Transitive properties of the ideal $\mathbb{S}_2$.

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Abstract

In this paper we compute transitive cardinal coefficients of the σ–ideal $\mathbb{S}_2$, the least nontrivial productive σ–ideal of subsets of the Cantor space 2^ω. We also apply transitive operations to $\mathbb{S}_2$. In particular, we show that σ–ideal of strongly $\mathbb{S}_2$ sets is equal to $\mathbb{B}_2$, one of Mycielski ideals.

0. Introduction. In this paper we investigate transitive properties of the σ–ideal $\mathbb{S}_2$. This ideal appeared for the first time in [10], but only incidentally. It was thoroughly investigated by Cichoń and Kraszewski in [5]. It turned out that cardinal characteristics of $\mathbb{S}_2$ are strongly connected with some intensively studied combinatorial properties of subsets of natural numbers (the splitting and reaping numbers). Namely,

$$\mathrm{add}(\mathbb{S}_2) = \omega_1, \quad \mathrm{cov}(\mathbb{S}_2) = \mathfrak{r}, \quad \mathrm{non}(\mathbb{S}_2) = \aleph_0\text{-}\mathfrak{s}, \quad \mathrm{cof}(\mathbb{S}_2) = \mathfrak{c}.$$

Moreover, $\mathbb{S}_2$ is the least nontrivial productive σ–ideal of subsets of the Cantor space 2^ω. The notion of productivity is a powerful tool for investigating properties of ideals on generalized Cantor spaces 2^κ. For more details see [9].

In the first part of this paper we completely describe all well-known transitive cardinal characteristics of $\mathbb{S}_2$. In the second part we apply transitive operations to $\mathbb{S}_2$. In particular, we show that the σ–ideal of strongly $\mathbb{S}_2$ sets is exactly $\mathbb{B}_2$, one of Mycielski ideals.

1. Definitions and basic properties. We use standard set-theoretical notation and terminology from [2]. Recall that the cardinality of the set of all real numbers is denoted by $\mathfrak{c}$. The cardinality of a set X is denoted by $|X|$. The power set of a set X is denoted by $\mathcal{P}(X)$. If κ is a cardinal number then $[X]^\kappa$ denotes the family of all subsets

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of the set X of cardinality κ. If $\varphi : X \to Y$ is a function then $\mathrm{rng}(\varphi)$ denotes the range of φ. If $A \subseteq Y$ then $\varphi^{-1}[A]$ denotes the pre-image of A.

Let $\mathcal{J}$ be an ideal of subsets of an abelian group G. We say that $\mathcal{J}$ is translation invariant if $A + g = \{x + g : x \in A\} \in \mathcal{J}$ for each $A \in \mathcal{J}$ and $g \in G$ and that $\mathcal{J}$ is symmetric if $-A = \{-x : x \in A\} \in \mathcal{J}$ for each $A \in \mathcal{J}$.

For an ideal $\mathcal{J}$ we consider the following cardinal numbers

$$\begin{aligned}
\mathrm{add}_t(\mathcal{J}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{J} \wedge \neg(\exists B \in \mathcal{J})(\forall A \in \mathcal{A})(\exists g \in G)\, A \subseteq B + g\},\\
\mathrm{add}_t^*(\mathcal{J}) &= \min\{|T| : T \subseteq G \wedge (\exists A \in \mathcal{J})\, A + T \notin \mathcal{J}\},\\
\mathrm{cov}_t(\mathcal{J}) &= \min\{|T| : T \subseteq G \wedge (\exists A \in \mathcal{J})\, A + T = G\},\\
\mathrm{cof}_t(\mathcal{J}) &= \min\{|\mathcal{B}| : \mathcal{B} \subseteq \mathcal{J} \wedge \mathcal{B} \text{ is a transitive base of } \mathcal{J}\},
\end{aligned}$$

where a family $\mathcal{B} \subseteq \mathcal{J}$ is called a *transitive base* if for each $A \in \mathcal{J}$ there exists $B \in \mathcal{B}$ and $g \in G$ such that $A \subseteq B + g$. The first two of these are both called *transitive additivity* and the last two are called *transitive covering number* and *transitive cofinality*, respectively. Let us notice that all definitions of cardinal coefficients mentioned above are valid also for an arbitrary family $\mathcal{A} \subseteq \mathcal{P}(G)$.

Let us also recall that by the *uniformity* of $\mathcal{J}$ we mean the following cardinal number

$$\mathrm{non}(\mathcal{J}) = \min\{|A| : A \subseteq G \wedge A \notin \mathcal{J}\}.$$

We say that an ideal $\mathcal{J}$ is $\kappa - translatable$ if

$$(\forall A \in \mathcal{J})(\exists B \in \mathcal{J})(\forall T \in [G]^\kappa)(\exists g \in G)\, A + T \subseteq B + g.$$

We define a *translatability number* of $\mathcal{J}$ as follows

$$\tau(\mathcal{J}) = \min\{\kappa : \mathcal{J} \textit{ is not } \kappa - -translatable\}.$$

In this paper we deal with the Cantor space 2^ω interpreted as the set of all functions from ω into the set $\{0, 1\}$. This space is endowed with the standard product topology. Moreover, we consider the standard product group structure on 2^ω.

We define

$$Pif = \{f : f \textit{ is a function} \wedge \mathrm{dom}(f) \in [\omega]^\omega \wedge \mathrm{rng}(f) \subseteq 2\}.$$

If $f \in Pif$ then we put

$$[f] = \{x \in 2^\omega : f \subseteq x\}.$$

Let $\mathbb{S}_2$ denotes the σ-ideal of subsets of 2^ω, which is generated by the family $\{[f] : f \in Pif\}$. We recall some properties of $\mathbb{S}_2$, which were proved in [5].

Fact 1.1 *(a) $\mathbb{S}_2$ is a proper σ-ideal, containing singletons, with a base consisting of Borel sets. Every $A \in \mathbb{S}_2$ is both meagre and null.*
(b) $\mathbb{S}_2$ is translation invariant.

(c) There exists a family of size $\mathfrak{c}$ of pairwise disjoint Borel subsets of 2^ω that do not belong to $\mathbb{S}_2$. $\square$

We call a family $\mathcal{F} \subseteq Pif$ *normal* if for each two different $f_1, f_2 \in \mathcal{F}$ we have $\mathrm{dom}(f_1) \cap \mathrm{dom}(f_2) = \emptyset$. Directly from the definition of $\mathbb{S}_2$ we can deduce that

$$A \in \mathbb{S}_2 \iff A \subseteq \bigcup_{f \in \mathcal{F}} [f],$$

for some countable normal family $\mathcal{F} \subseteq Pif$. In [5] the following useful lemma was proved.

Lemma 1.2 *Suppose that $\{f_i : i \in I\}$ is a normal family of functions from Pif, $f \in Pif$ and $[f] \subseteq \bigcup_{i \in I} [f_i]$. Then $[f] \subseteq [f_i]$ for some $i \in I$.* $\square$

Let A, S be two infinite subsets of ω. We say that S *splits* A if $|A \cap S| = |A \setminus S| = \omega$. Let us recall a cardinal number related with a notion of splitting, introduced by Malychin in [13], namely

$$\aleph_0\text{-}\mathfrak{s} = \min\{|\mathcal{S}| : \mathcal{S} \subseteq [\omega]^\omega \wedge (\forall \mathcal{A} \in [[\omega]^\omega]^\omega)(\exists S \in \mathcal{S})(\forall A \in \mathcal{A})(S \text{ } splits \text{ } A)\}.$$

More about cardinal numbers connected with the relation of splitting can be found in [8]. It was proved in [5] that $\mathrm{non}(\mathbb{S}_2) = \aleph_0\text{-}\mathfrak{s}$.

We will need one more σ–ideal. Let us define

$$\mathbb{B}_2 = \{A \subseteq 2^\omega : (\forall X \in [\omega]^\omega) \, A \restriction X \neq 2^X\},$$

where $A \restriction X = \{x \restriction X : x \in A\}$. This is one of the Mycielski ideals and was intensively studied by many authors (cf. [7], [15], [17]).

2. Transitive cardinal coefficients of $\mathbb{S}_2$. Let $\mathcal{J}$ be an ideal of subsets of an abelian group G. The first cardinal coefficient on the stage was the transitive covering number of $\mathcal{J}$ that appeared implicitly in 1938 in the famous Rothberger theorem, which was originally formulated for classical ideals of meagre and null subsets of the real line (cf. [18]). In his general version it says that if $\mathcal{J}$ and $\mathcal{I}$ are translation invariant ideals of subsets of G, orthogonal to each other (that is there exist $A \in \mathcal{J}$ and $B \in \mathcal{I}$ such that $A \cup B = G$) then $\mathrm{cov}_t(\mathcal{J}) \leq \mathrm{non}(\mathcal{I})$. It is worth observing that the transitive covering number may be different from the covering number and $\mathbb{S}_2$ is an example.

Theorem 2.1 $\mathrm{cov}_t(\mathbb{S}_2) = \mathfrak{c}$

Proof. It is obvious that $\mathrm{cov}_t(\mathbb{S}_2) \leq \mathfrak{c}$, so it is enough to show the other inequality. Let $T \subseteq 2^\omega$ and $A \in \mathbb{S}_2$. Without loss of generality we can assume that $A = \bigcup_{i<\omega} [f_i]$, where the family $\{f_i : i < \omega\} \subseteq Pif$ is normal. If $|T| < \mathfrak{c}$ then for every $i < \omega$ there exist a function $g_i : \mathrm{dom}(f_i) \to 2$ which is different from every function $f_i + t \restriction \mathrm{dom}(f_i)$, where

$t \in T$. Because the family $\{f_i : i < \omega\}$ is normal then there exists a function $x \in 2^\omega$ such that $\bigcup_{i<\omega} g_i \subseteq x$ and we have $x \notin (A+T)$ which ends the proof. $\square$

In 1985 Pawlikowski in [16] introduced the transitive cofinality and gave the complete description of transitive cofinalities of ideals of meagre and null subsets of the real line. He also mentioned a dual coefficient to the transitive cofinality. Following the way of describing cardinal characteristics of the continuum presented by Blass in [3] we will call it a transitive additivity and denote by $\mathrm{add}_t(\mathcal{J})$. Unfortunately, Pawlikowski (and then Bartoszyński and Judah in [2]) used this name and notation for yet another coefficient, introduced in [16]. In order not to make a mess we will call the latter coefficient the starred transitive additivity and denote by $\mathrm{add}_t^*(\mathcal{J})$.

Now we calculate these coefficients for $\mathbb{S}_2$. To begin with, we observe the following general property concerning starred transitive additivity.

Proposition 2.2 *Let $\mathcal{J}$ be a proper and translation invariant $\sigma-$ideal of subsets of a group G containing all singletons. Then $\mathrm{add}_t^*(\mathcal{J}) \leq \mathrm{non}(\mathcal{J})$.*

Proof. To prove that $\mathrm{add}_t^*(\mathcal{J}) \leq \mathrm{non}(\mathcal{J})$ it is enough to observe that for every set $T \subseteq G$ such that $T \notin \mathcal{J}$ we have $|T| \geq \mathrm{add}_t^*(\mathcal{J})$ because $\{0\} + T = T \notin \mathcal{J}$ and, of course, $\{0\} \in \mathcal{J}$. $\square$

Theorem 2.3 $\mathrm{add}_t^*(\mathbb{S}_2) = \aleph_0\text{-}\mathfrak{s}$.

Proof. As $\mathrm{non}(\mathbb{S}_2) = \aleph_0\text{-}\mathfrak{s}$ then thanks to Proposition 2.2 it is enough to show that $\mathrm{add}_t^*(\mathbb{S}_2) \geq \mathrm{non}(\mathbb{S}_2)$.

Suppose now that $T \subseteq 2^\omega$ and $A \in \mathbb{S}_2$. To finish the proof we show that if $|T| < \mathrm{non}(\mathbb{S}_2)$ then $A + T \in \mathbb{S}_2$. As in the proof of Theorem 2.1 we can assume that $A = \bigcup_{i<\omega}[f_i]$, where $f_i \in Pif$ form a normal family. Thus

$$A + T = \bigcup_{t\in T} A + t = \bigcup_{t\in T}\bigcup_{i<\omega}([f_i] + t) = \bigcup_{i<\omega}\bigcup_{t\in T}[f_i + t\restriction \mathrm{dom}(f_i)]$$

Fix $i < \omega$. Let $\iota : \mathrm{dom}(f_i) \to \omega$ be a bijection. It induces a bijection $\hat{\iota} : 2^{\mathrm{dom}(f_i)} \to 2^\omega$. The image of the set $\{f_i + t\restriction \mathrm{dom}(f_i) : t \in T\} \subseteq 2^{\mathrm{dom}(f_i)}$ by $\hat{\iota}$ has cardinality strictly smaller than $\mathrm{non}(\mathbb{S}_2)$. Consequently, it can be covered by a set $\bigcup_{j<\omega}[g_j]$, for some $\{g_j : j < \omega\} \subseteq Pif$. Hence

$$\bigcup_{t\in T}[f_i + t\restriction \mathrm{dom}(f_i)] \subseteq \bigcup_{j<\omega}[\hat{\iota}^{-1}(g_j)] \in \mathbb{S}_2,$$

which ends the proof. $\square$

In order to prove results about $\mathrm{add}_t(\mathbb{S}_2)$ and $\mathrm{cof}_t(\mathbb{S}_2)$ we introduce some extra notation. For a set $X \in [\omega]^\omega$ let $(X)^\omega_\omega$ denote the family of all infinite partitions of X into infinite parts. For $P_1, P_2 \in (\omega)^\omega_\omega$ we put $P_1 \preceq P_2$ if for every $p_1 \in P_1$ there exists $p_2 \in P_2$ such that $p_2 \subseteq p_1$ (we say that P_2 *dominates* P_1). It is not difficult to observe that

$\preceq$ is a partial ordering on $(\omega)^\omega_\omega$. Let us notice that if we consider $\preceq$ on the family (ω) of all partitions of ω (which is more common) then $\{\omega\}$ (one-element partition) is the smallest element of this ordering while the partition into singletons is the greatest one. Properties of relations on partitions of ω have been intensively studied lately by Matet, Majcher-Iwanow and others; for more information cf. [14], [6] or [12].

We define an unboundedness and dominating numbers $\mathfrak{b}_\preceq$ and $\mathfrak{d}_\preceq$ in a standard way.

$$\mathfrak{b}_\preceq = \min\{|\mathcal{R}| : R \subseteq (\omega)^\omega_\omega \wedge (\forall P \in (\omega)^\omega_\omega)(\exists R \in \mathcal{R}) R \not\preceq P\},$$
$$\mathfrak{d}_\preceq = \min\{|\mathcal{R}| : R \subseteq (\omega)^\omega_\omega \wedge (\forall P \in (\omega)^\omega_\omega)(\exists R \in \mathcal{R}) P \preceq R\}.$$

We have the following well-known lemma.

Lemma 2.4 $\mathfrak{b}_\preceq = \omega_1, \quad \mathfrak{d}_\preceq = \mathfrak{c}$.

Proof. Inequalities $\mathfrak{b}_\preceq \geq \omega_1$ and $\mathfrak{d}_\preceq \leq \mathfrak{c}$ are obvious. To show the other inequalities we first construct a family $\mathcal{P} \subseteq (\omega)^\omega_\omega$ of cardinality $\mathfrak{c}$ such that for every two partitions $P_1, P_2 \in \mathcal{P}$ if $p_1 \in P_1$ and $p_2 \in P_2$ then $p_1 \cap p_2$ is finite.

We deal with partitions of $\mathbb{Z} \times \mathbb{Z}$ instead of partitions of ω. Let $p_i^\alpha = \{(z_1, z_2) \in \mathbb{Z} \times \mathbb{Z} : i \leq z_2 - \alpha z_1 < i + 1\}$ for $i \in \mathbb{Z}$ and $\alpha \in [0, +\infty)$. Then $P^\alpha = \{p_i^\alpha : i \in \mathbb{Z}\}$ is a partition from $(\mathbb{Z} \times \mathbb{Z})^\omega_\omega$. It is not difficult to check that a family $\mathcal{P} = \{P^\alpha : \alpha \in [0, +\infty)\}$ has a needed property.

Now, if $\mathcal{R} \subseteq (\omega)^\omega_\omega$ is any subfamily of $\mathcal{P}$ of size ω_1 then $\mathcal{R}$ cannot be dominated by one partition. Indeed, if there exists a partition $P \in (\omega)^\omega_\omega$ such that for every $R \in \mathcal{R}$ and every $r \in R$ we have an element $p \in P$ such that $p \subseteq r$ then we get a contradiction as for different $R_1, R_2 \in \mathcal{R}$ and $r_1 \in R_1, r_2 \in R_2$ there is no $p \in P$ which is simultaneously contained in r_1 and r_2.

On the other hand, let us consider a family $\mathcal{R}$ such that every partition from $(\omega)^\omega_\omega$ is dominated by a partition from $\mathcal{R}$. For a given $R \in \mathcal{R}$ we define $\mathcal{P}_R = \{P \in \mathcal{P} : (\forall p \in P)(\exists r \in R) r \subseteq p\}$. Obviously $\mathcal{P} = \bigcup_{R \in \mathcal{R}} \mathcal{P}_R$. Moreover, every family $\mathcal{P}_R$ is at most countable because any element of R cannot be contained in elements of different partitions from $\mathcal{P}_R$. Therefore

$$\mathfrak{c} \leq |\mathcal{P}| \leq \omega \cdot |\mathcal{R}|$$

and we are done. □

Theorem 2.5 $\mathrm{add}_t(\mathbb{S}_2) = \omega_1, \quad \mathrm{cof}_t(\mathbb{S}_2) = \mathfrak{c}$.

Proof. As $\omega_1 \leq \mathrm{add}_t(\mathbb{S}_2)$ and $\mathrm{cof}_t(\mathbb{S}_2) \leq \mathfrak{c}$ then thanks to Lemma 2.4 we have to prove only $\mathrm{add}_t(\mathbb{S}_2) \leq \mathfrak{b}_\preceq$ and $\mathrm{cof}_t(\mathbb{S}_2) \geq \mathfrak{d}_\preceq$.

We observe the following useful fact. Let $\mathcal{P} \subseteq (\omega)^\omega_\omega$ be a family of partitions and $\mathcal{A} \subseteq \mathbb{S}_2$. Let us assume that for every partition $P \in \mathcal{P}$ there exist $A_P \in \mathcal{A}$ and $x_P \in 2^\omega$ such that $\bigcup_{p \in P} [\mathbf{0}_p] \subseteq A_P + x_P$, where $\mathbf{0}_p$ denotes a function constantly equal to 0 on its domain, which is the set p. Then there exists a family $\mathcal{R} \subseteq (\omega)^\omega_\omega$ of size $|\mathcal{A}|$ such that for every $P \in \mathcal{P}$ there exists $R \in \mathcal{R}$ such that $P \preceq R$. Indeed, without loss of generality we can assume that $A_P = \bigcup_{i<\omega} [f_i^P]$, where $\{f_i^P : i < \omega\} \subseteq Pif$ and

$\{\mathrm{dom}(f_i^P) : i < \omega\} \in (\omega)_\omega^\omega$ and by Lemma 1.2 we get that for every $p \in P$ there exists a natural number i_p such that $[\mathbf{0}_p] \subseteq [f_{i_p}^P + x_P \restriction \mathrm{dom}(f_{i_p}^P)]$. Thus $\mathrm{dom}(f_{i_p}^P) \subseteq p$ and, consequently, $P \preceq \{\mathrm{dom}(f_i^P) : i < \omega\}$. Hence $\mathcal{R} = \{\{\mathrm{dom}(f_i^P) : i < \omega\} : P \in \mathcal{P}\}$ is a family of the sort we are looking for.

Now, let $\mathcal{P} \subseteq (\omega)_\omega^\omega$ be an arbitrary family of partitions of size less than $\mathrm{add}_t(\mathbb{S}_2)$. From the definition of $\mathbb{S}_2$ we obtain that our assumption is fulfilled for a family $\mathcal{A}$ having one element. Thus $\mathcal{P}$ is bounded by one partition and we get $\mathrm{add}_t(\mathbb{S}_2) \leq \mathfrak{b}_\preceq$.

On the other hand, our assumption is fulfilled also for $\mathcal{P} = (\omega)_\omega^\omega$ and $\mathcal{A} \subseteq \mathbb{S}_2$ being a transitive base for $\mathbb{S}_2$. In this situation, the family $\mathcal{R}$ obtained from the fact mentioned above is a dominating family of partitions, so we have $\mathrm{cof}_t(\mathbb{S}_2) \geq \mathfrak{d}_\preceq$, which ends the proof. □

The last transitive property we deal with is translatability. In 1993 Carlson in [4] introduced the notion of κ–translatability and proved that the σ–ideal of meagre subsets of the real line and the σ–ideal generated by closed null subsets of the real line are ω–translatable. Bartoszyński in [1] proved that the σ–ideal of null subsets of the Cantor space is not 2–translatable. Kysiak in [11] introduced a natural notion of a translatability number.

As far as $\mathbb{S}_2$ is concerned, its translatability number can be computed precisely.

Theorem 2.6 $\tau(\mathbb{S}_2) = \omega_1$.

Proof. To begin with, we show that $\mathbb{S}_2$ is ω–translatable. Let $A \in \mathbb{S}_2$ be arbitrary. As usual, without loss of generality we can assume that $A = \bigcup_{i<\omega}[f_i]$, where $\{f_i : i < \omega\} \subseteq Pif$ and $\{\mathrm{dom}(f_i) : i < \omega\} \in (\omega)_\omega^\omega$. For every $i < \omega$ let us fix a partition $P_i = \{p_{ij} : j < \omega\} \in (\mathrm{dom}(f_i))_\omega^\omega$. Then $\{p_{ij} : i, j < \omega\} \in (\omega)_\omega^\omega$. We define

$$B = \bigcup_{i<\omega} \bigcup_{j<\omega} [\mathbf{0}_{p_{ij}}].$$

Obviously, $B \in \mathbb{S}_2$. For every $T = \{t_j : j < \omega\} \in [2^\omega]^\omega$ we define $g \in 2^\omega$ as follows:

$$(\forall i, j < \omega)\, g \restriction p_{ij} = (f_i + t_j) \restriction p_{ij}.$$

It is a routine calculation to show that $A + T \subseteq B + g$.

To show the other inequality, let us consider first a partition P of ω into infinite parts. We can observe that there exists a set $T \in [2^\omega]^{\omega_1}$ such that for every family $\{h_i : i < \omega\} \subseteq Pif$ if $\{\mathrm{dom}(h_i) : i < \omega\} = P$ then $T \not\subseteq \bigcup_{i<\omega}[h_i]$. Namely, it is enough to take T such that $(\forall p \in P)(\forall x, y \in T)(x \neq y \Rightarrow x \restriction p \neq y \restriction p)$.

Let $A = \{\mathbf{0}_\omega\}$. We claim that this set witnesses that $\mathbb{S}_2$ is not ω_1–translatable. So suppose $B = \bigcup_{i<\omega}[h_i]$ where $\{h_i : i < \omega\} \subseteq Pif$ and $\{\mathrm{dom}(h_i) : i < \omega\} = P \in (\omega)_\omega^\omega$. Consider the set T defined as above. Then no translation of B covers $T = A + T$. □

3. Transitive operations on $\mathbb{S}_2$. In this paragraph we apply transitive operations to the ideal $\mathbb{S}_2$. To begin with, let us recall some definitions.

Let us assume that $\mathcal{J}$ is a σ–ideal of subsets of an abelian group G which is proper, translation invariant, symmetric and contains all singletons. We define (cf. [19])

$$s(\mathcal{J}) = \{A \subseteq G : (\forall B \in \mathcal{J})\, A + B \neq G\},$$
$$g(\mathcal{J}) = \{A \subseteq G : (\forall B \in \mathcal{J})\, A + B \in \mathcal{J}\}$$

(Seredyński used $\mathcal{J}^*$ instead of $s(\mathcal{J})$). In [19] many basic properties of operations s and g can be found. If we apply these operations to the σ–ideals of meagre sets $\mathcal{M}$ and of null sets $\mathcal{N}$ we obtain strongly null sets $s(\mathcal{M})$, strongly meagre sets $s(\mathcal{N})$, meagre-additive sets $g(\mathcal{M})$ and null-additive sets $g(\mathcal{N})$ (see [2] for more information).

The following are well-known.

Fact 3.1 $\mathrm{non}(s(\mathcal{J})) = \mathrm{cov}_t(\mathcal{J}), \qquad \mathrm{non}(g(\mathcal{J})) = \mathrm{add}_t^*(\mathcal{J}).$ □

We can also observe other basic relations.

Proposition 3.2 $\mathrm{cov}_t(s(\mathcal{J})) \geq \mathrm{non}(\mathcal{J}), \qquad \mathrm{add}_t^*(g(\mathcal{J})) = \mathrm{non}(g(\mathcal{J})).$

Proof. Straightforward from definitions. □

We prove now that σ–ideals $\mathbb{S}_2$ and $\mathbb{B}_2$ are closely related to each other.

Theorem 3.3 $s(\mathbb{S}_2) = \mathbb{B}_2$.

Proof. Let us consider any $A \subseteq 2^\omega$. A standard calculation shows that if for some $X \in [\omega]^\omega$ we have $A \restriction X = 2^X$ then $A + [\mathbf{0}_X] = 2^\omega$. Hence if $A \notin \mathbb{B}_2$ then $A \notin s(\mathbb{S}_2)$.

On the other hand, let us consider any $C \subseteq 2^\omega$ such that $B + C = 2^\omega$ for some $B \in \mathbb{S}_2$. As in proofs in Paragraph 2, without loss of generality we can assume that $B = \bigcup_{i<\omega}[f_i]$, where $\{f_i : i < \omega\} \subseteq Pif$ and $\{\mathrm{dom}(f_i) : i < \omega\} \in (\omega)^\omega_\omega$. Then there exists $i < \omega$ such that $C \restriction \mathrm{dom}(f_i) = 2^{\mathrm{dom}(f_i)}$. Indeed, if we suppose that for all $i < \omega$ there exists $g_i \in 2^{\mathrm{dom}(f_i)} \setminus (C \restriction \mathrm{dom}(f_i))$ then we have $\bigcup_{i<\omega}(f_i + g_i) \in 2^\omega \setminus (B + C)$. Thus if $C \notin s(\mathbb{S}_2)$ then $C \notin \mathbb{B}_2$ which completes the proof. □

In [7] the authors showed that the covering number of $\mathbb{B}_2$ is a weird object and it is difficult to find reasonable estimations for it. In particular, it is relatively consistent that Martin's Axiom holds, $\mathfrak{c} = \omega_2$ and $\mathrm{cov}(\mathbb{B}_2) = \omega_1$. The following corollary shows that the situation for the transitive covering number of $\mathbb{B}_2$ is different.

Corollary 3.4 *If Martin's Axiom holds then* $\mathrm{cov}_t(\mathbb{B}_2) = \mathfrak{c}$.

Proof. From Theorem 3.3 and Proposition 3.2 we obtain that $\mathrm{cov}_t(\mathbb{B}_2) \geq \mathrm{non}(\mathbb{S}_2)$. It was proved in [5] that $\mathrm{non}(\mathbb{S}_2) = \aleph_0\text{-}\mathfrak{s}$ and it is well-known that under Martin's Axiom we have $\aleph_0\text{-}\mathfrak{s} = \mathfrak{c}$. □

In order to describe $g(\mathbb{S}_2)$ we need to introduce more definitions. By Inj we denote the set of all injections from ω into ω. For $A \subseteq 2^\omega$ and $\varphi \in Inj$ we put $\varphi * A = \{x \circ \varphi : x \in A\}$ and $A_\varphi = \{x \in 2^\omega : x \circ \varphi \in A\}$. It is easy to observe that we have $\varphi * A_\varphi = A$ and

$A \subseteq (\varphi * A)_\varphi$. Let $\mathcal{J}$ be a σ–ideal of subsets of 2^ω. We say that $\mathcal{J}$ is productive if for every $A \subseteq 2^\omega$ and $\varphi \in Inj$ if $\varphi * A$ is in $\mathcal{J}$ then so is A. We say that $\mathcal{J}$ has WFP (Weak Fubini Property) if for every $A \subseteq 2^\omega$ and $\varphi \in Inj$ if A_φ is in $\mathcal{J}$ then so is A. Straight from the definitions we obtain that $\mathbb{S}_2$, σ–ideals of meagre and null sets are productive and have WFP. For more discussion on these properties cf. [9].

We put

$$p(\mathcal{J}) = \{A \subseteq 2^\omega : (\forall \varphi \in Inj)\, \varphi * A \in \mathcal{J}\}.$$

In other words, $A \in p(\mathcal{J})$ if for every $T \in [\omega]^\omega$ the set $A \restriction T$ is in $\mathcal{J}(T)$, where $\mathcal{J}(T)$ denotes a version of $\mathcal{J}$ defined on 2^T instead of 2^ω.

Theorem 3.5 $g(\mathbb{S}_2) = p(\mathbb{S}_2)$.

Proof. Let us assume that $A \in g(\mathbb{S}_2)$ that is $(\forall B \in \mathbb{S}_2) A + B \in \mathbb{S}_2$. It is not difficult to observe that this condition is equivalent to $(\forall T \in [\omega]^\omega)[\mathbf{0}_T] + A \in \mathbb{S}_2$. But we can prove that if $\varphi \in Inj$ then $[\mathbf{0}_{\mathrm{rng}(\varphi)}] + A = (\varphi * A)_\varphi$. Hence, reformulating our condition we obtain $(\forall \varphi \in Inj)\, (\varphi * A)_\varphi \in \mathbb{S}_2$. Thus, as $\mathbb{S}_2$ is productive and has WFP, we show that this fact is equivalent to $(\forall \varphi \in Inj)\, \varphi * A \in \mathbb{S}_2$ and, consequently, to $A \in p(\mathbb{S}_2)$. $\square$

Finally, we will show that all operations that appeared in this paragraph are versions of one operation, defined in [19].

Let $\mathcal{A}, \mathcal{B}$ be translation invariant families of subsets of a group G. We put

$$\mathcal{G}_t(\mathcal{A}, \mathcal{B}) = \{A \subseteq G : (\forall B \in \mathcal{B})\, A + B \in \mathcal{A}\}.$$

Then we have the following results.

Proposition 3.6 *Let $\mathcal{J}$ be a translation invariant, symmetric σ– ideal of subsets of a group G. Then*
(a) $s(\mathcal{J}) = \mathcal{G}_t(\mathcal{P}(G) \setminus \{G\}, \mathcal{J})$,
(b) $g(\mathcal{J}) = \mathcal{G}_t(\mathcal{J}, \mathcal{J})$.
If $G = 2^\omega$ and $\mathcal{J}$ is productive and has WFP then
(c) $p(\mathcal{J}) = \mathcal{G}_t(\mathcal{J}, \mathbb{S}_2)$.

Proof. (a) and (b) are reformulations of definitions and were observed in [19]. To prove (c) it is enough to repeat carefully the proof of Theorem 3.5. $\square$

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