteq x\}.$$

Let $\mathbb{S}_2$ denotes the σ-ideal of subsets of the Cantor space 2^ω, which is generated by the family $\{[f] : f \in Pif\}$. It was thoroughly investigated in [2] and [4]. We recall some properties of $\mathbb{S}_2$, which were proved in [2].

Fact 1.1 *$\mathbb{S}_2$ is a proper, invariant σ-ideal which contains all singletons and has a Borel basis. Every $A \in \mathbb{S}_2$ is both meager and null. Moreover, there exists a family of size $\mathfrak{c}$ of pairwise disjoint Borel subsets of 2^ω that do not belong to $\mathbb{S}_2$. Hence $\mathbb{S}_2$ is not CCC.* $\square$

Let A, S be two infinite subsets of ω. We say that S *splits* A if $|A \cap S| = |A \setminus S| = \omega$. Let us recall a cardinal number related with a notion of splitting, introduced by Malychin in [5], namely

$$\aleph_0\text{-}\mathfrak{s} = \min\{|\mathcal{S}| : \mathcal{S} \subseteq [\omega]^\omega \wedge (\forall \mathcal{A} \in [[\omega]^\omega]^\omega)(\exists S \in \mathcal{S})(\forall A \in \mathcal{A})\, S\ splits\ A\}.$$

More about cardinal numbers connected with the relation of splitting can be found in [3].

2. Recław's proof revisited. In [6] Recław proved a theorem, which can be generalized as follows.

Theorem 2.1 *Let $\mathcal{I}$ and $\mathcal{J}$ be two $\sigma-$ideals of subsets of an abelian Polish group G, which are invariant and have Borel bases. If $\mathcal{I}$ is CCC then*

$$\mathcal{J} \cap s(\mathcal{J}) \subseteq \mathcal{I}.$$

Proof. *(Recław)* Let $X \in \mathcal{J} \cap s(\mathcal{J})$. Assume that $X \notin \mathcal{I}$. We construct a sequence $\{F_\alpha : \alpha < \omega_1\}$ of Borel sets from $\mathcal{J}$ and a sequence $\{t_\alpha : \alpha < \omega_1\}$ of elements of G. Let $t_0 = 0$ and F_0 be any Borel set from $\mathcal{J}$ containing X. Suppose that we have constructed F_β and t_β for $\beta < \alpha$. Then from the definition of $s(\mathcal{J})$ there exists $t_\alpha \in G$ such that

$$(X + t_\alpha) \cap \bigcup_{\beta<\alpha} F_\beta = \emptyset.$$

As F_α we take any Borel set from $\mathcal{J}$ containing $\bigcup_{\beta<\alpha} F_\beta \cup (X + t_\alpha)$.

Let $G_\alpha = F_\alpha \setminus \bigcup_{\beta<\alpha} F_\beta$. Thus $\{G_\alpha : \alpha < \omega_1\}$ is a family of pairwise disjoint Borel sets such that none of them belongs to $\mathcal{I}$, as $G_\alpha \supseteq X + t_\alpha$ and $\mathcal{I}$ is invariant. Hence $\mathcal{I}$ is not CCC, a contradiction. □

Corollary 2.2 *Let $\mathcal{I}$ and $\mathcal{J}$ be as above. If $\mathcal{I}$ is CCC then*

$$\min\{\mathrm{non}(\mathcal{J}), \mathrm{cov}_t(\mathcal{J})\} \leq \mathrm{non}(\mathcal{I}).$$

Proof. It is enough to observe that $\mathcal{J} \subseteq \mathcal{I}$ implies $\mathrm{non}(\mathcal{J}) \leq \mathrm{non}(\mathcal{I})$. □

Corollary 2.3 *Let $\mathcal{I}$ be a $\sigma-$ideal of subsets of the Cantor space 2^ω (endowed with a standard group structure), which is invariant and has a Borel basis. If $\mathcal{I}$ is CCC then*

$$\aleph_0\text{-}\mathfrak{s} \leq \mathrm{non}(\mathcal{I}).$$

Proof. In [2] it was proved that $\mathrm{non}(\mathbb{S}_2) = \aleph_0\text{-}\mathfrak{s}$ and in [4] it was proved that $\mathrm{cov}_t(\mathbb{S}_2) = \mathfrak{c}$. So it is enough to apply Corollary 2.2 for $G = 2^\omega$ and $\mathcal{J} = \mathbb{S}_2$. □

Question. Let $\mathcal{I}$ be an invariant CCC $\sigma-$ideal of subsets of the real line $\mathbb{R}$. Is the inequality $\aleph_0\text{-}\mathfrak{s} \leq \mathrm{non}(\mathcal{I})$ still true?

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