TRANSITIVE PROPERTIES OF IDEALS ON GENERALIZED CANTOR SPACES

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Abstract. In this paper we compute transitive cardinal coefficients of ideals on generalized Cantor spaces. In particular, we observe that there exists a null set $A \subseteq 2^{\omega_1}$ such that for every null set $B \subseteq 2^{\omega_1}$ we can find $x \in 2^{\omega_1}$ such that the set $A \cup (A + x)$ cannot be covered by any translation of the set B.

1. Introduction, definitions and basic properties

In 2001 Kraszewski in [5] defined a class of *productive* σ-ideals of subsets of the Cantor space 2^ω and observed that both σ-ideals of meagre sets and of null sets are in this class. Next, from every productive σ-ideal $\mathcal{J}$ one can produce a σ-ideal $\mathcal{J}_\kappa$ of subsets of the generalized Cantor space 2^κ. In particular, starting from meagre sets and null sets in 2^ω we obtain meagre sets and null sets in 2^κ, respectively. This description gives us a powerful tool for investigating combinatorial properties of ideals on 2^κ, which was done in [5]. In this paper we continue our research, focusing on transitive cardinal coefficients of ideals of subsets of 2^κ.

We use standard set-theoretical notation and terminology from [2]. Let $(G, +)$ be an infinite abelian group. We consider a σ-ideal $\mathcal{J}$ of subsets of G which is proper, contains all singletons and is invariant (under group operations).

For an ideal $\mathcal{J}$ we consider the following transitive cardinal numbers

$$\begin{aligned}
\mathrm{add}_t(\mathcal{J}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{J} \wedge \neg(\exists B \in \mathcal{J})(\forall A \in \mathcal{A})(\exists g \in G)\, A \subseteq B + g\},\\
\mathrm{add}_t^*(\mathcal{J}) &= \min\{|T| : T \subseteq G \wedge (\exists A \in \mathcal{J})\, A + T \notin \mathcal{J}\},\\
\mathrm{cov}_t(\mathcal{J}) &= \min\{|T| : T \subseteq G \wedge (\exists A \in \mathcal{J})\, A + T = G\},\\
\mathrm{cof}_t(\mathcal{J}) &= \min\{|\mathcal{B}| : \mathcal{B} \subseteq \mathcal{J} \wedge (\forall A \in \mathcal{J})(\exists B \in \mathcal{B})(\exists g \in G)\, A \subseteq B + g\}.
\end{aligned}$$

First two ones are both called *a transitive additivity*. The remaining two ones are called *a transitive covering number* and *a transitive cofinality*, respectively.

We say that an ideal $\mathcal{J}$ is *κ-translatable* if

$$(\forall A \in \mathcal{J})(\exists B_A \in \mathcal{J})(\forall S \in [G]^\kappa)(\exists t_S \in G)\, A + S \subseteq B_A + t_S.$$

We define a *translatability number* of $\mathcal{J}$ as follows

$$\tau(\mathcal{J}) = \min\{\kappa : \mathcal{J} \textit{ is not } \kappa - \textit{translatable}\}.$$

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For more information about relations between classical and transitive cardinal coefficients of ideals – see [2]. For more about translatability – see [1], [3] and [6].

From now on we deal with the generalized Cantor space 2^κ interpreted as the set of all functions from an infinite cardinal number κ into the set $\{0,1\}$. This space is endowed with the standard product topology. Moreover, we consider the standard product measure and the standard product group structure on 2^κ.

We introduce some extra notation in order to simplify further considerations. Let κ be an infinite cardinal number. We put $Inj(\omega,\kappa)=\{\varphi\in\kappa^\omega : \varphi \textit{ is an injection}\}$. For $A\subseteq 2^\kappa$, $B\subseteq 2^\omega$ and $\varphi\in Inj(\omega,\kappa)$ we put $\varphi * A=\{x\circ\varphi : x\in A\}$ and $B_\varphi=\{x\in 2^\kappa : x\circ\varphi\in B\}$.

Obviously, $\varphi * A\subseteq 2^\omega$ and $B_\varphi\subseteq 2^\kappa$. Another simple observation is that for $A\subseteq 2^\kappa, B\subseteq 2^\omega$ and $\varphi\in Inj(\omega,\kappa)$ we have $A\subseteq(\varphi * A)_\varphi$ and $\varphi * B_\varphi = B$.

Let $\mathcal{J}$ be a σ-ideal of subsets of 2^ω. We say that $\mathcal{J}$ is *productive* if

$$(\forall A\subseteq 2^\omega)(\forall\varphi\in Inj(\omega,\omega))(\varphi * A\in\mathcal{J}\Rightarrow A\in\mathcal{J}).$$

It is easy to show that $\mathcal{J}$ is productive if and only if for every $A\subseteq 2^\omega$ and $\varphi\in Inj(\omega,\omega)$ if $A\in\mathcal{J}$ then $A_\varphi\in\mathcal{J}$.

Directly from their definitions we deduce that the σ-ideals of meagre subsets and of null subsets of 2^ω are productive. Also the σ-ideal generated by closed null subsets of 2^ω is productive. Moreover, the ideal $\mathbb{S}_2$ investigated in [4] is the least non-trivial productive σ-ideal of subsets of the Cantor space.

For any productive σ-ideal $\mathcal{J}$ we define

$$\mathcal{J}_\kappa=\{A\subseteq 2^\kappa : (\exists\varphi\in Inj(\omega,\kappa))\,\varphi * A\in\mathcal{J}\}.$$

A standard consideration shows that $\mathcal{J}_\kappa$ is a σ-ideal of subsets of 2^κ. If $\mathcal{J}$ is invariant then so is $\mathcal{J}_\kappa$. If $A\in\mathcal{J}_\kappa$ then any $\varphi\in Inj(\omega,\kappa)$ such that $\varphi * A\in\mathcal{J}$ is called *a witness* for A.

Let us also recall one useful definition used in [5]. We say that an ideal $\mathcal{J}$ of subsets of 2^ω has *WFP (Weak Fubini Property)* if for every $\varphi\in Inj(\omega,\omega)$ and every $A\subseteq 2^\omega$ if A_φ is in $\mathcal{J}$ then so is A.

The σ-ideals of subsets of 2^ω mentioned above obviously have WFP. We will need the following technical lemma proved in [5].

Lemma 1.1. *If $\mathcal{J}$ is a productive ideal of subsets of 2^ω having WFP then for every $\varphi\in Inj(\omega,\kappa)$ and every $A\subseteq 2^\omega$ if $A_\varphi\in\mathcal{J}_\kappa$ then $A\in\mathcal{J}$.* □

2. Transitive cardinal coefficients of ideals on 2^κ

From now on we assume that $\mathcal{J}$ is a proper, invariant and productive σ-ideal of subsets of 2^ω containing all singletons and that $\kappa\geq\omega_1$. We investigate relations between transitive cardinal coefficients of $\mathcal{J}$ and those of $\mathcal{J}_\kappa$. Some of them are similar to relations between standard cardinal coefficients of $\mathcal{J}$ and $\mathcal{J}_\kappa$ proved in [5]. We omit the proofs, as they are also analogous.

Theorem 2.1. $\mathrm{add}_t(\mathcal{J}_\kappa) = \omega_1$. □

Theorem 2.2. $\mathrm{cof}_t(\mathcal{J}_\kappa) \leq \max\{\mathrm{cof}([\kappa]^{\leq\omega}), \mathrm{cof}_t(\mathcal{J})\}$. *Moreover, if* $\mathcal{J}$ *has WFP then* $\mathrm{cof}_t(\mathcal{J}_\kappa) \geq \mathrm{cof}_t(\mathcal{J})$. □

However, other transitive cardinal coefficients behave in a radically different way.

Theorem 2.3. *If* $\mathcal{J}$ *has WFP then* $\mathrm{add}^*_t(\mathcal{J}_\kappa) = \mathrm{add}^*_t(\mathcal{J})$.

Proof. Let $T \subseteq 2^\kappa$ be such that $A + T \notin \mathcal{J}_\kappa$ for some $A \in \mathcal{J}_\kappa$ and let φ be a witness for A. Then $\varphi * A \in \mathcal{J}$ and $\varphi * A + \varphi * T = \varphi * (A + T) \notin \mathcal{J}$. Hence $\mathrm{add}^*_t(\mathcal{J}_\kappa) \geq \mathrm{add}^*_t(\mathcal{J})$.

To show the other inequality, let us fix $T \subseteq 2^\omega$ such that $A + T \notin \mathcal{J}$ for some $A \in \mathcal{J}$. We have $A_{id_\omega} \in \mathcal{J}_\kappa$ (because $id_\omega \in Inj(\omega, \kappa)$ and $\mathcal{J}$ is productive). We define $T' = \{t \in 2^\kappa : t \restriction \omega \in T \wedge t \restriction (\kappa \setminus \omega) \equiv 0\}$. Then $A_{id_\omega} + T' = (A + T)_{id_\omega}$ and from Lemma 1.1 we get $(A + T)_{id_\omega} \notin \mathcal{J}_\kappa$, which ends the proof. □

Theorem 2.4. $\mathrm{cov}_t(\mathcal{J}_\kappa) = \mathrm{cov}_t(\mathcal{J})$.

Proof. Similar to the proof of Theorem 2.3. □

Theorem 2.5. *If* $\mathcal{J}$ *has WFP then* $\tau(\mathcal{J}_\kappa) = \tau(\mathcal{J})$.

Proof. Suppose that $\mathcal{J}$ is ξ-translatable. We consider any $A \in \mathcal{J}_\kappa$ and $\varphi \in Inj(\omega, \kappa)$ being its witness. Then $\varphi * A \in \mathcal{J}$ and we fix $B_{\varphi * A} \in \mathcal{J}$. If $S \in [2^\kappa]^\xi$ then without loss of generality we can assume that $\varphi * S \in [2^\omega]^\xi$ and thus there exists $t_{\varphi * S} \in 2^\omega$ such that $\varphi * A + \varphi * S \subseteq B_{\varphi * A} + t_{\varphi * S}$. Then

$$A + S \subseteq (\varphi * (A + S))_\varphi \subseteq (B_{\varphi * A} + t_{\varphi * S})_\varphi = (B_{\varphi * A})_\varphi + t$$

for some $t \in 2^\kappa$. Hence $\mathcal{J}_\kappa$ is ξ-translatable.

On the other hand, let us assume that $\mathcal{J}_\kappa$ is ξ-translatable and consider any $A \in \mathcal{J}$. Then $A' = A_{id_\omega} \in \mathcal{J}_\kappa$ and we fix $B_{A'} \in \mathcal{J}_\kappa$. If $T \in [2^\omega]^\xi$ then we define $T' \in [2^\kappa]^\xi$ like in the proof of Theorem 2.3. There exists an appropriate $t_{T'} \in 2^\kappa$ such that $A' + T' \subseteq B_{A'} + t_{T'}$. But $A' + T' = (A + T)_{id_\omega}$ and

$$(A + T + t_{T'} \restriction \omega)_{id_\omega} = (A + T)_{id_\omega} + t_{T'} \subseteq B.$$

Let us define

$$C = \bigcup_{T \in [2^\omega]^\xi} (A + T + t_{T'} \restriction \omega).$$

Then $C \subseteq 2^\omega$ and

$$C_{id_\omega} = \bigcup_{T \in [2^\omega]^\xi} (A + T + t_{T'} \restriction \omega)_{id_\omega} \subseteq B \in \mathcal{J}_\kappa.$$

Thus $C_{id_\omega} \in \mathcal{J}_\kappa$ and from Lemma 1.1 we know that $C \in \mathcal{J}$.

Let us consider any $S \in [2^\omega]^\xi$ and put $t_S = t_{T'} \restriction \omega$. Then

$$A + S = A + S + t_S + t_S \subseteq C + t_S$$

and we are done. □

As an immediate corollary we obtain the following interesting result.

Corollary 2.6. *There exists a null set $A \subseteq 2^{\omega_1}$ such that for every null set $B \subseteq 2^{\omega_1}$ we can find $x \in 2^{\omega_1}$ such that the set $A \cup (A + x)$ cannot be covered by any translation of the set B.*

Proof. From [1] we know that $\tau(\mathcal{N}) = 2$, where $\mathcal{N}$ states for the ideal of null subsets of 2^ω. In [5] is shown that $\mathcal{N}_{\omega_1}$ is exactly the ideal of null subsets of 2^{ω_1}. But from Theorem 2.5 we know that $\tau(\mathcal{N}_{\omega_1}) = 2$ and this is what we have been supposed to show. □

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