TRANSITIVE PROPERTIES OF IDEALS

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Abstract. In this paper we present a wide range of results connected with transitive properties of ideals. In particular, we present relations between standard and transitive coefficients of ideals and compute transitive cardinal coefficients of ideals on generalized Cantor spaces.

0. Introduction

In this paper we would like to give a description of cardinal characteristics connected with certain transitive properties of ideals. We begin our presentation from some history.

Let $\mathbb{G}$ be any abelian group and let $\mathcal{J}$ be any proper ideal of subsets of $\mathbb{G}$ which is *translation invariant* (that is $I+g\in\mathcal{J}$ for each $I\in\mathcal{J}$ and $g\in\mathbb{G}$). The first cardinal coefficient on the stage was a transitive covering number of $\mathcal{J}$ (denoted by $\mathrm{cov}_t(\mathcal{J})$) that appeared implicitely in 1938 in the famous Rothberger theorem, which was originally formulated for classical ideals of meagre and null subsets of the real line (cf. [21]). We can formulate this theorem more generally as follows.

Theorem 0.1. *Let $\mathcal{J}$ and $\mathcal{I}$ be translation invariant ideals of subsets of a group $\mathbb{G}$, orthogonal to each other (that is there exist $A\in\mathcal{J}$ and $B\in\mathcal{I}$ such that $A\cup B=\mathbb{G}$). Then*

$$\mathrm{cov}_t(\mathcal{J})\le\mathrm{non}(\mathcal{I}),$$

where $\mathrm{non}(\mathcal{I})$ is the minimal cardinality of the subset of $\mathbb{G}$ that do not belong to $\mathcal{I}$.

Proof. We fix $A\in\mathcal{J}$ and $B\in\mathcal{I}$ such that $A\cup B=\mathbb{G}$. Let $T\subseteq\mathbb{G}$ be the set of cardinality $\mathrm{non}(\mathcal{I})$, which is not in $\mathcal{I}$. One can notice that $A-T=\mathbb{G}$, which ends the proof. $\square$

In 1981 Carlson asked if it was possible to find a null subset B of the real line with a property that for every null subset A of the real line there exists a real number r such that $A\subseteq B+r$. We can reformulate this problem in the following way.

We call a family $\mathcal{B}\subseteq\mathcal{J}$ a transitive base of $\mathcal{J}$ if for each $A\in\mathcal{J}$ there exists $B\in\mathcal{B}$ and $g\in\mathbb{G}$ such that $A\subseteq B+g$. The minimal cardinality of a transitive base of $\mathcal{J}$ we call the transitive cofinality and denote by $\mathrm{cof}_t(\mathcal{J})$. Thus, the question was whether $\mathrm{cof}_t(\mathcal{N})=1$, where $\mathcal{N}$ denotes the $\sigma-$ideal of null subsets of the real line. The most general answer was obtained by Galvin in 1981 and independently by Brzuchowski, Cichoń and Węglorz in 1982 (cf. [7], p.166 or [22]).

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Theorem 0.2. *Let $\mathcal{J}$ be a translation invariant ideal of subsets of a group $\mathbb{G}$. If $\mathcal{J}$ is symmetric (that is $-A \in \mathcal{J}$ for each $A \in \mathcal{J}$), then* $\mathrm{cof}_t(\mathcal{J}) > 1$.

Proof. Suppose that there exists a set $B \in \mathcal{J}$ such that for every $A \in \mathcal{J}$ there exists $g \in \mathbb{G}$ such that $A \subseteq B + g$. If $b \notin B$ then $B \subsetneq B \cup \{b\} \subseteq B + g$ for some $g \in \mathbb{G}$. Thus $-B \subsetneq -B - g$. But we can assume that $B = -B$. Hence $B \subset B + g \subsetneq (B - g) + g = B$ which is a contradiction. □

The complete description of transitive cofinalities of ideals of meagre and null subsets of the real line was presented by Pawlikowski in [19] in 1984. He also mentioned a dual coefficient to a transitive cofinality. Following the way of describing cardinal caracteristics of the continuum presented by Blass in [4] we will call it a transitive additivity and denote by $\mathrm{add}_t(\mathcal{J})$. Unfortunately, Pawlikowski (and then Bartoszyński and Judah in [2]) used this name and notation for another coefficient. In order not to make a mess we will call it a starred transitive additivity and denote by $\mathrm{add}_t^*(\mathcal{J})$.

In 1989 Seredyński in [22] investigated properties of some transitive operations on ideals.

In 1993 Carlsson in [6] introduced the notion of κ-translatibility and proved that the $\sigma-$ideal of meagre substes of the real line and the $\sigma-$ideal generated by closed null subsets of the real line are ω-translatable. Bartoszyński in [1] proved that the $\sigma-$ideal of null substes of the Cantor space is not 2-translatable. Kysiak in [16] introduced a natural notion of a translatibility number.

In the second paragraph of this paper we present relations between standard and transitive coefficients of an ideal. In the fifth paragraph we discuss possibility of existence other relations. In the third paragraph we compute these characteristics for the $\sigma-$ideal $\mathbb{S}_2$. In the fourth paragraph we show that the transitive covering number can be totally different from the standard cofinality. The sixth paragraph is devoted to transitive operations. Finally, we focus our attention on transitive cardinal coefficients of ideals of subsets of generalized Cantor spaces.

1. Definitions and basic properties

We use standard set-theoretical notation and terminology derived from [15]. Let us remind that the cardinality of the set of all real numbers is denoted by $\mathfrak{c}$. The cardinality of a set X is denoted by $|X|$. A power set of a set X is denoted by $\mathcal{P}(X)$. If κ is a cardinal number then $[X]^\kappa$ ($[X]^{\leq\kappa}$) denotes the family of all subsets of the set X of cardinality κ (not greater than κ, respectively). $X^{<\omega}$ denotes the set of all finite sequences of elements of the set X. If $\varphi : X \to Y$ is a function then $\mathrm{rng}(\varphi)$ denotes the range of φ. If $A \subseteq Y$ then $\varphi^{-1}[A]$ denotes the pre-image of A.

Let $(G, +)$ be an infinite abelian group. We consider a $\sigma-$ideal $\mathcal{J}$ of subsets of G which is proper and contains all singletons (i.e. $\bigcup \mathcal{J} = G$). Moreover, we assume that $\mathcal{J}$ is translation invariant (i.e. $(\forall A \in \mathcal{J})(\forall g \in G)\, A+g = \{a+g : a \in A\} \in \mathcal{J}$) and symmetric (i.e. $(\forall A \in \mathcal{J}) - A = \{-a : a \in A\} \in \mathcal{J}$).

We say that a family $\mathcal{B} \subseteq \mathcal{J}$ is *cofinal* with $\mathcal{J}$ if for each $A \in \mathcal{J}$ there exists such $B \in \mathcal{B}$ that $A \subseteq B$. We also call such a family $\mathcal{B}$ a *base* of $\mathcal{J}$.

For an ideal $\mathcal{J}$ we consider the following cardinal numbers

$$\begin{aligned} \operatorname{add}(\mathcal{J}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{J} \ \ \& \ \ \bigcup \mathcal{A} \notin \mathcal{J}\}, \\ \operatorname{cov}(\mathcal{J}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{J} \ \ \& \ \ \bigcup \mathcal{A} = G\}, \\ \operatorname{non}(\mathcal{J}) &= \min\{|B| : B \subseteq G \ \ \& \ \ B \notin \mathcal{J}\}, \\ \operatorname{cof}(\mathcal{J}) &= \min\{|\mathcal{B}| : \mathcal{B} \subseteq \mathcal{J} \ \ \& \ \ \mathcal{B} \textit{ is a base of } \mathcal{J}\}. \end{aligned}$$

They are called the *additivity*, the *covering number*, the *uniformity* and the *cofinality* of $\mathcal{J}$, respectively. Note that the following relations hold:

$$\operatorname{add}(\mathcal{J}) \leq \operatorname{cov}(\mathcal{J}),\ \operatorname{add}(\mathcal{J}) \leq \operatorname{non}(\mathcal{J}),\ \operatorname{cov}(\mathcal{J}) \leq \operatorname{cof}(\mathcal{J}),\ \operatorname{non}(\mathcal{J}) \leq \operatorname{cof}(\mathcal{J}).$$

Moreover, $\operatorname{add}(\mathcal{J})$ is regular and $\operatorname{add}(\mathcal{J}) \leq \min\{\operatorname{cf}(\operatorname{non}(\mathcal{J})), \operatorname{cf}(\operatorname{cof}(\mathcal{J}))\}$.

We call a family $\mathcal{B} \subseteq \mathcal{J}$ a *transitive base* if for each $A \in \mathcal{J}$ there exists $B \in \mathcal{B}$ and $g \in G$ such that $A \subseteq B + g$.

For an ideal $\mathcal{J}$ we consider the following cardinal numbers

$$\begin{aligned} \operatorname{add}_t(\mathcal{J}) &= \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{J} \ \ \& \ \ \neg(\exists B \in \mathcal{J})(\forall A \in \mathcal{A})(\exists g \in G)\, A \subseteq B + g\}, \\ \operatorname{add}_t^*(\mathcal{J}) &= \min\{|T| : T \subseteq G \ \ \& \ \ (\exists A \in \mathcal{J})\, A + T \notin \mathcal{J}\}, \\ \operatorname{cov}_t(\mathcal{J}) &= \min\{|T| : T \subseteq G \ \ \& \ \ (\exists A \in \mathcal{J})\, A + T = G\}, \\ \operatorname{cof}_t(\mathcal{J}) &= \min\{|\mathcal{B}| : \mathcal{B} \subseteq \mathcal{J} \ \ \& \ \ \mathcal{B} \textit{ is a transitive base of } \mathcal{J}\}. \end{aligned}$$

First two ones are both called *transitive additivity*. The latter two ones are called *transitive covering number* and *transitive cofinality*, respectively. Let us notice that all definitions of cardinal coefficients mentioned above (both normal and transitive) are valid also for an arbitrary family $\mathcal{A} \subseteq P(G)$.

We say that an ideal $\mathcal{J}$ is $\kappa-$*translatable* if

$$(\forall A \in \mathcal{J})(\exists B \in \mathcal{J})(\forall T \in [G]^{\kappa})(\exists g \in G)\, A + T \subseteq B + g.$$

We define a *translatibility number* of $\mathcal{J}$ as follows

$$\tau(\mathcal{J}) = \min\{\kappa : \mathcal{J} \textit{ is not } \kappa-\textit{translatable}\}.$$

For a $\sigma-$ideal $\mathcal{J}$ of subsets of G we define the following families of subsets of G

$$\begin{aligned} s(\mathcal{J}) &= \{A \subseteq G : (\forall B \in \mathcal{J})\, A + G \neq G\}, \\ g(\mathcal{J}) &= \{A \subseteq G : (\forall B \in \mathcal{J})\, A + G \in \mathcal{J}\}, \end{aligned}$$

The following basic properties of operations s and g can be found e.g. in [22].

Proposition 1.1. *Let us assume that $\mathcal{J}$ is a proper, translation invariant, symmetric $\sigma-$ideal which contains singletons. Then (a) $g(\mathcal{J})$ is a proper, translation invariant, symmetric $\sigma-$ideal which contains singletons;*
(b) $s(\mathcal{J})$ is a proper, translation invariant, symmetric family of subsets of G which contains singletons;

(c) $g(\mathcal{J}) \subseteq \mathcal{J} \cap s(\mathcal{J})$;
(d) $s(s(s(\mathcal{J}))) = s(\mathcal{J})$;
(e) $g(g(\mathcal{J})) = g(\mathcal{J})$. □

If G is a fixed Polish locally compact group equipped with Haar measure then the $\sigma-$ideals of meagre subsets and of null subsets of G are denoted by $\mathcal{M}(G)$ and $\mathcal{N}(G)$, respectively. We will write $\mathcal{M}$ and $\mathcal{N}$ if it is clear which Polish group we consider or if the specification of a group is not necessary.

The Galvin-Mycielski-Solovay theorem [10] shows that $s(\mathcal{M})$ is a $\sigma-$ideal of strongly null sets (this theorem is used to be formulated for $G = 2^\omega$ or $G = \mathbb{R}$ but is true for every locally Polish group – see [16]). The family $s(\mathcal{N})$ is called strongly meager sets. Recently Bartoszyński and Shelah show [3] that under CH strongly meager sets do not form an ideal.

Sets from σ–ideals $g(\mathcal{M})$ and $g(\mathcal{N})$ are called meager-additive and null-additive, respectively (see [2] for more information).

From now on we deal with the generalized Cantor space 2^κ interpreted as the set of all functions from an infinite cardinal number κ into the set $\{0,1\}$. This spaces are endowed with the standard product topology. Moreover, we consider the standard product measure on 2^κ.

We define

$$Pif = \{f : f\ is\ a\ function\ \ \&\ \ \mathrm{dom}(f) \in [\omega]^\omega\ \ \&\ \ \mathrm{rng}(f) \subseteq 2\}.$$

If $f \in Pif$ then we put

$$[f] = \{x \in 2^\omega : f \subseteq x\}.$$

Let $\mathbb{S}_2$ denotes the σ-ideal of subsets of 2^ω, which is generated by a family $\{[f] : f \in Pif\}$. We recall some properties of $\mathbb{S}_2$, which were proved in [8].

Fact 1.2. *(a) $\mathbb{S}_2$ is a proper σ-ideal, containig singletons, with a base consisting of Borel sets. Every $A \in \mathbb{S}_2$ is both meager and null.*
(b) $\mathbb{S}_2$ is translation invariant ans symmetric.
(c) There exist a family of size $\mathfrak{c}$ of pairwise disjoint Borel subsets of 2^ω that do not belong to $\mathbb{S}_2$. □

We call a family $\mathcal{F} \subseteq Pif$ *normal* if for each two different $f_1, f_2 \in \mathcal{F}$ we have $\mathrm{dom}(f_1) \cap \mathrm{dom}(f_2) = \emptyset$. Directly from the definition of $\mathbb{S}_2$ we can deduce that

$$A \in \mathbb{S}_2 \Longleftrightarrow A \subseteq \bigcup_{f \in \mathcal{F}} [f],$$

for some countable normal family $\mathcal{F} \subseteq Pif$. In [8] the following useful lemma was proved.

Lemma 1.3. *Suppose that $\{f_i : i \in I\}$ is a normal family of functions from Pif, $f \in Pif$ and $[f] \subseteq \bigcup_{i \in I} [f_i]$. Then $[f] \subseteq [f_i]$ for some $i \in I$.* □

Let A, S be two infinite subsets of ω. We say that S *splits* A if $|A \cap S| = |A \setminus S| = \omega$. Let us recall a cardinal number related with a notion of splitting, introduced by Malychin in [17], namely

$$\aleph_0\text{-}\mathfrak{s} = \min\{|\mathcal{S}| : \mathcal{S} \subseteq [\omega]^\omega\ \ \&\ \ (\forall \mathcal{A} \in [[\omega]^\omega]^\omega)(\exists S \in \mathcal{S})(\forall A \in \mathcal{A})(S\ splits\ A)\}.$$

More about cardinal numbers connected with the relation of splitting can be found in [12].

We define also a reaping number

$$\mathfrak{r} = \min\{|\mathcal{R}| : \mathcal{R} \subseteq [\omega]^\omega \ \& \ (\forall A \in [\omega]^\omega)(\exists R \in \mathcal{R})(A \textit{ does not split} R)\}.$$

(cf. [23] for more details). In [8] the following fact was proved.

Fact 1.4. $\mathrm{add}(\mathbb{S}_2) = \omega_1$, $\mathrm{non}(\mathbb{S}_2) = \aleph_0\text{-}\mathfrak{s}$, $\mathrm{cov}(\mathbb{S}_2) = \mathfrak{r}$, $\mathrm{cof}(\mathbb{S}_2) = \mathfrak{c}$. □

We introduce some extra notation in order to simplify further considerations. Let λ and κ be any infinite cardinal numbers. We put $Inj(\lambda,\kappa) = \{\varphi \in \kappa^\lambda : \varphi \textit{ is an injection}\}$.

Definition. For $A \subseteq 2^\kappa$ and $\varphi \in Inj(\lambda,\kappa)$ we put $\varphi * A = \{x \circ \varphi : x \in A\}$.
For $B \subseteq 2^\omega$ and $\varphi \in Inj(\omega,\kappa)$ we put $B_\varphi = \{x \in 2^\kappa : x \circ \varphi \in B\}$.

Obviously, $\varphi * A \subseteq 2^\lambda$ and $B_\varphi \subseteq 2^\kappa$. Another simple observation is that for $B \subseteq 2^\omega$ and $\varphi \in Inj(\omega,\kappa)$ we have $\varphi * B_\varphi = B$. Similarly, for $A \subseteq 2^\kappa$ and $\varphi \in Inj(\omega,\kappa)$ we have $A \subseteq (\varphi * A)_\varphi$.

From now on let $\mathcal{J}$ be a $\sigma-$ideal of subsets of 2^ω and κ be any infinite cardinal number. We define

$$\kappa(\mathcal{J}) = \{A \subseteq 2^\kappa : (\exists \varphi \in Inj(\omega,\kappa))\, \varphi * A \in \mathcal{J}\}.$$

If $A \subseteq 2^\kappa$ then any $\varphi \in Inj(\omega,\kappa)$ such that $\varphi * A \in \mathcal{J}$ we called *a witness* for A. If $\mathcal{I} \subseteq \mathcal{J}$ then $\kappa(\mathcal{I}) \subseteq \kappa(\mathcal{J})$. Moreover, we have $\mathcal{J} \subseteq \omega(\mathcal{J})$, because for every $A \in \mathcal{J}$ the identity on ω is a witness for A. The $\sigma-$ideal generated by the family $\kappa(\mathcal{J})$ we denote by $\mathcal{J}_\kappa$.

Definition. The $\sigma-$ideal $\mathcal{J}$ is *productive* if $\omega(\mathcal{J}) \subseteq \mathcal{J}$.

We formulate some equivalent versions of this property now, proved in [13].

Fact 1.5. *For a σ–ideal $\mathcal{J}$ of subsets of 2^ω the following conditions are equivalent:*
(a) $\mathcal{J}$ is productive,
(b) $(\forall A \subseteq 2^\omega)(\forall \varphi \in Inj(\omega,\omega))(\varphi * A \in \mathcal{J} \Rightarrow A \in \mathcal{J})$,
(c) $(\forall A \subseteq 2^\omega)(\forall \varphi \in Inj(\omega,\omega))(A \in \mathcal{J} \Rightarrow A_\varphi \in \mathcal{J})$. □

Directly from their definitions we deduce that the $\sigma-$ideals of meagre subsets and of null subsets of 2^ω are productive. Also the $\sigma-$ideal generated by closed null subsets of 2^ω is productive. Moreover, $\mathbb{S}_2$ is the least non-trivial productive $\sigma-$ideal of subsets of the Cantor space.

If $\mathcal{J}$ is productive then $\kappa(\mathcal{J}) = \mathcal{J}_\kappa$ for any infinite cardinal number κ. $\sigma-$ideals $\mathcal{J}_\kappa$ for a certain productive $\sigma-$ideal $\mathcal{J}$ of subsets of 2^ω were intensively studied in [13].

We shall use in our further considerations the following simple lemma.

Lemma 1.6. *If $A, B \subseteq 2^\omega$, $\varphi \in Inj(\omega,\kappa)$, $s \in 2^\omega$ and $t \in 2^\kappa$ then*
(a) $A_\varphi + t = (A + t \circ \varphi)_\varphi$;
(b) $(B + s)_\varphi = B_\varphi + s'$ for some $s' \in 2^\kappa$ such that $s' \circ \varphi = s$;
(c) $(A + B)_\varphi = A_\varphi + B_\varphi$.

If $A, B \subseteq 2^\kappa$, $\varphi \in Inj(\lambda, \kappa)$ *then*
(d) $\varphi * (A + B) = \varphi * A + \varphi * B$.

Proof. Straightforward from the definitions. □

Let us also recall one useful definition used in [13].

The ideal $\mathcal{J}$ of subsets of 2^ω has WFP (Weak Fubini Property) if for every $\varphi \in Inj(\omega, \omega)$ and every $A \subseteq 2^\omega$ if A_φ is in $\mathcal{J}$ then so is A.

The $\sigma-$ideals mentioned previously, i.e. $\sigma-$ideals of meagre sets and of null sets of 2^ω, the $\sigma-$ideal generated by closed null subsets of 2^ω and $\mathbb{S}_2$ obviously have WFP.

We will need the following technical lemma proved in [13].

Lemma 1.7. *If* $\mathcal{J}$ *is a productive ideal of subsets of* 2^ω *having WFP then for every* $\varphi \in Inj(\omega, \kappa)$ *and every* $A \subseteq 2^\omega$ *if* $A_\varphi \in \mathcal{J}_\kappa$ *then* $A \in \mathcal{J}$. □

We introduce a notion which is in a sense dual to the notion of productivity. Let $\mathcal{J}$ be a $\sigma-$ideal of subsets of 2^ω. We put

$$p(\mathcal{J}) = \{A \subseteq 2^\kappa : (\forall \varphi \in Inj(\omega, \omega))\, \varphi * A \in \mathcal{J}\}.$$

The following fact holds.

Fact 1.8. *(a)* $p(\mathcal{J}) \subseteq \mathcal{J}$.
(b) If $\mathcal{J}$ *is proper, translation invariant, symmetric and contains singletons then so is* $p(\mathcal{J})$.

Proof. (a) is obvious as $id_\omega \in Inj(\omega, \omega)$.
Let $\mathcal{J}$ be a proper, translation invariant and symmetric $\sigma - ideal$ of subsets of 2^ω. Then $p(\mathcal{J})$ is a $\sigma-$ideal because of the fact that for $\varphi \in Inj(\omega, \omega)$ and sets $A_i \subseteq 2^\omega$ we have $\varphi * \bigcup_{i<\omega} A_i = \bigcup_{i<\omega} \varphi * A_i$. Properness is straight from (a) and containing singletons is straight from the definition and the assumption. To get translation invariance and symmetry it is enough to notice that for $\varphi \in Inj(\omega, \omega)$ and $A \subseteq 2^\omega, x \in 2^\omega$ we have $\varphi * (-A) = -\varphi * A$ and $\varphi * (A + x) = \varphi * A + x \circ \varphi)$. □

The next theorem shows the duality mentioned above.

Theorem 1.9. *Let us consider functions* $\omega, p : P(P(2^\omega)) \to P(P(2^\omega))$ *defined as follows*

$$\omega(\mathcal{A}) = \{A \subseteq 2^\kappa : (\exists \varphi \in Inj(\omega, \omega))\, \varphi * A \in \mathcal{A}\}$$

$$p(\mathcal{A}) = \{A \subseteq 2^\kappa : (\forall \varphi \in Inj(\omega, \omega))\, \varphi * A \in \mathcal{A}\}.$$

Then ω *is a topological closure operator (in a sense of Kuratowski),* p *is a topological interior operator and they determine the same topology.*

Proof. As far as an operation ω is concerned, we obtain straight from the definition that $\omega(\emptyset) = \emptyset$, $\mathcal{A} \subseteq \omega(\mathcal{A})$ and $\omega(\mathcal{A} \cup \mathcal{B}) = \omega(\mathcal{A}) \cup \omega(\mathcal{B})$. Furthemore, using the fact that for $\varphi, \psi \in Inj(\omega, \omega)$ and $A \subseteq 2^\omega$ we have $\varphi \circ \psi \in Inj(\omega, \omega)$ and $\psi * (\varphi * A) = (\varphi \circ \psi) * A$ we get by simple calculations $\omega(\omega(\mathcal{A})) = \omega(\mathcal{A})$, which implies that ω is a topological closure operation.

Similarly, we obtain from the definition that $p(2^\omega) = 2^\omega$, $p(\mathcal{A}) \subseteq \mathcal{A})$ and $p(\mathcal{A} \cap \mathcal{B}) = p(\mathcal{A}) \cap p(\mathcal{B})$. Also the same argument as above helps us to show that $p(p(\mathcal{A})) = p(\mathcal{A})$. Consequently, p is a topological interior operation.

Let $\mathcal{T}_\omega$ and $\mathcal{T}_p$ denote topologies on $P(P(2^\omega))$ determined by operations ω and p, respectively. We prove that $\mathcal{T}_\omega = \mathcal{T}_p$. First, let us observe that for every family $\mathcal{A}$ we have $p(\mathcal{A})^c = \omega(\mathcal{A}^c)$. We know that $\mathcal{A} \in \mathcal{T}_\omega \iff \mathcal{A}^c = \omega(\mathcal{A}^c)$ and $\mathcal{A} \in \mathcal{T}_p \iff \mathcal{A} = p(\mathcal{A})$. To finish the proof it is enough to notice that $\mathcal{A} \in \mathcal{T}_p$ implies $\omega(\mathcal{A}^c) = p(\mathcal{A})^c = \mathcal{A}^c$ and $\mathcal{A} \in \mathcal{T}_\omega$ implies $\mathcal{A} = (\mathcal{A}^c)^c = (\omega(\mathcal{A}^c))^c = (p(\mathcal{A})^c)^c = p(\mathcal{A})$. $\square$

We will need one more $\sigma-$ideal. Let us define

$$\mathbb{B}_2 = \{A \subseteq 2^\omega : (\forall X \in [\omega]^\omega)\, A \restriction X \neq 2^X\},$$

where $A \restriction X = \{x \restriction X : x \in A\}$. This is one of the Mycielski ideals and was intensively studied by many authors (cf. [9], [18], [20]). It is an easy observation that $\mathbb{B}_2 = \{A \subseteq 2^\omega : (\forall \varphi \in Inj(\omega,\omega))\, \varphi * A \neq 2^\omega\}$ and, consequently, $\mathbb{B}_2 = p(P(2^\omega) \setminus \{2^\omega\})$.

2. Transitive cardinal coefficients of ideals

Let $(G,+)$ be an infinite abelian group. We consider a $\sigma-$ideal $\mathcal{J}$ of subsets of G which is proper and contains all singletons. Moreover, we assume that $\mathcal{J}$ is translation invariant and symmetric.

In this section we present relations between standard and transitive cardinal characteristics of $\mathcal{J}$. First of all, we have the following diagram.

Theorem 2.1.

$$\begin{tikzcd}
 & \mathrm{add}_t(\mathcal{J}) \arrow[r] & \mathrm{cof}_t(\mathcal{J}) \arrow[dr] & \\
\mathrm{add}(\mathcal{J}) \arrow[ur] \arrow[r] \arrow[dr] & \mathrm{cov}(\mathcal{J}) \arrow[rr] \arrow[dr] & & \mathrm{cof}(\mathcal{J}) \\
 & \mathrm{add}_t^*(\mathcal{J}) \arrow[r] \arrow[dr] & \mathrm{cov}_t(\mathcal{J}) & \\
 & & \mathrm{non}(\mathcal{J}) \arrow[uur] &
\end{tikzcd}$$

where $\kappa \to \lambda$ means $\kappa \leq \lambda$. Moreover, every inequality may be strict.

Proof. Left to the reader. For possibility of strict inequalities – cf. Paragraph 5. $\square$

There are also some extra connections between these coefficients.

Proposition 2.2. $\mathrm{add}(\mathcal{J}) = \min\{\mathrm{add}_t(\mathcal{J}), \mathrm{add}_t^*(\mathcal{J})\}$.

Proof. Let us cosider $\mathcal{A} \subseteq \mathcal{J}$ such that $|\mathcal{A}| < \min\{\mathrm{add}_t(\mathcal{J}), \mathrm{add}_t^*(\mathcal{J})\}$. Then there exists $B \in \mathcal{J}$ such that for every $A \in \mathcal{A}$ there exists $g_A \in G$ such that $A \subseteq B + g_A$. Let $T = \{g_A : A \in \mathcal{A}\}$. Then $|T| < \mathrm{add}_t^*(\mathcal{J})$ so $B + T \in \mathcal{J}$. Furthemore, $\bigcup \mathcal{A} \subseteq B + T$, which ends the proof. $\square$

Proposition 2.3. $\mathrm{cov}(\mathcal{J}) \geq \min\{\mathrm{add}_t(\mathcal{J}), \mathrm{cov}_t(\mathcal{J})\}$.

Proof. Let us cosider $\mathcal{A} \subseteq \mathcal{J}$ such that $|\mathcal{A}| < \min\{\mathrm{add}_t(\mathcal{J}), \mathrm{cov}_t(\mathcal{J})\}$. Then there exists $B \in \mathcal{J}$ and $T \in [G]^{|\mathcal{A}|}$ such that $\bigcup \mathcal{A} \subseteq B + T$. But $B + T \neq G$ and, consequently, $\bigcup \mathcal{A} \neq G$ and we are done. □

This result gives an interesting corollary concerning an ideal of meagre sets.

Corollary 2.4. *If* $\mathfrak{b} = \mathfrak{c}$ *then* $\mathrm{cov}(\mathcal{M}) = \mathrm{cov}_t(\mathcal{M})$.

Proof. Pawlikowski in [19] proved that $\mathrm{add}_t(\mathcal{M}) = \mathfrak{b}$. So the assumption $\mathrm{cov}(\mathcal{M}) < \mathrm{cov}_t(\mathcal{M}) \leq \mathfrak{c}$ leads to a contradiction. □

Fact 2.5. *If* $\mathrm{cof}(\mathcal{J}) > |G|$ *then* $\mathrm{cof}_t(\mathcal{J}) = \mathrm{cof}(\mathcal{J})$.

Proof. Straightforward from the definitions. □

Our next observation is connected with cofinalities of transitive coefficients.

Proposition 2.6. *(a)* $\mathrm{cf}(\mathrm{add}_t(\mathcal{J})) \geq \mathrm{add}(\mathcal{J})$,
(b) $\mathrm{cf}(\mathrm{add}_t^*(\mathcal{J})) \geq \mathrm{add}(\mathcal{J})$,
(c) $\mathrm{cf}(\mathrm{cof}_t(\mathcal{J})) \geq \mathrm{add}(\mathcal{J})$.

Proof. To prove (a) let us consider $\mathcal{A} \subseteq \mathcal{J}$ such that for every $B \in \mathcal{J}$ there exists $A \in \mathcal{A}$ which cannot be covered by any translation of the set B. Let us assume that $\lambda = \mathrm{cf}(\mathrm{add}_t(\mathcal{J})) < \mathrm{add}(\mathcal{J})$. Then there exist $\mathcal{A}_\xi \subseteq \mathcal{J}$ for $\xi < \lambda$ such that

$$\mathcal{A} = \bigcup_{\xi<\lambda} \mathcal{A}_\xi \quad \text{and} \quad |\mathcal{A}_\xi| < \mathrm{add}_t(\mathcal{J}).$$

Thus for every $\xi < \lambda$ there exists B_ξ such that $(\forall A \in \mathcal{A}_\xi)(\exists t \in G)\, A \subseteq B_\xi + t$. But then every set $A \in \mathcal{A}$ can be covered by some translation of a set $B = \bigcup_{\xi<\lambda} B_\xi \in \mathcal{J}$, which leads to a contradiction.

Proofs of (b) and (c) are analogous. □

Finally, we show some interactions between a translatibility number and the diagram from Theorem 2.1.

Proposition 2.7. $\tau(\mathcal{J}) \leq \mathrm{add}_t^*(\mathcal{J}) \leq \max\{\tau(\mathcal{J}), \mathrm{cof}_t(\mathcal{J})\}$.

Proof. First inequality is an immediate consequence of the definitons. To prove the other one, let us assume that $\mathcal{B}$ is a transitive base and $\tau(\mathcal{J}) = \kappa$, i.e.

$$(\exists A \in \mathcal{J})(\forall B \in \mathcal{J})(\exists T_B \in [G]^\kappa)(\forall g \in G)\, A + T \not\subseteq B + g.$$

We define $T = \bigcup_{B\in\mathcal{B}} T_B$. It is not difficult to show that $A + T \notin \mathcal{J}$, which ends the proof. □

Corollary 2.8. $\mathrm{add}_t^*(\mathcal{J}) = \tau(\mathcal{J})$ *or* $\mathrm{add}_t^*(\mathcal{J}) \leq \mathrm{cof}_t(\mathcal{J})$. □

3. Transitive cardinal coefficients of $\mathbb{S}_2$

In this section we compute transitive cardinal coefficients for the $\sigma-$ideal $\mathbb{S}_2$.

Theorem 3.1. $\mathrm{add}_t^*(\mathbb{S}_2) = \mathrm{non}(\mathbb{S}_2)$

Proof. To prove that $\mathrm{add}_t^*(\mathbb{S}_2) \leq \mathrm{non}(\mathbb{S}_2)$ it is enough to observe that for every set $T \subseteq 2^\omega$ such that $T \notin \mathbb{S}_2$ we have $|T| \geq \mathrm{add}_t^*(\mathbb{S}_2)$ because $\{0\} + T = T \notin \mathbb{S}_2$ and, of course, $\{0\} \in \mathbb{S}_2$.

Suppose now that $T \subseteq 2^\omega$ and $A \in \mathbb{S}_2$. To finish the proof we show that if $T \in \mathbb{S}_2$ then $A + T \in \mathbb{S}_2$. Without loss of generality we can assume that $A = \bigcup_{i<\omega}[f_i]$, where the family $\{f_i : i < \omega\} \subseteq Pif$ is normal. Thus

$$A + T = \bigcup_{t\in T} A + t = \bigcup_{t\in T}\bigcup_{i<\omega}([f_i] + t) = \bigcup_{i<\omega}\bigcup_{t\in T}[f_i + t\restriction \mathrm{dom}(f_i)]$$

Fix $i < \omega$. Let $\iota : \mathrm{dom}(f_i) \to \omega$ be an isomorphism. It induces an isomorphism $\hat{\iota} : 2^{\mathrm{dom}(f_i)} \to 2^\omega$. The image of the set $\{f_i + t\restriction \mathrm{dom}(f_i) : t \in T\} \subseteq 2^{\mathrm{dom}(f_i)}$ by $\hat{\iota}$ has cardinality strictly smaller than $\mathrm{non}(\mathbb{S}_2)$. Consequently, it can be covered by a set $\bigcup_{j<\omega}[g_j]$, for some $\{g_j : j < \omega\} \subseteq Pif$. Hence

$$\bigcup_{t\in T}[f_i + t\restriction \mathrm{dom}(f_i)] \subseteq \bigcup_{j<\omega}[\hat{\iota}^{-1}(g_j)] \in \mathbb{S}_2,$$

which ends the proof. □

Theorem 3.2. $\mathrm{cov}_t(\mathbb{S}_2) = \mathfrak{c}$

Proof. It is obvious that $\mathrm{cov}_t(\mathbb{S}_2) \leq \mathfrak{c}$, so it is enough to show the other inequality. Let $T \subseteq 2^\omega$ and $A \in \mathbb{S}_2$. We can assume as in the proof of Theorem 3.1 that

$$A + T = \bigcup_{i<\omega}\bigcup_{t\in T}[f_i + t\restriction \mathrm{dom}(f_i)],$$

where $f_i \in Pif$ form a normal family.
If $|T| < \mathfrak{c}$ then for every $i < \omega$ there exist a function $g_i : \mathrm{dom}(f_i) \to 2$ which is different from every function $f_i + t \restriction \mathrm{dom}(f_i)$, where $t \in T$. Because the family $\{f_i : i < \omega\}$ is normal then there exists a function $x \in 2^\omega$ such that $\bigcup_{i<\omega} g_i \subseteq x$ and we have $x \notin (A + T)$ which ends the proof. □

For a set $X \in [\omega]^\omega$ let $(X)^\omega_\omega$ denotes the family of all infinite partitions of X into infinite parts. Let $\mathcal{R}$ be a family of partitions from $(\omega)^\omega_\omega$. We say that $\mathcal{R}$ has a property $(*)_1$ if

$$(\forall P \in (\omega)^\omega_\omega)(\exists R \in \mathcal{R})(\forall p \in P)(\exists r \in R)\, r \subseteq p\}$$

and a property $(*)_2$ if

$$(\forall P \in (\omega)^\omega_\omega)(\exists R \in \mathcal{R})(\exists r \in R)(\forall p \in P)\, p \not\subseteq r\}.$$

We introduce new cardinal numbers connected with these properties:

$$\lambda_1 = \min\{|\mathcal{R}| : \mathcal{R} \subseteq (\omega)^\omega_\omega \;\&\; \mathcal{R}\; has\; (*)_1\}$$
$$\lambda_2 = \min\{|\mathcal{R}| : \mathcal{R} \subseteq (\omega)^\omega_\omega \;\&\; \mathcal{R}\; has\; (*)_2\}.$$

The following theorem justifies the introduction of λ_1 and λ_2.

Theorem 3.3. $\mathrm{cof}_t(\mathbb{S}_2) = \lambda_1, \quad \mathrm{add}_t(\mathbb{S}_2) = \lambda_2.$

Proof. We prove only the first part of this theorem. The proof of the second part is analogous.

Let $\mathcal{R} \subseteq (\omega)^\omega_\omega$ be a family of partitions having $(*)_1$. Let

$$\mathcal{R}^* = \{\bigcup_{r\in R}[\mathbf{0}_r] : R \in \mathcal{R}\} \subseteq \mathbb{S}_2$$

where $\mathbf{0}_r$ denotes a function constantly equal to 0 on its domain, which is the set r. Let $A \in \mathbb{S}_2$. We can assume that $A = \bigcup_{i<\omega}[f_i]$, where $\{f_i : i < \omega\} \subseteq Pif$ and $\{\mathrm{dom}(f_i) : i < \omega\} \in (\omega)^\omega_\omega$. From the definition of $\mathcal{R}$ we know that there exist $R \in \mathcal{R}$ such that if i is a natural number then $r_i \subseteq \mathrm{dom}(f_i)$ for some $r_i \in R$. We define a function $x \in 2^\omega$ in the following way:

$x(n) = f_i(n)$ if $n \in r_i$ for some $i < \omega$

$x(n) = 0$ elsewhere

It is a routine to check that

$$A \subseteq \bigcup_{r\in R}[\mathbf{0}_r] + x.$$

Hence the family $\mathcal{R}^*$ is a transitive base for the ideal $\mathbb{S}_2$ and, consequently, $\mathrm{cof}_t(\mathbb{S}_2) \leq \lambda_1$.

Now let $\mathcal{B} \subseteq \mathbb{S}_2$ be a transitive base for $\mathbb{S}_2$. We can assume that if $B \in \mathcal{B}$ then $B = \bigcup_{i<\omega}[f_i]$ where $\{f_i : i < \omega\}$ is a normal family from Pif and $\{\mathrm{dom}(f_i) : i < \omega\} \in (\omega)^\omega_\omega$. Let $P \in (\omega)^\omega_\omega$ and let $A = \bigcup_{p\in P}[\mathbf{0}_p] \in \mathbb{S}_2$. Then

$$A \subseteq \bigcup_{i<\omega}[f_i] + f = \bigcup_{i<\omega}[f_i + f \restriction \mathrm{dom}(f_i)]$$

for some $\bigcup_{i<\omega}[f_i] \in \mathcal{B}$ and $f \in 2^\omega$. By the Lemma 1.2 we obtain that for every $p \in P$ there exists a natural number i_p such that $[\mathbf{0}_p] \subseteq [f_{i_p} + f\restriction \mathrm{dom}(f_{i_p})]$. Thus $\mathrm{dom}(f_{i_p}) \subseteq p$. Hence the family

$$\mathcal{R} = \{\{\mathrm{dom}(f_i) : i < \omega\} : \bigcup_{i<\omega}[f_i] \in \mathcal{B}\} \subseteq (\omega)^\omega_\omega$$

has the property $(*)_1$. It is easy to check that $\lambda_1 \leq |\mathcal{R}| \leq |\mathcal{B}|$ which ends the proof. □

To reach the final result we need a simple lemma:

Lemma 3.4. *There exists a family $\mathcal{P} \subseteq (\omega)^\omega_\omega$ of cardinality $\mathfrak{c}$ such that for every two partitions $P_1, P_2 \in \mathcal{R}$ if $p_1 \in P_1$ and $p_2 \in P_2$ then $p_1 \cap p_2$ is finite.*

Proof. We deal with partitions of $\mathbb{Z} \times \mathbb{Z}$ except for partitions of ω. Let $p^\alpha_i = \{(z_1, z_2) \in \mathbb{Z} \times \mathbb{Z} : i \leq z_2 - \alpha z_1 < i + 1\}$ for $i \in \mathbb{Z}$ and $\alpha \geq 0$. Then $P^\alpha = \{p^\alpha_i : i \in \mathbb{Z}\}$ is a partition from $(\mathbb{Z} \times \mathbb{Z})^\omega_\omega$. It is not difficult to check that a family $\mathcal{P} = \{P^\alpha : \alpha \geq 0\}$ has a needed property. □

Theorem 3.5. $\mathrm{cof}_t(\mathbb{S}_2) = \mathfrak{c}, \quad \mathrm{add}_t(\mathbb{S}_2) = \omega_1$.

Proof. Directly from the definition and Proposition 1.2(a) we obtain $\mathrm{cof}_t(\mathbb{S}_2) \leq \mathrm{cof}(\mathbb{S}_2) \leq \mathfrak{c}$ and $\omega_1 \ \leq \mathrm{add}(\mathbb{S}_2)\,\mathrm{add}_t(\mathbb{S}_2)$, so it is enough to show that $\lambda_1 \geq \mathfrak{c}$ and $\lambda_2 \leq \omega_1$.

Let $\mathcal{P} \subseteq (\omega)^\omega_\omega$ be a family the existence of which we proved in the previous lemma. Let $\mathcal{R} \subseteq (\omega)^\omega_\omega$ be a family with the $(*)_1$ property. For a given $R \in \mathcal{R}$ we define $\mathcal{P}_R = \{P \in \mathcal{P} : (\forall p \in P)(\exists r \in R) r \subseteq p\}$. Obviously $\mathcal{P} = \bigcup_{R\in\mathcal{R}} \mathcal{P}_R$. Moreover, every family $\mathcal{P}_R$ is at most countable because any element of R cannot be contained in elements of different partitions from $\mathcal{P}_R$. Therefore

$$\mathfrak{c} \leq |\mathcal{P}| \leq \omega \cdot |\mathcal{R}|.$$

On the other hand, if we consider a family $\mathcal{R}$ to be any subfamily of $\mathcal{P}$ of size ω_1 then $\mathcal{R}$ has the property $(*)_2$. Actually, if there exists a partition $P \in (\omega)^\omega_\omega$ such that for every $R \in \mathcal{R}$ and every $r \in R$ we have an element $p \in P$ such that $p \subseteq r$ then we get a contradiction as for different $R_1, R_2 \in \mathcal{R}$ and $r_1 \in R_1, r_2 \in R_2$ there is no $p \in P$ which simultaneously contained in r_1 and r_2. □

As a matter of fact, the proof of Theorem 3.5 could be shortened if we observe that the family $\mathcal{P}$ defined in Lemma 3.4 is a $(\mathfrak{c}, \omega_1)$-Lusin set for a certain relation (see [14] for more disscusion).

Finally, we compute the transatibility number of $\mathbb{S}_2$.

Theorem 3.6. $\tau(\mathbb{S}_2) = \omega_1$.

Proof. To begin with, we show that $\mathbb{S}_2$ is ω–translatable. Let $A \in \mathbb{S}_2$ be arbitrary. As usual, without loss of generality we can assume that $A = \bigcup_{i<\omega}[f_i]$, where $\{f_i : i < \omega\} \subseteq Pif$ and $\{\mathrm{dom}(f_i) : i < \omega\} \in (\omega)^\omega_\omega$. For every $i < \omega$ let us fix a partition $P_i = \{p_{ij} : j < \omega\} \in (\mathrm{dom}(f_i))^\omega_\omega$. Then $\{p_{ij} : i, j < \omega\} \in (\omega)^\omega_\omega$. We define

$$B = \bigcup_{i<\omega} \bigcup_{j<\omega} [\mathbf{0}_{p_{ij}}].$$

Obviously, $B \in \mathbb{S}_2$. For every $T = \{t_j : j < \omega\} \in [2^\omega]^\omega$ we define $g \in 2^\omega$ as follows:

$$(\forall i, j < \omega)\, g \restriction p_{ij} = (f_i + t_j) \restriction p_{ij}.$$

It is a routine calculation to show that $A + T \subseteq B + g$.

To show the other inequality, let us consider first a partition P of ω into infinite parts. We can observe that there exists a set $T_P \in [2^\omega]^{\omega_1}$ such that for every family $\{h_i : i < \omega\} \subseteq Pif$ if $\{\mathrm{dom}(h_i) : i < \omega\} = P$ then $T \not\subseteq \bigcup_{i<\omega}[h_i]$. Namely, it is enough to take T such that $(\forall p \in P)(\forall x, y \in T)(x \neq y \Rightarrow x \restriction p \neq y \restriction p)$.

Let us assume that $\mathbb{S}_2$ is ω_1–translatable and let us fix any $A \in \mathbb{S}_2$ of the form as in the first part of the proof. Hence there exists $B \in \mathbb{S}_2$ (we can assume again that $B = \bigcup_{j<\omega}[h_j]$, where $\{h_j : j < \omega\} \subseteq Pif$ and $\{\mathrm{dom}(h_j) : j < \omega\} \in (\omega)^\omega_\omega$) such that for every $T \in [2^\omega]^{\omega_1}$ there exists $g \in 2^\omega$ such that $A + T \subseteq B + g$. But then from Lemma 1.3 we obtain that $(\forall j < \omega)(\exists i < \omega)\ \mathrm{dom}(h_j) \subseteq \mathrm{dom}(f_i)$ and, consequently, for every $i < \omega$ we have $Z_i \in [\omega]^\omega$ such that $\{\mathrm{dom}(h_j) : j \in Z_i\}$ is a partition of $\mathrm{dom}(f_i)$. Thus for every $i < \omega$ we have

$$[f_i] + T \subseteq \bigcup_{j \in Z_i} [h_j] + g. \qquad (*)$$

Now, identyfying 2^ω with $2^{\mathrm{dom}(f_i)}$ as in the proof of Theorem 3.1, we obtain for every $i < \omega$ a set $T_i = T_{\{\mathrm{dom}(h_j) : j \in Z_i\}}$. Let $T_i = \{t^i_\alpha : \alpha < \omega_1\}$. Then the set

$$T = \{\bigcup_{i<\omega} t^i_\alpha : \alpha < \omega_1\}$$

contradicts the condition $(*)$, which ends the proof. □

4. Cofinality versus transitive covering

In this section we show that transitive covering of an ideal may be totally different from its cofinality.

Theorem 4.1. *Let λ be a cardinal number of an uncountable cofinality and let $\langle G_\alpha : \alpha < \lambda\rangle$ be an increasing sequence of subgroups of a group G such that $G = \bigcup_{\alpha<\lambda} G_\alpha$. If $\mathcal{J}$ is a $\sigma-$ideal of subsets of G generated by a family $\{G_\alpha : \alpha < \lambda\}$ then* $\mathrm{cof}(\mathcal{J}) = \mathrm{cf}(\lambda)$ *and*

$$\mathrm{cov}_t(\mathcal{J}) = \inf\{|\,{}^G/_{G_\alpha}| : \alpha < \lambda\}.$$

Proof. Straight from the fact, that the sequence $\langle G_\alpha : \alpha < \lambda\rangle$ is increasing we can deduce that

$$\mathcal{J} = \{A \subseteq G : (\exists \xi < \lambda)\, A \subseteq G_\xi\}.$$

It is a simple observation that $\mathcal{J}$ is a translation invariant, symmetric $\sigma-$ideal containing sigletons. It is also proper as the cofinality of λ is uncountable.

Let us fix a given sequence of cardinal numbers $\langle \xi_\alpha : \alpha < \mathrm{cf}(\lambda)\rangle$, cofinal in λ. Then the family $\{G_{\xi_\alpha} : \alpha < \mathrm{cf}(\lambda)\}$ is a base for $\mathcal{J}$. Moreover, no family of elements of $\mathcal{J}$ of cardinality strictly less than $\mathrm{cf}(\lambda)$ can be a base of $\mathcal{J}$ as all elements of such a family are contained in G_ξ for some $\xi < \lambda$. Hence $\mathrm{cof}(\mathcal{J}) = \mathrm{cf}(\lambda)$.

Let us observe that a sequence of cardinal numbers $\langle|\,{}^G/_{G_\alpha}| : \alpha < \lambda\rangle$ is decreasing so there exists $\zeta < \lambda$ such that $|\,{}^G/_{G_\alpha}| = |\,{}^G/_{G_\zeta}|$ for $\alpha \geq \zeta$. Let us consider now a set $T \subseteq G$ such that $|T| = \mathrm{cov}_t(\mathcal{J})$ and there exists $A \in \mathcal{J}$ such that $A + T = G$. Without loss of generality we may assume that $A = G_\xi$ for some $\zeta \leq \xi < \lambda$. Then we may get $T' \subseteq T$ such that $(\forall t \in T')\, T' \cap (G_\xi + t) = \{t\}$ and $G_\xi + T' = G$. Thus $\mathrm{cov}_t(\mathcal{J}) = |T'| = |\,{}^G/_{G_\xi}|$ and, consequently,

$$\mathrm{cov}_t(\mathcal{J}) = |\,{}^G/_{G_\xi}| = |\,{}^G/_{G_\zeta}| = \inf\{|\,{}^G/_{G_\alpha}| : \alpha < \lambda\},$$

which ends the proof. □

As an application of Theorem 4.1 we construct a $\sigma-$ideal, the transitive covering of which is in general radically bigger than its cofinality. First, we introduce some necessary notation.

A set $\mathcal{H} \subseteq \mathbb{R}$ is called *a Hamel basis* if it is a basis of $(\mathbb{R}, +)$ treated as a linear space over a field $\mathbb{Q}$ of rational numbers.

From now on let us fix a Hamel basis $\mathcal{H}$ and its enumeration $\mathcal{H} = \{h_\alpha : \alpha < \mathfrak{c}\}$. Then every real number x has the unique representation in this basis, i.e.

$$(\forall x \in \mathbb{R})(\exists!\, r_x \in \mathbb{Q}^{\mathfrak{c}})(|\operatorname{supp}(r_x)| < \omega \ \ \& \ \ x = \sum_{\alpha<\mathfrak{c}} r_x(\alpha) h_\alpha),$$

where $\operatorname{supp}(r_x) = \{\alpha : r_x(\alpha) \neq 0\}$. In order to simplify the notation we replace $\operatorname{supp}(r_x)$ by $\operatorname{supp}(x)$.

Definition. Let $\{P_\xi : \xi < \omega_1\}$ be a fixed partition of $\mathfrak{c}$ into parts of cardinality $\mathfrak{c}$. Let A be any set. We say that a function $f \in \mathbb{R}^A$ is *Hamel-bounded* if

$$(\exists \xi < \omega_1)(\forall a \in A)(\operatorname{supp}(f(a)) \subseteq \bigcup_{\beta<\xi} P_\beta).$$

Then we put $HB(A) = \{f \in \mathbb{R}^A : f \ is \ Hamel - bounded\}$. One can check that $HB(A)$ is a subgroup of $\mathbb{R}^A$ with a standard addition of functions.
For any function $f \in HB(A)$ its *Hamel-bound* $hb(f)$ is defined as follows:

$$hb(f) = \min\{\xi < \omega_1 : (\forall a \in A)(\operatorname{supp}(f(a)) \subseteq \bigcup_{\beta<\xi} P_\beta)\}.$$

Let κ be an infinite cardinal number. Let $B_\xi = \{f \in HB(\kappa) : hb(f) \leq \xi\}$. Of course, $\langle B_\xi : \xi < \omega_1\rangle$ is an increasing sequence of subgroups of a group $HB(\kappa)$ and $HB(\kappa) = \bigcup_{\xi<\omega_1} B_\xi$. we define $T(\kappa)$ as a $\sigma-$ideal generated by the family $\{B_\xi : \xi < \omega_1\}$.

Lemma 4.2. $|{}^{HB(\kappa)}/_{B_\xi}| = 2^\kappa$ *for every* $\xi < \omega_1$.

Proof. Let us fix B_ξ for some $\xi < \omega_1$. We consider a set $T \subseteq HB(\kappa)$ such that $(\forall t \in T)\, T \cap (B_\xi + t) = \{t\}$ and $B_\xi + T = HB(\kappa)$.

Let us fix $P \subseteq \kappa$ and a real number x such that $x \in \mathcal{H} \setminus \{h_\alpha : \alpha \in \bigcup_{\beta<\xi} P_\beta\}$. We define a function $f_P \in HB(\kappa)$ as follows:

$$f_P(\alpha) = \chi_P(\alpha) \cdot x$$

where χ_P denotes the characteristic function of a set P. Then there exists $t_P \in T$ and $g \in b_\xi$ such that $f_P = g + t_P$. In particular, for each $\alpha \in P$ we have

$$x = f_P(\alpha) = g(\alpha) + t_P(\alpha).$$

But we know from the assumption that $\operatorname{supp}(x) \not\subseteq \bigcup_{\beta<\xi} P_\beta$, so we have $\operatorname{supp}(x) \subseteq \operatorname{supp}(t_P(\alpha))$ for each $\alpha \in P$. On the other hand, if $\alpha \notin P$ then $f_P(\alpha) = 0$ and, consequently, $\operatorname{supp}(t_P(\alpha)) = \operatorname{supp}(g(\alpha)) \subseteq \bigcup_{\beta<\xi} P_\beta$ for such α's.

Let P_1 and P_2 be two different subsets of κ and $\alpha \in P_1 \triangle P_2$. Suppose that $t_{P_1} = t_{P_2} = t$. Then

$$\operatorname{supp}(x) \subseteq \operatorname{supp}(t(\alpha)) \subseteq \bigcup_{\beta<\xi} P_\beta,$$

which is a contradition. Hence $t_{P_1} \neq t_{P_2}$ and, consequently,

$$|{}^{HB(\kappa)}/_{B_\xi}| = |T| \geq |\mathcal{P}(\kappa)| = 2^\kappa,$$

which ends the proof, as $|HB(\kappa)| = 2^\kappa$. □

Corollary 4.3. *For every infinite cardinal number κ we have* $\mathrm{cof}(T(\kappa)) = \omega_1$ *and* $\mathrm{cov}_t(T(\kappa)) = 2^\kappa$.

Proof. It is enough to apply Theorem 4.1 for $\lambda = \omega_1$, $G = HB(\kappa)$, $G_\xi = B_\xi$ and $\mathcal{J} = T(\kappa)$. $\square$

5. Possible diagrams

After reading Paragraph 2 it occurs a natural question whether there are any other relations between coefficients in the diagram form Theorem 2.1. One of possible ways of solving this problem (negatively) is to put ω_1 and $\omega_2 = \mathfrak{c}$ in nodes of the diagram and try to find a σ–ideal $\mathcal{J}$ and a model for every legal configuration.

There are 23 legal configurations (taking under consideration Theorem 2.1 and Propositions 2.2 and 2.3). In this section we shall present models for 12 of them. From now on, $\circ$ will stand for ω_1 and $\bullet$ will stand for $\omega_2 = \mathfrak{c}$.

Theorem 5.1. *If $G = 2^\omega$ and $\mathcal{J} = \mathbb{S}_2$ then there are the following possibilities for the diagram from Theorem 2.1.*

(a)

$$\begin{tikzcd}
 & \circ \arrow[r] & \bullet \arrow[dr] & \\
\circ \arrow[ur] \arrow[r] \arrow[dr] & \bullet \arrow[rr] \arrow[dr] & & \bullet \\
 & \bullet \arrow[r] \arrow[dr] & \bullet & \\
 & & \bullet \arrow[uur] &
\end{tikzcd}$$

(b)

$$\begin{tikzcd}
 & \circ \arrow[r] & \bullet \arrow[dr] & \\
\circ \arrow[ur] \arrow[r] \arrow[dr] & \bullet \arrow[rr] \arrow[dr] & & \bullet \\
 & \circ \arrow[r] \arrow[dr] & \bullet & \\
 & & \circ \arrow[uur] &
\end{tikzcd}$$

(c)

$$\begin{tikzcd}
 & \circ \arrow[r] & \bullet \arrow[dr] & \\
\circ \arrow[ur] \arrow[r] \arrow[dr] & \circ \arrow[rr] \arrow[dr] & & \bullet \\
 & \bullet \arrow[r] \arrow[dr] & \bullet & \\
 & & \bullet \arrow[uur] &
\end{tikzcd}$$

(d)

$$\begin{tikzcd}
 & \circ \arrow[r] & \bullet \arrow[dr] & \\
\circ \arrow[ur] \arrow[r] \arrow[dr] & \circ \arrow[rr] \arrow[dr] & & \bullet \\
 & \circ \arrow[r] \arrow[dr] & \bullet & \\
 & & \circ \arrow[uur] &
\end{tikzcd}$$

Proof. Fact 1.4 and Theorems 3.1, 3.2 and 3.5 give us a complete description of a diagram from Theorem 2.1 for $\mathcal{J} = \mathbb{S}_2$.

To obtain the diagram (a) it is enough to assume Martin's Axiom and $\mathfrak{c} = \omega_2$, as it well-known that we have then $\aleph_0\text{-}\mathfrak{s} = \mathfrak{r} = \mathfrak{c}$. For the diagram (b) we need $\aleph_0\text{-}\mathfrak{s} = \omega_1$ and $\mathfrak{r} = \omega_2 = \mathfrak{c}$. This sitiuation takes place in a model obtained by adding ω_2 random reals to a model of CH, as $\aleph_0\text{-}\mathfrak{s} \leq \max\{\mathfrak{b}, \mathfrak{s}\}$ (cf. [12]) and $\mathrm{cov}(\mathcal{N}) \leq \mathfrak{r}$ (cf. [23] or [8]). To obtain the diagram (c) we need $\mathfrak{r} = \omega_1$ and $\aleph_0\text{-}\mathfrak{s} = \omega_2 = \mathfrak{c}$. To get such a model we use a countable support iteration of length ω_2 (over CH) of the Blass-Shelah forcing introduced in [5] (see also [2]). This iteration preserves p-points, so a p-point from the ground model causes $\mathfrak{r} = \omega_1$. On the other hand on every step we add a subset of ω which is either almost contained or almost disjoint with every subset of ω from the ground model. A standard argument shows that

$\mathfrak{s} = \omega_2$ where $\mathfrak{s}$ is a classical splitting number (cf. [23]). But $\mathfrak{s} \leq \aleph_0\text{-}\mathfrak{s}$ which ends the proof. Finally, the diagram (d) holds in the iterated Sacks model, because we have then $\aleph_0\text{-}\mathfrak{s} = \mathfrak{r} = \omega_1 < \mathfrak{c} = \omega_2$. □

Theorem 5.2. *If $G = \mathbb{R}$ and $\mathcal{J} = \mathcal{M}$ then there are the following possibilities for the diagram from Theorem 2.1.*

(a)

$$\begin{tikzcd}
& \bullet \arrow[r] & \bullet \arrow[dr] & \\
\bullet \arrow[ur] \arrow[r] \arrow[dr] & \bullet \arrow[rr] \arrow[dr] & & \bullet \\
& \bullet \arrow[r] \arrow[dr] & \bullet & \\
& & \bullet \arrow[uur] &
\end{tikzcd}$$

(b)

$$\begin{tikzcd}
& \bullet \arrow[r] & \bullet \arrow[dr] & \\
\circ \arrow[ur] \arrow[r] \arrow[dr] & \circ \arrow[rr] \arrow[dr] & & \bullet \\
& \circ \arrow[r] \arrow[dr] & \circ & \\
& & \bullet \arrow[uur] &
\end{tikzcd}$$

(c)

$$\begin{tikzcd}
& \circ \arrow[r] & \bullet \arrow[dr] & \\
\circ \arrow[ur] \arrow[r] \arrow[dr] & \circ \arrow[rr] \arrow[dr] & & \bullet \\
& \circ \arrow[r] \arrow[dr] & \circ & \\
& & \circ \arrow[uur] &
\end{tikzcd}$$

(d)

$$\begin{tikzcd}
& \circ \arrow[r] & \circ \arrow[dr] & \\
\circ \arrow[ur] \arrow[r] \arrow[dr] & \circ \arrow[rr] \arrow[dr] & & \bullet \\
& \circ \arrow[r] \arrow[dr] & \circ & \\
& & \bullet \arrow[uur] &
\end{tikzcd}$$

Proof. We will use well-known Cichoń's diagram (cf. [2]), Theorem 0.1 and Pawlikowski's results: $\mathrm{add}_t(\mathcal{M}) = \mathfrak{b}$ and $\mathrm{cof}_t(\mathcal{M}) = \mathfrak{d}$ (cf. [19]).

The diagram (a) holds assuming Martin's Axiom and $\mathfrak{c} = \omega_2$. For the diagram (b) we need $\mathrm{non}(\mathcal{N}) = \omega_1$ and $\mathfrak{b} = \omega_2 = \mathfrak{c}$ which is true after adding ω_1 random reals to the model of $\mathrm{MA} + \mathfrak{c} = \omega_2$. To obtain the diagram (c) we need $\mathrm{non}(\mathcal{M}) = \mathrm{non}(\mathcal{N})\omega_1$ and $\mathfrak{d} = \omega_2 = \mathfrak{c}$ so it is enough to iterate ω_2 times Miller forcing, with countable support (over CH). Finally, the diagram (d) requires $\mathrm{non}(\mathcal{N}) = \mathfrak{d} = \omega_1$ and $\mathrm{non}(\mathcal{M}) = \omega_2 = \mathfrak{c}$ and adding ω_2 random reals to the model of CH will do. □

Theorem 5.3. *If $G = \mathbb{R}$ and $\mathcal{J} = \mathcal{N}$ then there are the following possibilities for the diagram from Theorem 2.1.*

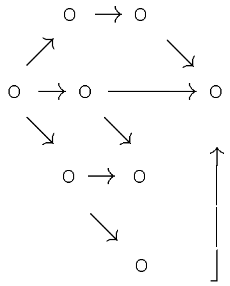
(a)

$$\begin{tikzcd}
& \circ \arrow[r] & \bullet \arrow[dr] & \\
\circ \arrow[ur] \arrow[r] \arrow[dr] & \bullet \arrow[rr] \arrow[dr] & & \bullet \\
& \circ \arrow[r] \arrow[dr] & \bullet & \\
& & \bullet \arrow[uur] &
\end{tikzcd}$$

(b)

$$\begin{tikzcd}
& \circ \arrow[r] & \bullet \arrow[dr] & \\
\circ \arrow[ur] \arrow[r] \arrow[dr] & \circ \arrow[rr] \arrow[dr] & & \bullet \\
& \circ \arrow[r] \arrow[dr] & \circ & \\
& & \bullet \arrow[uur] &
\end{tikzcd}$$

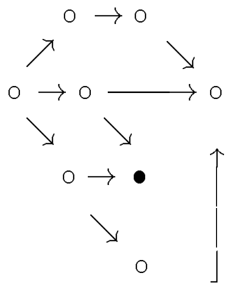
(c)



Proof. Again, we will use Cichoń's diagram, Theorem 0.1 and Pawlikowski's results: $\mathrm{add}_t(\mathcal{N}) = \mathrm{add}(\mathcal{N})$, $\mathrm{cof}_t(\mathcal{N}) = \mathrm{cof}(\mathcal{N})$ and $\mathrm{add}(\mathcal{N}) = \min\{\mathfrak{b}, \mathrm{add}_t^*(\mathcal{N})\}$.

The diagram (a) requires $\mathrm{add}(\mathcal{N}) = \omega_1$ and $\mathfrak{b} = \mathrm{non}(\mathcal{N}) = \mathrm{cov}(\mathcal{N}) = \omega_2 = \mathfrak{c}$. A suitable model can be found in [2] or in [11]. For the diagram (b) we need $\mathrm{non}(\mathcal{M}) = \omega_1$ and $\mathrm{non}(\mathcal{N}) = \omega_2 = \mathfrak{c}$ so adding ω_2 Cohen reals to a model of CH will do. Finally, the diagram (c) holds in iterated Sacks model, as we have $\mathrm{cof}(\mathcal{N}) = \omega_1 < \mathfrak{c} = \omega_2$. □

Theorem 5.4. *If $G = HB(\omega)$ and $\mathcal{J} = T(\omega)$ then there is the following possibility for the diagram from Theorem 2.1.*



Proof. It is enough to take any model of $\mathfrak{c} = \omega_2$ and apply Corollary 4.3 □

Remark. Of course, some of diagrams mentioned above can be also obtained for other ideals, e.g. diagrams 5.1(b), 5.2(a) or 5.2(c) can be obtained for $\sigma-$ideal of null sets as well.

6. Transitive operations on ideals

In this paragraph we prove some results connected with operations s and g. As we have assumed at the beginning of this paper, $\mathcal{J}$ is a $\sigma-$ideal of subsets of G which is proper, translation invariant, symmetric and contains all singletons.

Fact 6.1. $\mathrm{non}(s(\mathcal{J})) = \mathrm{cov}_t(\mathcal{J})$, $\qquad \mathrm{non}(g(\mathcal{J})) = \mathrm{add}_t^*(\mathcal{J})$.

Proof. Straightforward from definitions. □

Proposition 6.2. $\mathrm{cov}_t(s(\mathcal{J})) \geq \mathrm{non}(\mathcal{J})$.

Proof. Let us consider $T \subseteq G$ such that $A + T = G$ for some $A \in s(\mathcal{J})$. But we know that for every $B \in \mathcal{J}$ we have $A + B \neq G$. Hence $T \notin \mathcal{J}$. □

Proposition 6.3. $\mathrm{add}_t^*(g(\mathcal{J})) = \mathrm{non}(g(\mathcal{J}))$.

Proof. It occurs from Theorem 2.1 and Fact 7.1 that it is enough to show that $\mathrm{add}_t^*(g(\mathcal{J})) \geq \mathrm{add}_t^*(\mathcal{J})$. Let us consider $T \subseteq G$ such that there exists $A \in g(\mathcal{J})$ such that $A + T \notin g(\mathcal{J})$. This means that for some $B \in \mathcal{J}$ we have $(A+T)+B = (A+B)+T \notin \mathcal{J}$. But $A + B \in \mathcal{J}$ which ends the proof. □

Corollary 6.4. $\mathrm{cov}_t(g(\mathcal{J})) \geq \max\{\mathrm{non}(\mathcal{J}), \mathrm{cov}_t(\mathcal{J})\}$.

Proof. Straightforward from definitions, Proposition 1.1(c) and Proposition 6.2. □

In the next part of the paragraph we show that $\sigma-$ideals $\mathbb{S}_2$ and $\mathbb{B}_2$ are closely related to each other.

Theorem 6.5. $s(\mathbb{S}_2) = \mathbb{B}_2$.

Proof. Let us consider any $A \subseteq 2^\omega$. It is a standard calculation which shows that if for some $X \in [\omega]^\omega$ we have $A \restriction X = 2^X$ then $A + [\mathbf{0}_X] = 2^\omega$. Hence if $A \notin \mathbb{B}_2$ then $A \notin s(\mathbb{S}_2)$.

On the other hand, let us consider any $C \subseteq 2^\omega$ such that $B + C = 2^\omega$ for some $B \in \mathbb{S}_2$. As in proofs in Paragraph 3, without loss of generality we can assume that $B = \bigcup_{i<\omega}[f_i]$, where $\{f_i : i < \omega\} \subseteq Pif$ and $\{\mathrm{dom}(f_i) : i < \omega\} \in (\omega)^\omega_\omega$. It occurs that there exists $i < \omega$ such that $C \restriction \mathrm{dom}(f_i) = 2^{\mathrm{dom}(f_i)}$. Actually, if we suppose that for all $i < \omega$ there exists $g_i \in 2^{\mathrm{dom}(f_i)} \setminus C \restriction \mathrm{dom}(f_i)$ then we have $\bigcup_{i<\omega}(f_i + g_i) \in 2^\omega \setminus B + C$. Thus if $C \notin s(\mathbb{S}_2)$ then $C \notin \mathbb{B}_2$ which completes the proof. □

In [9] the authors showed that the covering number of $\mathbb{B}_2$ is a weird object and it is difficult to find reasonable estimations for it. In particular, it is relatively consistent that Martin's Axiom holds, $\mathfrak{c} = \omega_2$ and $\mathrm{cov}(\mathbb{B}_2) = \omega_1$. The following corollary shows that the situation for the transitive covering number of $\mathbb{B}_2$ is different.

Corollary 6.6. *If Martin's Axiom holds then* $\mathrm{cov}_t(\mathbb{B}_2) = \mathfrak{c}$.

Proof. From Theorem 6.5 and Proposition 6.2 we obtain that $\mathrm{cov}_t(\mathbb{B}_2) = \mathrm{non}(\mathbb{S}_2)$. It was proved in [8] that $\mathrm{non}(\mathbb{S}_2) = \aleph_0\text{-}\mathfrak{s}$ and it is well-known that under Martin's Axiom we have $\aleph_0\text{-}\mathfrak{s} = \mathfrak{c}$. □

It is a natural question to ask what we know about $g(\mathbb{S}_2)$. The next theorem partially answers it.

Theorem 6.7. $g(\mathbb{S}_2) = p(\mathbb{S}_2)$.

Proof. Let us assume that $A \in g(\mathbb{S}_2)$ that is $(\forall B \in \mathbb{S}_2) A + B \in \mathbb{S}_2$. It is not difficult to observe that this condition is equivalent to $(\forall T \in [\omega]^\omega)[\mathbf{0}_T] + A \in \mathbb{S}_2$. But we can prove that if $\varphi \in Inj(\omega,\omega)$ then $[\mathbf{0}_{\mathrm{dom}(\varphi)}] + A = (\varphi * A)_\varphi$. Hence, reformulating our condition we obtain $(\forall \varphi \in Inj(\omega,\omega))\,(\varphi * A)_\varphi \in \mathbb{S}_2$. Thus, as $\mathbb{S}_2$ is productive and has WFP, we show that this fact is equivalent to $(\forall \varphi \in Inj(\omega,\omega))\,\varphi * A \in \mathbb{S}_2$ and, consequently, to $A \in p(\mathbb{S}_2)$. □

Finally, we will show that all operations that appeared in this paragraph are versions of one operation, defined in [22].

Let $\mathcal{A}, \mathcal{B}$ be translation invariant families of subsets of a group G. We put

$$\mathcal{G}_t(\mathcal{A}, \mathcal{B}) = \{A \subseteq G : (\forall B \in \mathcal{B})\, A + B \in \mathcal{A}\}.$$

The following proposition takes place.

Proposition 6.8. *Let $\mathcal{J}$ be a translation invariant, symmetric $\sigma-$ideal of subsets of a group G. Then*
(a) $s(\mathcal{J}) = \mathcal{G}_t(\mathcal{P}(G) \setminus \{G\}, \mathcal{J})$,
(b) $g(\mathcal{J}) = \mathcal{G}_t(\mathcal{J}, \mathcal{J})$.
If $G = 2^\omega$ and $\mathcal{J}$ is productive and has WFP then
(c) $p(\mathcal{J}) = \mathcal{G}_t(\mathcal{J}, \mathbb{S}_2)$.

Proof. (a) and (b) are reformulations of definitions and was observed in [22]. To prove (c) it is enough to repeat carefully the proof of Theorem 6.7 $\square$

7. Transitive cardinal coefficients of ideals on 2^κ

From now on we assume that $\mathcal{J}$ is a proper and productive $\sigma-$ideal of subsets of 2^ω containing all singletons (i.e. $\bigcup \mathcal{J} = 2^\omega$) and that $\kappa \geq \omega_1$. We investigate relations between transitive cardinal coefficients of $\mathcal{J}$ and those of $\mathcal{J}_\kappa$. Some of them are parallel to relations between standard cardinal coefficients of $\mathcal{J}$ and $\mathcal{J}_\kappa$ proved in [13].

Theorem 7.1. $\mathrm{add}_t(\mathcal{J}_\kappa) = \omega_1$.

Proof. The proof is actually the same as the one of Theorem 2.1 from [13] which states that $\mathrm{add}(\mathcal{J}) = \omega_1$. $\square$

Theorem 7.2. $\mathrm{cof}_t(\mathcal{J}_\kappa) \leq \max\{\mathrm{cof}([\kappa]^{\leq\omega}), \mathrm{cof}_t(\mathcal{J})\}$. *Moreover, if $\mathcal{J}$ has WFP then* $\mathrm{cof}_t(\mathcal{J}_\kappa) \geq \mathrm{cof}_t(\mathcal{J})$.

Proof. Similarly as above, constructions are analogous to those from Theorems 2.2 and 2.5 from [13] which state that $\mathrm{cof}(\mathcal{J}_\kappa) \leq \max\{\mathrm{cof}([\kappa]^{\leq\omega}), \mathrm{cof}(\mathcal{J})\}$ and if $\mathcal{J}$ has WFP then $\mathrm{cof}(\mathcal{J}_\kappa) \geq \mathrm{cof}(\mathcal{J})$. The only addition is making use of Lemma 1.6(a) and (b). $\square$

Theorem 7.3. *If $\omega_1 \leq \lambda \leq \kappa$ then* $\mathrm{add}^*_t(\mathcal{J}_\lambda) \leq \mathrm{add}^*_t(\mathcal{J}_\kappa)$.

Proof. Let us fix $T \subseteq 2^\kappa$ such that $A + T \notin \mathcal{J}_\kappa$ for some $A \in \mathcal{J}_\kappa$. We take $Phi \in Inj(\lambda, \kappa)$ such that $\mathrm{rng}(\varphi) \subseteq \mathrm{rng}(\Phi)$, where φ is a witness for A. Let us consider $\Phi * A, \Phi * T \subseteq 2^\lambda$. Obviously, $|\varphi * T| \leq |T|$. Furthermore $\psi = \Phi^{-1} \circ \varphi \in Inj(\omega, \lambda)$ is a witness for $\Phi * A \in \mathcal{J}_\lambda$. From Lemma 1.6(d) we know that $\Phi * (A + T) = \Phi * A + \Phi * T$.

Now, if $\chi \in Inj(\omega, \lambda)$ was a witness for $\Phi * A + \Phi * T \in \mathcal{J}_\lambda$ then $\Phi \circ \chi \in Inj(\omega, \kappa)$ would be a witness for $A + T \in \mathcal{J}_\kappa$ which contradicts the assumption. Hence $\Phi * A + \Phi * T \notin \mathcal{J}_\lambda$ which ends the proof. $\square$

Theorem 7.4. *If $\mathcal{J}$ has WFP then* $\mathrm{add}^*_t(\mathcal{J}_\kappa) = \mathrm{add}^*_t(\mathcal{J})$.

Proof. It is enough to prove that $\mathrm{add}^*_t(\mathcal{J}_\kappa) \leq \mathrm{add}^*_t(\mathcal{J})$ and apply Theorem 7.3.

Let us fix $T \subseteq 2^\omega$ such that $A + T \notin \mathcal{J}$ for some $A \in \mathcal{J}$. We define $T' = \{t \in 2^\kappa : t \restriction \omega \in T \;\; \& \;\; t \restriction (\kappa \setminus \omega) \equiv 0\}$. Obviously, $|T'| = |T|$. From Lemma 1.6(a) we have

$$A_{id_\omega} + T' = \bigcup_{t' \in T'} (A + t' \restriction \omega)_{id_\omega} = \bigcup_{t \in T} (A + t)_{id_\omega} = (\bigcup_{t \in T} A + t)_{id_\omega} = (A + T)_{id_\omega}$$

and $A_{id_\omega} \in \mathcal{J}_\kappa$ as $\mathcal{J}$ is productive. But $id_\omega \in Inj(\omega, \kappa)$ and $\mathcal{J}$ has WFP and from Lemma 1.7 we get $(A + T)_{id_\omega} \notin \mathcal{J}_\kappa$, which ends the proof. $\square$

Theorem 7.5. $\mathrm{cov}_t(\mathcal{J}_\kappa) = \mathrm{cov}_t(\mathcal{J})$.

Proof. Let us fix $T \subseteq 2^\omega$ such that $A + T = 2^\omega$ for some $A \in \mathcal{J}$. We define T' in the same way as in the proof of Theorem 7.4. Similarly, we get $A_{id_\omega} + T' = (A+T)_{id_\omega} = (2^\omega)_{id_\omega} = 2^\kappa$ and, consequently, $\mathrm{cov}_t(\mathcal{J}_\kappa) \leq \mathrm{cov}_t(\mathcal{J})$.

To show the other inequality, let us fix $S \subseteq 2^\kappa$ such that $A + S = 2^\kappa$ for some $A \in \mathcal{J}_\kappa$. Let $\varphi \in Inj(\omega, \kappa)$ be a witness of A. We consider $\varphi * S \subseteq 2^\omega$. Then $|\varphi * S| \leq |S|$ and from Lemma 1.6(d)

$$\varphi * A + \varphi * S = \varphi * (A + S) = \varphi * 2^\kappa = 2^\omega.$$

But $\varphi * A \in \mathcal{J}$, which ends the proof. □

Theorem 7.6. $\tau(\mathcal{J}) \leq \tau(\mathcal{J}_\kappa)$.

Proof. Let us assume that $\mathcal{J}$ is ξ-transitive i.e. for every $A \in \mathcal{J}$ there exists $B_A \in \mathcal{J}$ such that for every $S \in [2^\omega]^\xi$ there exists $t_S \in 2^\omega$ such that $A + S \subseteq B_A + t_S$. We show that $\mathcal{J}_\kappa$ is also ξ-transitive.

Let us consider any $A \in \mathcal{J}_\kappa$ and let $\varphi \in Inj(\omega,\kappa)$ be its witness. Then $\varphi * A \in \mathcal{J}$ and we fix $B_{\varphi * A} \in \mathcal{J}$. If $S \in [2^\kappa]^\xi$ then without loss of generality we can assume that $\varphi * S \in [2^\omega]^\xi$ and thus there exists $t_{\varphi * S} \in 2^\omega$ such that $\varphi * A + \varphi * S \subseteq B_{\varphi * A} + t_{\varphi * S}$. Using Lemma 1.6(b) and (d) we obtain

$$A + T \subseteq (\varphi * (A+T))_\varphi \subseteq (B_{\varphi * A} + t_{\varphi * S})_\varphi = (B_{\varphi * A})_\varphi + t$$

for some $t \in 2^\kappa$. Hence $\mathcal{J}_\kappa$ is ξ-transitive as we can take $(B_{\varphi * A})_\varphi$ for B_A and t for t_S from the definition of ξ-transitivity of $\mathcal{J}_\kappa$. □

Theorem 7.7. *If $\mathcal{J}$ has WFP then $\tau(\mathcal{J}_\kappa) = \tau(\mathcal{J})$.*

Proof. It is enough to prove that $\tau(\mathcal{J}_\kappa) \leq \tau(\mathcal{J})$ and apply Theorem 7.6. As in the proof of Theorem 7.6, we assume that $\mathcal{J}_\kappa$ is ξ-transitive and show that $\mathcal{J}$ is also ξ-transitive.

Let us consider any $A \in \mathcal{J}$. Then $A' = A_{id_\omega} \in \mathcal{J}_\kappa$ and we fix $B_{A'} \in \mathcal{J}_\kappa$. If $T \in [2^\omega]^\xi$ then we define $T' \subseteq 2^\kappa$ like in the proof of Theorem 7.4. Obviously, $T' \in [2^\kappa]^\xi$ and there exists appropriate $t_{T'} \in 2^\kappa$ such that $A' + T' \subseteq B_{A'} + t_{T'}$. But $A' + T' = (A+T)_{id_\omega}$. From Lemma 1.6(b) we obtain

$$(A + T + t_{T'} \restriction \omega)_{id_\omega} = (A+T)_{id_\omega} + t_{T'} \subseteq B.$$

Let us define

$$C = \bigcup_{T \in [2^\omega]^\xi} (A + T + t_{T'} \restriction \omega).$$

Then $C \subseteq 2^\omega$ and

$$C_{id_\omega} = \bigcup_{T \in [2^\omega]^\xi} (A + T + t_{T'} \restriction \omega)_{id_\omega} \subseteq B \in \mathcal{J}_\kappa.$$

Thus $C_{id_\omega} \in \mathcal{J}_\kappa$ and from Lemma 1.7 we know that $C \in \mathcal{J}$. To finish the proof it is enough to show that from every $S \in [2^\omega]^\xi$ there exists $t_S \in 2^\omega$ such that $A + S \subseteq C + t_S$.

Let us consider any $S \in [2^\omega]^\xi$ and put $t_S = t_{T'} \restriction \omega$. Then

$$A + S = A + S + t_{T'} \restriction \omega + t_{T'} \restriction \omega \subseteq C + t_{T'} \restriction \omega = C + t_S$$

and we are done. □

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