Math 132 - Week 1
Textbook sections: 1.1-6, 2.1-2.4
Topics covered:

- Overview of course
- Review of complex arithmetic
- Polar and Cartesian forms
- Powers and roots
- Subsets of the complex plane
- Complex limits
- Complex derivatives
- The Cauchy-Riemann equations

Technical details of course

- The assessment is 1% on a quiz, 9% on 9 homework assignments, 20% each on two mid-terms on Feb 4 and Feb 25, and 50% on the final at March 21.
- The quiz is a 10 minute test on lower division material, and will be given in the first tutorial session (i.e. this Thursday). A sample quiz can be found at the class web page.
- The first homework assignment is due in the second tutorial session (i.e. next Thursday). Questions are from the textbook, and can be found on the web or outside my door (MS 5622).
- My office hours are Tu 11-12 and Fri 1-3 in my office (MS 5622).
- More information can be found in the accompanying class handout, or on the class web page at

`http://www.math.ucla.edu/~ tao/132.1.00w`

Overview of course

- In *real analysis*, we study functions $y = f(x)$ which take one real number as input and one real number as output. Examples:

  $$f(x) = x^2 + 2x + 1, \quad f(x) = e^x + \pi \sin(x) - \log(x).$$

- We can do many things with these functions, e.g. differentiation, integration, algebraic manipulation, locating zeroes, expanding as power series, etc.

- In *complex analysis*, we study functions $w = f(z)$ which take one complex number as input and one complex number as output. Examples:

  $$f(z) = z^2 + 2iz + i, \quad f(z) = e^z + \pi i \sin(z) - \log(iz).$$

- In the first half of this course, we will be learning how manipulate complex functions in the usual ways. I.e. we will be differentiating, integrating, and algebraically manipulating these functions, finding their zeroes, writing them out as power series, etc.

- Many of these things will be familiar, but there are some surprises too. Example: if you want to integrate $1/z$ from i to $2i$, the answer may be $\ln(2)$, $\ln(2) + 2\pi i$, or $\ln(2) - 2\pi i$, depending on what route you take from i to $2i$.

- In the second half of the course, we'll develop some beautiful and powerful tools in complex analysis, such as contour integration and residue calculus, which allow us to solve problems which are extremely difficult using standard techniques. In fact, we can even use these tools to solve problems which have no complex numbers in them at all!

- Example: the integral

$$\int_0^\infty \frac{dx}{x^4+1}$$

  is very hard to compute using real-analytic techniques, but can be evaluated using residue calculus to be $\pi/\sqrt{8}$.

- We'll also develop some of the theory of complex functions, such as classification of zeroes and singularities. But most of the emphasis of the course will be on computation.

Cartesian form of complex numbers - review

- Complex numbers can be written in either Cartesian form or polar form.

- A complex number z can be written uniquely in Cartesian form as $z = x + yi$, where x and y are real numbers. (Note: in EE j is sometimes used instead of i).

- If $z = x + yi$, then $x = \mathrm{Re}(z)$ is called the *real part* of z, and $y = \mathrm{Im}(z)$ is called the *imaginary part*. E.g. $\mathrm{Im}(3 - 4i) = -4$.

- Two complex numbers are equal if and only if they have the same real part and the same imaginary part. If $a + bi = c + di$, then $a = c$ and $b = d$.

- Numbers with positive real part and zero imaginary part are called *positive*. Numbers with negative real part and zero imaginary part are called *negative*. Numbers with a non-zero imaginary part are *neither positive or negative!* In general, complex numbers should not be compared against each other, it only causes confusion.

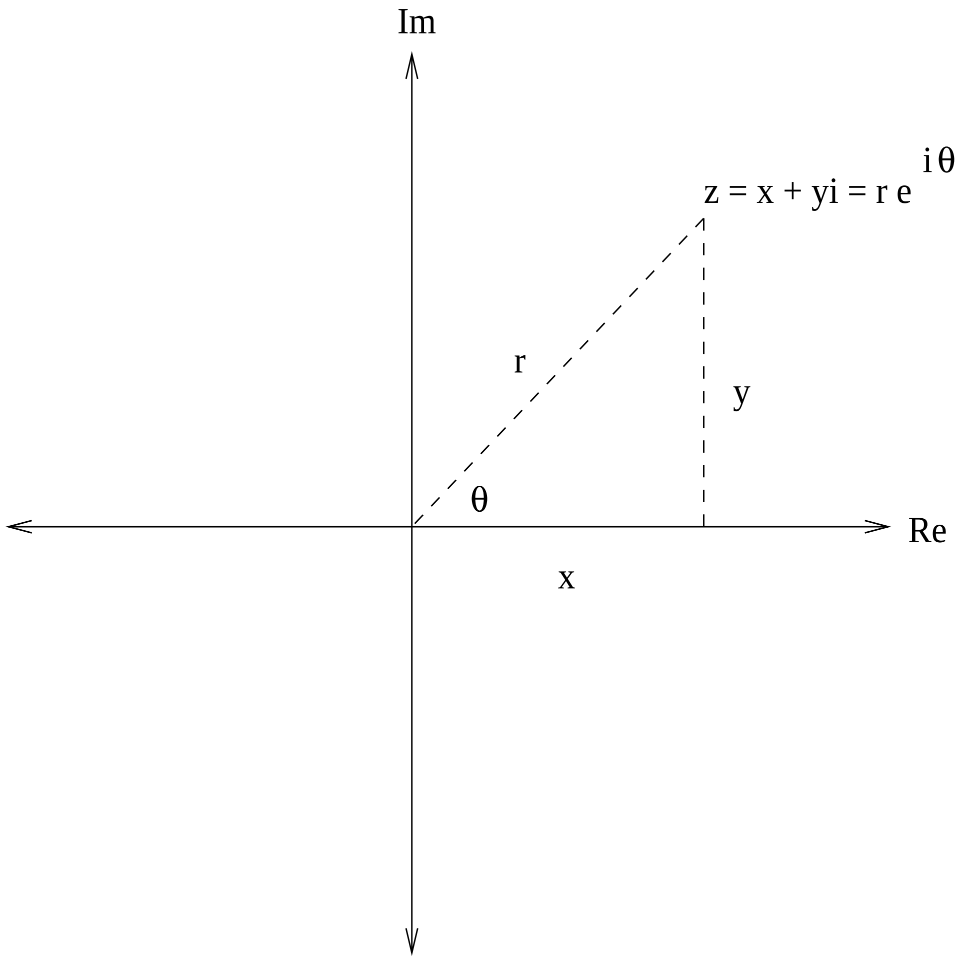
Polar form of complex numbers - review

- A complex number z can be written in polar form as $re^{i\theta}$, where θ is real and r is a non-negative real. For any z, there is only one choice of r, but θ is only determined up to a multiple of 2π.

- The relationship between Cartesian and polar forms is given by Euler's formula

  $$e^{i\theta} = \cos\theta + i\sin\theta.$$

  Thus we have $x = r\cos\theta$ and $y = r\sin\theta$.



- If $z = re^{i\theta}$, then $r = |z|$ is called the *magnitude*, *modulus*, or *absolute value* of z, and θ is called an *argument* or *phase* of z. The set of all possible arguments of z is denoted $\arg(z)$. E.g. $|4+3i| = 5$, and $\arg(4+3i) = \sin^{-1}(\frac{3}{5}) + 2k\pi$. By convention, $\arg(0)$ is undefined.
- If $z = x + yi$, then $r = \sqrt{x^2+y^2}$, $\cos\theta = \frac{x}{r}$, and

$\sin\theta = \frac{y}{r}$.

- If $re^{i\theta} = se^{i\alpha} \neq 0$, then $r = s$ and $\theta = \alpha + 2k\pi$ for some integer k.

- There are infinitely many choices for the argument of a complex number. To cut down the number of choices we often restrict the argument to lie in the interval $(-\pi, \pi]$. The *standard argument* $\mathrm{Arg}(z)$ is defined as the unique argument of z that lies in this interval. E.g. $\mathrm{Arg}(4+3i) = \sin^{-1}(\frac{3}{5})$. By convention, $\mathrm{Arg}(0)$ is undefined.

- It is nice that $\mathrm{Arg}(z)$ has only a single value (as opposed to $\arg(z)$), but it has one unsightly flaw: $\mathrm{Arg}(z)$ is discontinuous on the negative real axis (it jumps from $+\pi$ to $-\pi$). This will cause some difficulties later on.

Complex arithmetic

- Arithmetic operations $(+, -, *, /)$ follow the usual rules of algebra, but with the additional rule that $i^2 = -1$:

$$\begin{aligned}
(a+bi)+(c+di) &= (a+c)+(b+d)i \\
(a+bi)-(c+di) &= (a-c)+(b-d)i \\
(a+bi)(c+di) &= (ac-bd)+(ad+bc)i \\
(a+bi)/(c+di) &= [(a+bi)(c-di)]/[(c+di)(c-di)] \\
&= [(ac+bd)+(bc-ad)i]/(c^2+d^2) \\
&= \tfrac{ac+bd}{c^2+d^2} + \tfrac{bc-ad}{c^2+d^2} i.
\end{aligned}$$

- Note: division by zero is undefined.

- Addition and subtraction can be interpreted geometrically via the parallelogram law.

- All the usual laws of algebra (commutativity, associativity, distributivity(can be proven to hold for the complex numbers. (I.e. the complex numbers form a field).

- Multiplication and division are quite messy in Cartesian co-ordinates, but become much nicer in polar form:

  $(re^{i\theta})(se^{i\alpha}) = rse^{i(\theta+\alpha)} \quad (re^{i\theta})/(se^{i\alpha}) = r/se^{i(\theta-\alpha)}$.

Powers and roots

- If one wants to raise a complex number z to an integer n, or take the n^{th} root of z, it's best to work in polar co-ordinates.

- For instance, to compute $(1+i)^{20}$, write $1+i$ in polar form as $\sqrt{2}e^{i\pi/4}$, and compute
$$\begin{aligned}(1+i)^{20} &= (\sqrt{2}e^{i\pi/4})^{20} \\ &= 2^{10}e^{5\pi i} \\ &= -1024.\end{aligned}$$
Note that it doesn't matter which polar form you start with, because there is only one possible answer to $(1+i)^{20}$.

- To compute n^{th} roots is similar, except that different polar forms can give you different roots:
$$\begin{aligned}(1+i)^{1/2} &= (\sqrt{2}e^{i(\frac{\pi}{4}+2k\pi)})^{1/2} \\ &= 2^{1/4}e^{i(\frac{\pi}{8}+k\pi)} \\ &= 2^{1/4}e^{\pi i/8} \text{ or } 2^{1/4}e^{9\pi i/8}.\end{aligned}$$

- Every complex number z has exactly n n^{th} roots. More generally, any polynomial of degree n has exactly n roots (the Fundamental Theorem of Algebra; we'll prove this later). This is one advantage that complex numbers have over the reals.

Basic complex operations - review

- The magnitude $|z|$ behaves well with respect to multiplication or division:

$$|zw| = |z|\,|w|, \quad |z/w| = |z|/|w|, \quad |z^n| = |z|^n.$$

- With respect to addition or subtraction, we have the triangle inequality:

$$|z + w| \leq |z| + |w|.$$

  More generally we have

$$||z| - |w|| \leq |z \pm w| \leq |z| + |w|.$$

  E.g. if $|z| = 9$ and $|w| = 2$, then we know that $7 \leq |z + w| \leq 11$, $7 \leq |z - w| \leq 11$ and that $|zw| = 18$.

- If $z = x + yi$, the *complex conjugate* of z is defined by $\overline{z} = x - yi$. In polar form we have

$$\overline{re^{i\theta}} = re^{-i\theta}.$$

  As a rule of thumb, to conjugate a complicated expression one conjugates each term separately, so that all is are replaced by $-i$s. E.g.

$$\overline{(z + we^{i\theta})(w - ize^{-i\theta})} = (\overline{z} + \overline{w}e^{-i\theta})(\overline{w} + i\overline{z}e^{i\theta}).$$

An important identity is

$$z\overline{z} = |z|^2.$$

- If $x+iy$ is a complex number, the exponential $e^{x+iy} = exp(x+iy)$ is defined by Euler's formula

$$e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y).$$

One of the basic properties of the exponential is that

$$e^z e^w = e^{z+w}.$$

(Exercise!)

- We'll explain why Euler's formula works a bit later in the course.

Subsets of the complex plane

- In real analysis, functions often have a domain which is a subset of the real line such as an interval or half-line. For instance, $f(x) = \sqrt{x}$ is defined in the interval $[0, \infty)$, while $\sin^{-1}(x)$ is defined on $[-1, 1]$. If, say, we are integrating $f(x) = e^x$ from 1 to 2 then the domain of interest is only $[1, 2]$ even though e^x is defined on all of $\mathbf{R}$.

- These domains may be open, closed, or half-open; bounded or unbounded; connected or disconnected (e.g. a function defined on $[1, 2] \cup [3, 4]$ has a disconnected domain).

- In complex analysis, we often consider functions $w = f(z)$ which are defined on subsets of the complex plane, such as the open unit disk $\{z : |z| < 1\}$, the closed unit disk $\{z : |z| \leq 1\}$, the upper half-plane $\{z : \mathrm{Im}(z) > 0\}$, the punctured plane $\mathbf{C} - \{0\}$, and so forth. Of course, these sets can become very complicated.

- We can classify these sets as open, closed, connected, etc. as with subsets of the real line. Sets which are particularly nice are called *domains*.

Open and closed sets

- Let S be a subset of the complex plane. We say that a point z_0 is in the *interior* of S if one can find a radius $\varepsilon > 0$ such that the ball $\{z : |z - z_0| < \varepsilon\}$ is contained in S.

- Similarly, we say that z_0 is in the *exterior* of S if one can find a radius $\varepsilon > 0$ such that the ball $\{z : |z - z_0| < \varepsilon\}$ is disjoint from S.

- A point which is neither in the interior or exterior is said to be on the *boundary* of S.

- If S contains all its boundary, it is said to be *closed*; if it contains none of its boundary, it is said to be *open*. If it only contains some of its boundary, then it is neither open nor closed.

- By convention, the empty set $\emptyset$ and the complex plane $\mathbf{C}$ are said to have no boundary, and are thus both open and closed.

- Rule of thumb: sets defined using $<$ are open; sets defined using $=$ or $\leq$ are closed.

Examples:

- Let $S = \{z : |z| < 1\}$. If $|z_0| < 1$, then z_0 is in the interior of S; if $|z_0| > 1$, then z_0 is in the exterior of S; of $|z_0| = 1$, then z_0 is in the boundary of S. Since S does not contain any part of its boundary, it is open.

- The set $S = \{z : |z| \leq 1\}$ has the same interior, exterior, and boundary, but it contains all its boundary, so it is closed.

- The set $S = \mathbf{C} - \{x \in \mathbf{R} : x \leq 0\}$ has $\{x \in \mathbf{R} : x \leq 0\}$ as its boundary, so it is open. Note that this set has no exterior.

- The set $S = \{0\} \cup \{z \in \mathbf{R} : \mathrm{Arg}(z) = \pi/4\}$ has no interior, and the exterior is just $\mathbf{C} - S$. So S is its own boundary, so this set is closed.

- Rule of thumb: sets drawn with solid lines are closed, those drawn with dotted lines are open.

- Open sets have the property that for any point inside the set, you can move a little bit in every direction and stay inside the set. This is handy for differentiation.

Connected sets

- We can divide open sets into two types: connected sets and disconnected sets.
- If S is an open set, we say that S is *connected* if every two points z_1, z_2 in S can be connected by a polygonal path. (A polygonal path is a finite sequence of line segments such that each segment starts where the previous segment ends).
- If an open set is not connected, we say it is disconnected. (It is possible to classify non-open sets as being connected or not connected, but that's a little trickier to do rigorously).
- For instance, the open set $\{z : |z| > 1\}$ is connected despite having a "hole" in it; any two points in this set can be connected by a polygonal path. However, the open set $\{z : \mathrm{Re}(z) \neq 0\}$ is not connected.
- Connected sets are good for integration, because you can get from any point to any other point in the set. Sets which are both open and connected are known as *domains*, and are the best class of sets for doing analysis (because you can differentiate AND integrate).

Bounded sets

- There is one final distinction we will use; we shall divide sets into bounded and unbounded sets.
- A set is *bounded* if it can be contained in some ball $\{z : |z| < R\}$ for some $0 < R < \infty$. If a set cannot be contained in any such ball, it is said to be *unbounded*.
- For instance, the closed unit square $\{z : 0 \leq \mathrm{Re}(z) \leq 1, 0 \leq \mathrm{Im}(z) \leq 1\}$ is bounded because it can be contained in the ball $\{z : |z| < 10\}$ (for instance). However, the open right half-plane $\{z : \mathrm{Re}(z) > 0\}$ cannot be contained in any such ball, and is therefore unbounded.
- Another way of saying this is that unbounded sets go off to infinity, whereas bounded sets do not.

Complex differentiation - overview

- If $y = f(x)$ is a real-valued function, the derivative $f'(x)$ is defined as the limit

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}.$$

  Not all functions are differentiable everywhere; for instance, there might be a point where the right limit $x \to x_0^+$ differs from the left limit $x \to x_0^-$.

- In these lectures we'll work out how to differentiate complex functions $w = f(z)$. Of course, to use the above definition we must first define complex limits.

- Complex differentiation is very similar to real differentiation, but there is one big difference: most complex functions are *not* differentiable! In order to be differentiable one must satisfy certain compatibility conditions, called the Cauchy-Riemann equations.

- Fortunately, almost all the standard functions (polynomials, e^z, etc.) satisfy the Cauchy-Riemann equations and are differentiable.

Complex functions

- A *complex function* is a function which takes one complex number as input (usually called z) and spits out one complex number as output (usually called $f(z)$, or w). Examples:

$$f(z) = z + 2 - i$$
$$f(z) = z^2$$
$$f(z) = 3$$

- A real function $y = f(x)$ can be easily graphed on a two-dimensional piece of paper, with one axis for the domain and one for the range. But a complex function cannot - one needs two axes for the domain and two for the range! (This hasn't stopped people from trying, though, using color or animation to get the extra dimensions). We'll describe some other ways to visualize complex functions later on, but for now let's think of functions just as abstract input-output devices.

- The above examples of functions were described in terms of a single complex variable z. But there are other ways to describe these functions. As with complex numbers, complex functions can be written in Cartesian or polar form.

- In Cartesian form, we write complex functions in terms of a variable $x + iy$, where x and y are reals. In some sense, we are turning a function of one complex variable into a function of two real variables. For instance, the function $f(z) = z + 2 - i$ in Cartesian form becomes

$$f(x + iy) = (x + 2) + i(y - 1),$$

and the function $f(z) = z^2$ in Cartesian form becomes

$$f(x + iy) = (x^2 - y^2) + i(2xy).$$

- In general, any complex function $f(z)$ can be written in Cartesian form

$$f(x + iy) = u(x + iy) + iv(x + iy)$$

where u and v are real-valued functions. $u(x + iy)$ and $v(x+iy)$ are called the real and imaginary parts of f. For instance, if $f(z) = |z|$, then $u(x + iy) = \sqrt{x^2 + y^2}$ and $v(x + iy) = 0$.

- One can always convert Cartesian form back to standard form, but it isn't always pretty. For instance,

$$f(x + iy) = x^2 + iy^2$$

becomes

$$f(z) = \mathrm{Re}(z)^2 + i\mathrm{Im}(z)^2.$$

- Functions can also be written in polar form by replacing z with $re^{i\theta}$, although this is less common. For example, $f(z) = z^2$ becomes

$$f(re^{i\theta}) = r^2 e^{2i\theta}.$$

- These three ways of writing a complex function are all equally valid - they all describe the same function even though they do look very different. (Form and Function are two different things!)

Limits of a function

- In order to make sense of a derivative, we're going to have to give meaning to expressions such as

$$\lim_{z \to z_0} f(z).$$

- The formal definition is a bit unwieldy, unfortunately:

  Definition. Let $f(z)$ be a complex function whose domain contains a punctured ball $\{z : 0 < |z - z_0| < r\}$. We say that

$$\lim_{z \to z_0} f(z) = L$$

  or

$$f(z) \to L \text{ as } z \to z_0$$

  if, for every $\varepsilon > 0$, one can find a $\delta > 0$ such that

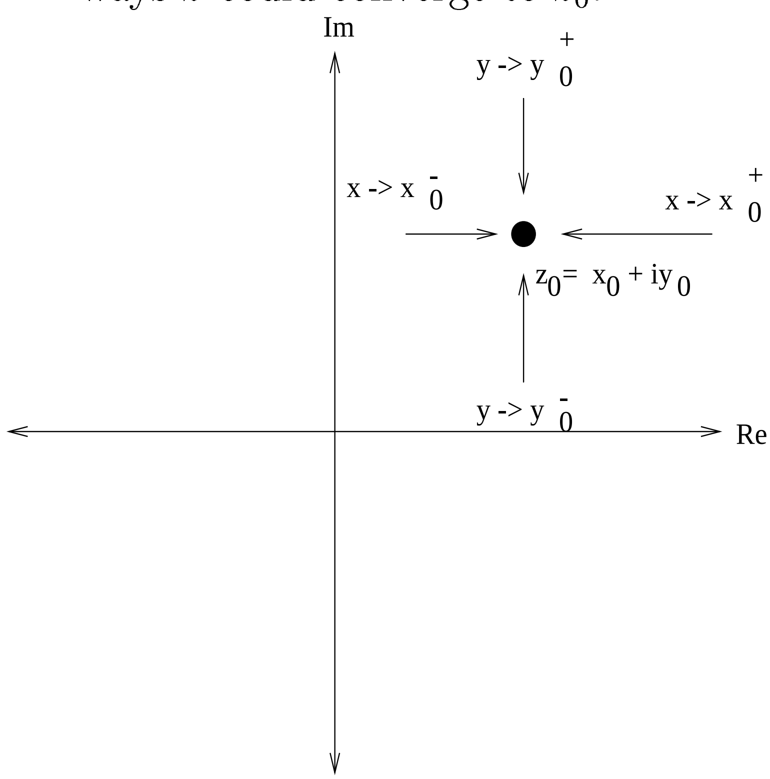
$$|f(z) - L| < \varepsilon \text{ whenever } 0 < |z - z_0| < \delta.$$

- Informally, this is saying that if you can keep z very close to z_0, then this forces $f(z)$ to be close to L. We won't be using this definition in this course.

Partial limits

- Another way of thinking about the statement $\lim_{z \to z_0} f(z) = L$ is that no matter which way you choose to move z toward z_0, $f(z)$ is forced to converge to L.

- On the real line, there are only two ways for a variable x to converge to x_0: from the right and from the left. On the complex plane, there are *infinitely* many ways z could converge to z_0.



- For instance, we could let z approach z_0 from the right and compute the partial limit. If $z_0 = x_0 + iy_0$, we write the right partial limit as

$$\lim_{x \to x_0^+} f(x + iy_0).$$

This is just an ordinary real-variable limit and can be computed by the usual techniques. Similarly, we have the left partial limit

$$\lim_{x \to x_0^-} f(x + iy_0).$$

the partial limit from above

$$\lim_{y \to y_0^+} f(x_0 + iy).$$

and the partial limit from below

$$\lim_{y \to y_0^-} f(x_0 + iy).$$

These are not the only partial limits; there are infinitely many others. For instance, we can approach from the upper right diagonal:

$$\lim_{t \to 0^+} f((x_0 + t) + i(y_0 + t))$$

or even from a parabola

$$\lim_{t \to 0^+} f((x_0 + t) + i(y_0 + t^2))$$

etc.

- If the full limit $\lim_{z\to z_0} f(z)$ equals L, then every single partial limit must also equal L. However, if even just two of the partial limits do not agree, or if one of the partial limits diverges, then the full limit must diverge as well.

- For instance, consider the limit $\lim_{z\to 0} z/|z|$. If we let z approach 0 from the right, the partial limit is 1:

$$\lim_{x\to 0^+} x/|x| = \lim_{x\to 0^+} x/x = 1.$$

  If we approach 0 from the left, the partial limit is -1:

$$\lim_{x\to 0^-} x/|x| = \lim_{x\to 0^-} x/(-x) = -1.$$

  If we approach 0 from above, we get i:

$$\lim_{y\to 0^+} iy/|iy| = \lim_{y\to 0^+} iy/y = i.$$

  And so forth. Since the partial limits do not match, the limit as a whole does not exist, and we say that $z/|z|$ diverges at $z = 0$.

- Thus if you suspect a limit to not exist, a good way to test it is to compute two partial limits (e.g. a horizontal and vertical limit) and see if they disagree. If two partial limits do agree, though, this does not guarantee that the full limit exists, because other partial limits might still disagree.

Continuity

- In many cases we can find a limit just by substitution:

$$\lim_{z \to z_0} f(z) = f(z_0).$$

This does not happen for all functions though - only those which are continuous.

- **Definition** Let $f(z)$ be a complex function on a domain (an open connected set), and let z_0 be a point in this domain. We say that f is *continuous* at z_0 if we have $\lim_{z \to z_0} f(z) = f(z_0)$. If f is continuous at every point in its domain, we say simply that f is *continuous*.

- Not every function is continuous. For instance, the standard argument function $\mathrm{Arg}(z)$ is continuous except at 0 (where it is undefined) and on the negative real axis (where the partial limits from above and below don't match).

- Intuitively, a continuous function is *stable*: small changes in the input do not translate to large changes in the output.

A partial list of continuous functions

- Every polynomial in z or in x and y is continuous. For instance $f(x+iy) = x^3 - 3ix^2y$ is continuous, as is the real part function $\mathrm{Re}(x+iy) = x$, the imaginary part function $\mathrm{Im}(x+iy) = y$, and the conjugtation function $\overline{x+iy} = x - iy$.
- The absolute value function $f(z) = |z|$ is continuous everywhere, even at zero.
- The exponential function $f(z) = e^z$ is continuous.
- If $f(z)$ and $g(z)$ are continuous at z_0, then so are $f(z)+g(z)$, $f(z)-g(z)$, and $f(z)g(z)$. If $g(z_0) \neq 0$, then $f(z)/g(z)$ is also continuous at z_0.
- If g is continuous at z_0, and f is continuous at $g(z_0)$, then the composition $f \circ g(z) = f(g(z))$ is continuous at z_0. E.g. $f(z) = e^{z^2+iz}$ is continuous as it is the composition of two continuous functions.
- If $u(x+iy)$ and $v(x+iy)$ are continuous functions of x and y, then $f(x+iy) = u(x+iy) + iv(x+iy)$ are continuous functions of $x+iy$.

Limit laws

- The limit laws for complex limits are much the same as for real limits:

$$\lim_{z\to z_0} f(z) \pm g(z) = \lim_{z\to z_0} f(z) \pm \lim_{z\to z_0} g(z)$$

$$\lim_{z\to z_0} cf(z) = c \lim_{z\to z_0} f(z)$$

$$\lim_{z\to z_0} f(z) = \lim_{z\to z_0} \mathrm{Re}(f(z)) + i \lim_{z\to z_0} \mathrm{Im}(f(z))$$

$$\lim_{z\to z_0} f(z)g(z) = \lim_{z\to z_0} f(z) \lim_{z\to z_0} g(z)$$

$$\lim_{z\to z_0} f(z)/g(z) = \lim_{z\to z_0} f(z)/ \lim_{z\to z_0} g(z) \text{ if denom.} \neq 0$$

- There is also a L'Hôpital's rule: if $f(z_0) = g(z_0) = 0$, and f and g are continuously differentiable at z_0, then

$$\lim_{z\to z_0} f(z)/g(z) = \lim_{z\to z_0} f'(z)/g'(z).$$

(We'll define what it means to be differentiable shortly).

- As mentioned already, $\lim_{z\to z_0} f(z) = f(z_0)$ if f is continuous at z_0.

- The squeeze test: if h is a real-valued function such that $\lim_{z\to z_0} h(z) = 0$ and $|f(z) - L| < h(z)$ for all

$z \neq z_0$, then $\lim_{z \to z_0} f(z) = L$. For instance,

$$\lim_{z \to 0} \frac{\overline{z}^2}{z} = 0$$

because

$$\left| \frac{\overline{z}^2}{z} \right| = \frac{|z|^2}{|z|} = |z|$$

and $\lim_{z \to 0} |z| = 0$.

Partial derivatives

- Just as there are partial limits and full limits in complex analysis, there are partial derivatives and full derivatives.

- A complex function $f(z)$ can be thought of as a function $f(x+yi)$ of two real variables x and y. As such, one can form partial derivatives

$$\frac{\partial f}{\partial x}(x_0+iy_0) = \lim_{x\to x_0} \frac{f(x+iy_0)-f(x_0+iy_0)}{x-x_0}$$

and

$$\frac{\partial f}{\partial y}(x_0+iy_0) = \lim_{y\to y_0} \frac{f(x_0+iy)-f(x_0+iy_0)}{y-y_0}$$

These partial derivatives work exactly like they do in real analysis. For instance, if f is the squaring function

$$f(x+iy) = (x^2-y^2)+i(2xy),$$

then

$$\frac{\partial f}{\partial x}(x+iy) = 2x+i(2y)$$

and

$$\frac{\partial f}{\partial y}(x+iy) = -2y+i(2x)$$

The complex derivative

- The complex derivative f' or df/dz is defined as

$$\frac{df}{dz}(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

If this limit exists, we say that f is complex differentiable at z_0, or simply that f is differentiable at z_0.

- Let's look at the partial limits of this definition. If z approaches $z_0 = x_0 + iy_0$ horizontally, so $z = x + iy_0$, then the above limit becomes

$$\lim_{x \to x_0} \frac{f(x + iy_0) - f(x_0 + iy_0)}{(x + iy_0) - (x_0 + iy_0)} = \frac{\partial f}{\partial x}(z_0).$$

If instead z approaches z_0 vertically, so $z = x_0 + iy$, then the above limit becomes

$$\lim_{y \to y_0} \frac{f(x_0 + iy) - f(x_0 + iy_0)}{(x_0 + iy) - (x_0 + iy_0)} = \frac{1}{i}\frac{\partial f}{\partial y}(z_0).$$

- We conclude: if f is differentiable at z_0, then

$$\frac{df}{dz}(z_0) = \frac{\partial f}{\partial x}(z_0) = \frac{1}{i}\frac{\partial f}{\partial y}(z_0).$$

In particular, in order for f to be complex differentiable at z_0, the x derivative and y derivative must be related by the equation

$$\frac{\partial f}{\partial x}(z_0) = \frac{1}{i}\frac{\partial f}{\partial y}(z_0).$$

This equation is known as the Cauchy-Riemann equation(s). These equations must be satisfied in order for the complex derivative to exist.

- In fact, this is pretty much a necessary and sufficient condition. Theorem: If the partial derivatives of f exist and are continuous, then f is complex differentiable exactly when the Cauchy-Riemann equations are satisfied.

- Informal proof (optional material): Start with

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}.$$

Writing $z = z_0 + \Delta x + i\Delta y$, we get

$$f'(z_0) = \lim_{\Delta x, \Delta y \to 0} \frac{f(z_0 + \Delta x + i\Delta y) - f(z_0)}{\Delta x + i\Delta y}.$$

Now use Taylor's theorem with remainder (which works when the partial derivatives are continuous):

$$f(z_0+\Delta x+i\Delta y) = f(z_0)+\Delta x\frac{\partial f}{\partial x}(z_0)+\Delta y\frac{\partial f}{\partial y}(z_0)+error.$$

If we use the Cauchy-Riemann equations, this becomes

$$f(z_0+\Delta x+i\Delta y) = f(z_0)+(\Delta x+i\Delta y)\frac{\partial f}{\partial x}(z_0)+error.$$

So the limit converges to $\frac{\partial f}{\partial x}(z_0)$ (from Taylors the orem we can check that the error goes to zero).

Examples

- Take the squaring function again:

$$f(x+iy) = (x^2 - y^2) + i(2xy).$$

The partial derivatives

$$\frac{\partial f}{\partial x} = 2x + 2yi, \quad \frac{\partial f}{\partial y} = -2y + 2ix$$

satisfy the Cauchy-Riemann equations $\frac{\partial f}{\partial x} = 1/i \frac{\partial f}{\partial y}$ for all x and y, and so this function is differentiable everywhere. The derivative is

$$\frac{df}{dz} = \frac{\partial f}{\partial x} = 2x + 2iy.$$

In other words, the function z^2 is differentiable and has derivative $2z$.

- Now take the exponential function

$$f(x+iy) = exp(x+iy) = e^x \cos y + ie^x \sin y.$$

The partial derivatives are

$$\frac{\partial f}{\partial x} = e^x \cos y + ie^x \sin y, \quad \frac{\partial f}{\partial y} = -e^x \sin y + ie^x \cos y.$$

Again, the Cauchy-Riemann equations are always satisfied, so this function is differentiable everywhere, with derivative $e^x \cos y + ie^x \sin y$. In other words, e^z is differentiable with derivative e^z.

- The conjugate function

$$f(x+iy) = x - iy$$

has partial derivatives

$$\frac{\partial f}{\partial x} = 1, \quad \frac{\partial f}{\partial y} = -i.$$

The Cauchy-Riemann equations $\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y}$ are never satisfied, so this function is never differentiable.

- Let's take the function $f(z) = |z|^2$, which in Cartesian form is

$$f(x+iy) = x^2 + y^2.$$

The partial derivatives are

$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = 2y.$$

The Cauchy-Riemann equations read $2x = 1/i2y$, or $2xi = 2y$. Taking real and imaginary parts we get $x = 0$ and $y = 0$. So this function is only differentiable at the origin 0, and is not differentiable otherwise.

Cartesian form of Cauchy-Riemann equations

- The Cauchy-Riemann equations are

$$\frac{\partial f}{\partial x} = \frac{1}{i}\frac{\partial f}{\partial y}$$

If we write f in Cartesian form as $f(x+iy) = u(x+iy) + iv(x+iy)$, then these equations become

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{1}{i}(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}).$$

Simplifying, we get

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Taking real and imaginary parts we get the Cartesian form of the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x}.$$

This illustrates that in order for a function to be differentiable, its real and imaginary parts must be closely related.

Some differentiable functions

- All polynomials in z, such as $z^2 - 4iz + 2 + i$, are differentiable, and they differentiate just as you'd expect (in this case, it's $2z - 4i$). Polynomials in x and y are generally not differentiable, as we've seen - even though they are continuous.

- The exponential function e^z is differentiable, and is its own derivative. The functions $\text{Arg}(z)$, $\text{Re}(z)$, $\text{Im}(z)$, $|z|$, and $\overline{z}$ are not differentiable anywhere.

- The sum, difference, product, or composition of two differentiable functions is again differentiable, and one uses the product rule, chain rule, etc. to compute the derivative of the combined expression. e.g. e^{2iz^2} is differentiable with derivative $4ize^{2iz^2}$.

- The quotient of two differentiable functions is also differentiable as long as the denominator is non-zero. The derivative is worked out using the quotient rule. Thus e^z/z is differentiable everywhere except at $z = 0$.

- As a rule of thumb: if it is obvious what the derivative should be (e.g. e^{z^2}, then the function is differentiable, but if it is something more exotic (e.g. $\text{Arg}(z)$) then chances are the function is not differentiable.