TRANSITIVE PROPERTIES OF IDEALS ON GENERALIZED CANTOR SPACES

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ABSTRACT. In this paper we compute transitive cardinal coefficients of ideals on generalized Cantor spaces. In particular, we observe that there exists a null set \( A \subseteq 2^{\omega_1} \) such that for every null set \( B \subseteq 2^{\omega_1} \) we can find \( x \in 2^{\omega_1} \) such that the set \( A \cup (A + x) \) cannot be covered by any translation of the set \( B \).

1. INTRODUCTION, DEFINITIONS AND BASIC PROPERTIES

In 2001 Kraszewski in [5] defined a class of productive \( \sigma \)-ideals of subsets of the Cantor space \( 2^{\omega} \) and observed that both \( \sigma \)-ideals of meagre sets and of null sets are in this class. Next, from every productive \( \sigma \)-ideal \( J \) one can produce a \( \sigma \)-ideal \( J_{\kappa} \) of subsets of the generalized Cantor space \( 2^{\kappa} \). In particular, starting from meagre sets and null sets in \( 2^{\omega} \) we obtain meagre sets and null sets in \( 2^{\kappa} \), respectively. This description gives us a powerful tool for investigating combinatorial properties of ideals on \( 2^{\kappa} \), which was done in [5]. In this paper we continue our research, focusing on transitive cardinal coefficients of ideals of subsets of \( 2^{\kappa} \).

We use standard set-theoretical notation and terminology from [2]. Let \((G,+)\) be an infinite abelian group. We consider a \( \sigma \)-ideal \( J \) of subsets of \( G \) which is proper, contains all singletons and is invariant (under group operations).

For an ideal \( J \) we consider the following transitive cardinal numbers:

\[
\text{add}_t(J) = \min\{|A| : A \subseteq J \land \neg(\exists B \in J)(\forall A \in A)(\exists g \in G) A \subseteq B + g\},
\]

\[
\text{add}^*_t(J) = \min\{|T| : T \subseteq G \land (\exists A \in J) A + T \notin J\},
\]

\[
\text{cov}_t(J) = \min\{|T| : T \subseteq G \land (\exists A \in J) A + T = G\},
\]

\[
\text{cof}_t(J) = \min\{|B| : B \subseteq J \land (\forall A \in J)(\exists B \in B)(\exists g \in G) A \subseteq B + g\}.
\]

First two ones are both called a transitive additivity. The remaining two ones are called a transitive covering number and a transitive cofinality, respectively.

We say that an ideal \( J \) is \( \kappa \)-translatable if

\[
(\forall A \in J)(\exists B_A \in J)(\forall S \in [G]^{\kappa})(\exists t_S \in G) A + S \subseteq B_A + t_S.
\]

We define a translatability number of \( J \) as follows

\[
\tau(J) = \min\{\kappa : J \text{ is not } \kappa - \text{translatable}\}.
\]

2000 Mathematics Subject Classification. 03E05, 03E17.

Key words and phrases. generalized Cantor spaces, transitive cardinal coefficients.
For more information about relations between classical and transitive cardinal coefficients of ideals – see [2]. For more about translatability – see [1], [3] and [6].

From now on we deal with the generalized Cantor space $2^\kappa$ interpreted as the set of all functions from an infinite cardinal number $\kappa$ into the set $\{0,1\}$. This space is endowed with the standard product topology. Moreover, we consider the standard product measure and the standard product group structure on $2^\kappa$.

We introduce some extra notation in order to simplify further considerations. Let $\kappa$ be an infinite cardinal number. We put $\text{Inj}(\omega, \kappa) = \{\varphi \in \kappa^\omega : \varphi \text{ is an injection}\}$. For $A \subseteq 2^\kappa$, $B \subseteq 2^\omega$ and $\varphi \in \text{Inj}(\omega, \kappa)$ we put $\varphi \ast A = \{x \circ \varphi : x \in A\}$ and $B_\varphi = \{x \in 2^\kappa : x \circ \varphi \in B\}$.

Obviously, $\varphi \ast A \subseteq 2^\omega$ and $B_\varphi \subseteq 2^\kappa$. Another simple observation is that for $A \subseteq 2^\kappa$, $B \subseteq 2^\omega$ and $\varphi \in \text{Inj}(\omega, \kappa)$ we have $A \subseteq (\varphi \ast A)_\varphi$ and $\varphi \ast B_\varphi = B$.

Let $J$ be a $\sigma$-ideal of subsets of $2^\omega$. We say that $J$ is productive if

$$(\forall A \subseteq 2^\omega)(\forall \varphi \in \text{Inj}(\omega, \omega))(\varphi \ast A \in J \Rightarrow A \in J).$$

It is easy to show that $J$ is productive if and only if for every $A \subseteq 2^\omega$ and $\varphi \in \text{Inj}(\omega, \omega)$ if $A \in J$ then $A_\varphi \in J$.

Directly from their definitions we deduce that the $\sigma$-ideals of meagre subsets and of null subsets of $2^\omega$ are productive. Also the $\sigma$-ideal generated by closed null subsets of $2^\omega$ is productive. Moreover, the ideal $S_2$ investigated in [4] is the least non-trivial productive $\sigma$-ideal of subsets of the Cantor space.

For any productive $\sigma$-ideal $J$ we define

$$J_\kappa = \{A \subseteq 2^\kappa : (\exists \varphi \in \text{Inj}(\omega, \kappa))(\varphi \ast A \in J)\}.$$ 

A standard consideration shows that $J_\kappa$ is a $\sigma$-ideal of subsets of $2^\kappa$. If $J$ is invariant then so is $J_\kappa$. If $A \in J_\kappa$ then any $\varphi \in \text{Inj}(\omega, \kappa)$ such that $\varphi \ast A \in J$ is called a witness for $A$.

Let us also recall one useful definition used in [5]. We say that an ideal $J$ of subsets of $2^\omega$ has WFP (Weak Fubini Property) if for every $\varphi \in \text{Inj}(\omega, \omega)$ and every $A \subseteq 2^\omega$ if $A_\varphi$ is in $J$ then so is $A$.

The $\sigma$-ideals of subsets of $2^\omega$ mentioned above obviously have WFP. We will need the following technical lemma proved in [5].

**Lemma 1.1.** If $J$ is a productive ideal of subsets of $2^\omega$ having WFP then for every $\varphi \in \text{Inj}(\omega, \kappa)$ and every $A \subseteq 2^\omega$ if $A_\varphi \in J_\kappa$ then $A \in J$.  

2. Transitive cardinal coefficients of ideals on $2^\kappa$

From now on we assume that $J$ is a proper, invariant and productive $\sigma$-ideal of subsets of $2^\omega$ containing all singletons and that $\kappa \geq \omega_1$. We investigate relations between transitive cardinal coefficients of $J$ and those of $J_\kappa$. Some of them are similar to relations between standard cardinal coefficients of $J$ and $J_\kappa$ proved in [5]. We omit the proofs, as they are also analogous.
Theorem 2.1. \( \text{add}_T(J_\kappa) = \omega_1. \)

\[ \text{Theorem 2.2.} \quad \text{cof}_T(J_\kappa) \leq \max\{\text{cof}([\kappa]^{\leq \omega}), \text{cof}(J)\}. \quad \text{Moreover, if } J \text{ has WFP then } \text{cof}_T(J_\kappa) \geq \text{cof}_T(J). \]

However, other transitive cardinal coefficients behave in a radically different way.

\[ \text{Theorem 2.3.} \quad \text{If } J \text{ has WFP then } \text{add}_T(J_\kappa) = \text{add}_T(J). \]

\[ \text{Proof.} \quad \text{Let } T \subseteq 2^\kappa \text{ be such that } A + T \not\in J_\kappa \text{ for some } A \in J_\kappa \text{ and let } \varphi \text{ be a witness for } A. \quad \text{Then } \varphi \ast A \in J \text{ and } \varphi \ast A + \varphi \ast T = \varphi \ast (A + T) \not\in J. \quad \text{Hence } \text{add}_T(J_\kappa) \geq \text{add}_T(J). \]

\[ \text{To show the other inequality, let us fix } T \subseteq 2^\omega \text{ such that } A + T \not\in J \text{ for some } A \in J. \quad \text{We have } A_{id_\omega} \in J_\kappa \text{ (because } id_\omega \in Inj(\omega, \kappa) \text{ and } J \text{ is productive).} \quad \text{We define } T' = \{ t \in 2^\kappa : t \upharpoonright \omega \in T \land t \upharpoonright (\kappa \setminus \omega) \equiv 0 \}. \quad \text{Then } A_{id_\omega} + T' = (A + T)_{id_\omega} \text{ and from Lemma 1.1 we get } (A + T)_{id_\omega} \not\in J_\kappa, \text{ which ends the proof.} \]

\[ \text{Theorem 2.4.} \quad \text{cov}_T(J_\kappa) = \text{cov}_T(J). \]

\[ \text{Proof.} \quad \text{Similar to the proof of Theorem 2.3.} \]

\[ \text{Theorem 2.5.} \quad \text{If } J \text{ has WFP then } \tau(J_\kappa) = \tau(J). \]

\[ \text{Proof.} \quad \text{Suppose that } J \text{ is } \xi\text{-translatable. We consider any } A \in J_\kappa \text{ and } \varphi \in Inj(\omega, \kappa) \text{ being its witness. Then } \varphi \ast A \in J \text{ and we fix } B_{\varphi \ast A} \in J. \quad \text{If } S \in [2^\kappa]^{\xi} \text{ then without loss of generality we can assume that } \varphi \ast S \in [2^\omega]^{\xi} \text{ and thus there exists } t_{\varphi \ast S} \in 2^\omega \text{ such that } \varphi \ast A + \varphi \ast S \subseteq B_{\varphi \ast A} + t_{\varphi \ast S}. \quad \text{Then} \]

\[ A + S \subseteq (\varphi \ast (A + S)) \varphi \subseteq (B_{\varphi \ast A} + t_{\varphi \ast S}) \varphi = (B_{\varphi \ast A}) \varphi + t \]

\[ \text{for some } t \in 2^\omega. \quad \text{Hence } J_\kappa \text{ is } \xi\text{-translatable.} \]

\[ \text{On the other hand, let us assume that } J_\kappa \text{ is } \xi\text{-translatable and consider any } A \in J. \quad \text{Then } A' = A_{id_\omega} \in J_\kappa \text{ and we fix } B_{A'} \in J_\kappa. \quad \text{If } T \in [2^\omega]^{\xi} \text{ then we define } T' \in [2^\omega]^{\xi} \text{ like in the proof of Theorem 2.3. There exists an appropriate } t_{T'} \in 2^\omega \text{ such that } A' + T' \subseteq B_{A'} + t_{T'}. \quad \text{But } A' + T' = (A + T)_{id_\omega} \text{ and} \]

\[ (A + T + t_{T'} \upharpoonright \omega)_{id_\omega} = (A + T)_{id_\omega} + t_{T'} \subseteq B. \]

\[ \text{Let us define} \]
\[ C = \bigcup_{T \in [2^\omega]^{\xi}} (A + T + t_{T'} \upharpoonright \omega). \]
\[ \text{Then } C \subseteq 2^\omega \text{ and} \]
\[ C_{id_\omega} = \bigcup_{T \in [2^\omega]^{\xi}} (A + T + t_{T'} \upharpoonright \omega)_{id_\omega} \subseteq B \in J_\kappa. \]

\[ \text{Thus } C_{id_\omega} \in J_\kappa \text{ and from Lemma 1.1 we know that } C \in J. \quad \text{Let us consider any } S \in [2^\omega]^{\xi} \text{ and put } t_S = t_{T'} \upharpoonright \omega. \quad \text{Then} \]
\[ A + S = A + S + t_S + t_S \subseteq C + t_S \]

\[ \text{and we are done.} \]

\[ \text{As an immediate corollary we obtain the following interesting result.} \]
Corollary 2.6. There exists a null set \( A \subseteq \mathcal{2}^{\omega_1} \) such that for every null set \( B \subseteq \mathcal{2}^{\omega_1} \) we can find \( x \in \mathcal{2}^{\omega_1} \) such that the set \( A \cup (A + x) \) cannot be covered by any translation of the set \( B \).

Proof. From [1] we know that \( \tau(\mathcal{N}) = 2 \), where \( \mathcal{N} \) states for the ideal of null subsets of \( \mathcal{2}^{\omega_1} \). In [5] is shown that \( \mathcal{N}_{\omega_1} \) is exactly the ideal of null subsets of \( \mathcal{2}^{\omega_1} \). But from Theorem 2.5 we know that \( \tau(\mathcal{N}_{\omega_1}) = 2 \) and this is what we have been supposed to show.

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References