Conditional gambler's ruin problem with arbitrary winning and losing probabilities with applications

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Abstract

In this paper we provide formulas for the expectation of a conditional game duration in a finite state-space one-dimensional gambler's ruin problem with arbitrary winning $p(n)$ and losing $q(n)$ probabilities (*i.e.,* they depend on the current fortune). The formulas are stated in terms of the parameters of the system. Beyer and Waterman [*Mathematics Magazine*, **50**(1), 1977] showed that for the classical gambler's ruin problem the distribution of a conditional absorption time is symmetric in p and q. Our formulas imply that for non-constant winning/losing probabilities the expectation of a conditional game duration is symmetric in these probabilities (*i.e.,* it is the same if we exchange $p(n)$ with $q(n)$) as long as a ratio $q(n)/p(n)$ is constant.

Most of the formulas are applied to a non-symmetric random walk on a circle/polygon. Moreover, for a symmetric random walk on a circle we construct an optimal strong stationary dual chain – which turns out to be an absorbing, non-symmetric, birth and death chain. We apply our results and provide a formula for its expected absorption time, which is a fastest strong stationary time for the aforementioned symmetric random walk on a circle. This way we improve upon a result of Diaconis and Fill [*The Annals of Probability*, **18**(4), 1990], where strong stationary time – however not the fastest – was constructed. Expectations of the fastest strong stationary time and the one constructed by Diaconis and Fill differ by 3/4, independently of a circle's size.

Keywords: Gambler's ruin problem, conditional absorption time, random walk on a polygon, random walk on a circle, birth and death chain, strong stationary dual chain, Möbius monotonicity

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1. Introduction

The classical gambler's ruin problem is following. Having initially i dollars, $1 \leq i \leq N-1$, in one step we either win one dollar (*i.e.,* we move to $i+1$) with probability $p \in (0,1)$, or we lose one dollar (*i.e.,* we move to $i-1$) with probability $q = 1-p$. The game ends when the player reaches N (wins the game) or 0 (goes broke). The typical questions one can ask are:

- What is the probability of winning (*i.e.,* reaching N before 0)?
- What is the (expected) game duration?
- What is the (expected) conditional game duration (*i.e.,* game duration given we win or given we lose)?
- Is the (expected) conditional game duration symmetric in p and q?

Similarly, one can consider random walk on $\mathbf{Z}_{m+1} = \{0, \ldots, m\}$: being at state i we either move clockwise with a probability $p \in (0,1)$ (*i.e.,* from i to $i+1 \bmod (m+1)$) or we move counterclockwise with a probability $1-p$ (*i.e.,* we move from i to $i-1 \bmod (m+1)$). We will refer to this as to *the classical random walk on a polygon* (cf. [Sar06]). Assuming we start at i, the typical questions one can ask are:

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- What is the probability that all vertices have been visited before the particle returns to i?
- What is the probability that the last vertex visited is j ?
- What is the expected number of moves needed to visit all the vertices?
- What is the expected additional number of moves needed to return to i after visiting all the vertices?

All above questions were answered in the classical settings. Several generalizations were studied. The probability of winning in a gambler's ruin problem with general winning and losing probabilities (*i.e.,* $p(i)$ being probability of moving from i to $i+1$ and $q(i)$ being the probability of moving from i to $i-1$, with $p(i)+q(i)\leq 1$, $i\in\{1,\ldots,N-1\}$) goes back to Parzen [Par62], revisited in [ES09]. Siegmund duality based proof is given in [Lor17] (where more general, multidimensional, game is considered). In [Len09] the questions related to the conditional game duration are answered for the classical gambler's ruin problem with ties allowed, *i.e.,* $p+q\leq 1$ (with probability $1-(p+q)$ we can stay at a given state). In [Lef08] author considers specific generalization, namely $p(i)=q(i)=\frac{1}{2(2ci+1)}, c\geq 0$ (thus the probability of staying is $1-\frac{1}{2ci+1}$) and answers the question about the winning probability and the expected game duration (and also considers the corresponding diffusion process). In this paper we present formulas for the expected (conditional) absorption time in terms of parameters of the system (*i.e.,* winning/losing probabilities $p(i),q(i)$). Similar problem was considered in [ES00], the recursion for the expected conditional game duration is given therein (equations (3.4) and (3.5)), however it is not solved in its general form – later on author considers only constant winning/losing probabilities. In [GMZ12] (similar results with different proofs are presented in [MZ16]) the generating function of absorption time (including a conditional one) is given in terms of eigenvalues of a transition matrix and eigenvalues of a truncated transition matrix. The questions for the classical random walk on a polygon were answered in [Sar06]. Some generalizations (rather then allowing arbitrary winning/losing probabilities, symmetric random walks on tetrahedra, octahedra, and hexahedra, are considered) are studied in [SM17].

In 1977 in [BW77] it was shown that for a classical gambler's ruin problem with $p(n)=p=1-q(n)=1-q$, the distribution of a conditional game duration is symmetric in p and q, *i.e.,* it is the same as in a game with $p'=q$ and $q'=p$. In 2009 in [Len09] it was extended to a case $p+q<1$ (*i.e.,* the classical case with ties allowed). In this paper we show that that the expected conditional game duration is symmetric also for non-constant winning/losing probabilities $p(n),q(n)$ as long as $q(n)/p(n)$ is constant (thus, including for example the spatially non-homogeneous case).

In Section 2 we introduce gambler's ruin problem with arbitrary winning and losing probabilities $p(i),q(i)$ together with main results. In Section 2.1 the main result is applied to constant $r(i)=r=q(i)/p(i)$, in Section 2.2 it is applied to non-homogeneous case, whereas the classical case is recalled in Section 2.3. The main example is given in Section 2.4. The results are applied to a random walk on polygon in Section 4. Last Section 5 contains proofs of main results.

2. Gambler's ruin problem

Fix an integer $N\geq 2$. Let

$$\mathbf{p}=(p(0),p(1),\ldots,p(N)),\quad \mathbf{q}=(q(0),q(1),\ldots,q(N)),$$

where $p(0)=q(0)=p(N)=q(N)=0$ and $p(i),q(i)>0,p(i)+q(i)\leq 1$ for $i\in\{1,2\ldots,N-1\}$. Consider a Markov chain $\mathbf{X}=\{X_k\}_{k\geq 0}$ on $\mathbb{E}=\{0,1,\ldots,N\}$ with transition probabilities

$$\mathbf{P}_X(i,j)=\begin{cases} p(i) & \text{if} \quad j=i+1,\\ q(i) & \text{if} \quad j=i-1,\\ 1-(p(i)+q(i)) & \text{if} \quad j=i.\end{cases}$$

We will refer to $\mathbf{X}$ starting at i as to the (gambler's ruin) *game* $G(\mathbf{p},\mathbf{q},0,i,N)$. Note that the chain will eventually end up in either in N (the *winning* state) or in 0 (the *losing* state). To simplify some notation, let $r(i)=\frac{q(i)}{p(i)}$ for $i\in\{1,\ldots,N-1\}$.

Define $\tau_j=\inf\{k: X_k=j\}$. We will study the following *smaller* games $G(\mathbf{p},\mathbf{q},j,i,k)$ with k as the *winning* state and j as the *losing* state ($j\le i\le k$), *i.e.,* $p(j)=q(j)=p(k)=q(k)=0$. Let us define:

$$\begin{aligned}
\rho_{j:i:k} &= P(\tau_k<\tau_j|X_0=i),\\
T_{j:i:k} &= \inf\{n\ge 0: X_n=j \text{ or } X_n=k,\ X_0=i\},\\
W_{j:i:k} &= T_{j:i:k} \text{ conditioned on } X_{T_{j:i:k}}=k,\\
B_{j:i:k} &= T_{j:i:k} \text{ conditioned on } X_{T_{j:i:k}}=j.
\end{aligned}$$

In other words: $\rho_{j:i:k}$ is the probability that a gambler starting with i dollars wins in the smaller game; $T_{j:i:k}$ is the distribution of a game duration (time till gambler either wins or goes broke); $W_{j:i:k}$ is the distribution of $T_{j:i:k}$ conditioned on $X_{T_{j:i:k}}=k$ (winning) and similarly $B_{j:i:k}$ is the distribution of $T_{j:i:k}$ conditioned on $X_{T_{j:i:k}}=j$ (losing).

Notation. For given rates $\mathbf{p},\mathbf{q}$ by $\mathbf{p}\leftrightarrow\mathbf{q}$ we understand new rates $\mathbf{p}'=\mathbf{q},\mathbf{q}'=\mathbf{p}$. For some random variable R (one of ρ,T,W,B) for a game with rates $\mathbf{p},\mathbf{q}$, by $R(\mathbf{p}\leftrightarrow\mathbf{q})$ we understand the random variable defined for a game with rates $\mathbf{p}'=\mathbf{q},\mathbf{q}'=\mathbf{p}$ (and similarly, e.g., $ER(\mathbf{p}\leftrightarrow\mathbf{q})$ is an expectation of R defined for such a game). We say that R (ER) is *symmetric in* $\mathbf{p}$ and $\mathbf{q}$ if $R\overset{distr}{=}R(\mathbf{p}\leftrightarrow\mathbf{q})$ ($ER=ER(\mathbf{p}\leftrightarrow\mathbf{q})$).

By $f(n)=\Theta(g(n))$ we mean $\exists(c_1,c_2>0)\ \exists(n_0)\ \forall(n>n_0)\ c_1g(n)\le f(n)\le c_2g(n)$. In this section we use the convention: empty sum equals 0, empty product equals 1; however in Section 4 we use some nonstandard notation, see details on page 13.

In next theorem we provide formulas for expected game duration, for completeness (and since we will need them later) we also include known results for $\rho_{j:i:k}$.

Theorem 2.1. *Consider the gambler's ruin problem on* $\mathbb{E}=\{0,1,\ldots,N\}$ *described above. We have*

$$\begin{aligned}
\rho_{j:i:k} &= \frac{\sum_{n=j+1}^{i}\prod_{s=j+1}^{n-1}\left(\frac{q(s)}{p(s)}\right)}{\sum_{n=j+1}^{k}\prod_{s=j+1}^{n-1}\left(\frac{q(s)}{p(s)}\right)} = \frac{\sum_{n=j+1}^{i}\prod_{s=j+1}^{n-1} r(s)}{\sum_{n=j+1}^{k}\prod_{s=j+1}^{n-1} r(s)},\\
ET_{j:i:k} &= \frac{\sum_{n=j+1}^{k-1}[d_n\sum_{s=j+1}^{n}\frac{1}{p(s)d_s}]}{\sum_{n=j}^{k-1}d_n}\sum_{n=j}^{i-1}d_n-\sum_{n=j+1}^{i-1}\left[d_n\sum_{s=j+1}^{n}\frac{1}{p(s)d_s}\right],
\end{aligned}\tag{2.1}$$

where $d_s=\prod_{i=j+1}^{s}\frac{q(i)}{p(i)}=\prod_{i=j+1}^{s}r(i)$ *(with convention* $d_j=1$*).*

The proof of Theorem 2.1 is postponed to Section 5.1.1. We will also need a formula for $ET_{j:i:k}$ in case when k is the only absorbing state.

Theorem 2.2. *Fix* $j\le i\le k$ *and consider a birth and death chain on* $\{j,\ldots,k\}$ *with rates* $p(s),q(s),s=j,\ldots,k$ *with* $q(j)=p(k)=q(k)=0$ *and* $q(s)>0$ *for* $s=j+1,\ldots,k-1$ *and* $p(s)>0$ *for* $s=j,\ldots,k-1$ *(i.e.,* k *is the only absorbing state). Then, the expectation of absorption time, starting from* i *is given by*

$$ET_{j:i:k}=\sum_{n=i}^{k-1}\left[d_n\sum_{s=j}^{n}\frac{1}{p(s)d_s}\right].$$

Now we go back to situation with two absrobing states, *i.e.,* also $p(j) = 0$. Next theorem (our main contribution) gives the formulas for $EW_{0:i:k}$ and $EB_{0:i:k}$. First, let us introduce some necessary notation. With some abuse of notation let us extend

$$\rho_{j:i:k} = \frac{\sum_{n=j+1}^{i} \prod_{s=j+1}^{n-1} \left(\frac{q(s)}{p(s)}\right)}{\sum_{n=j+1}^{k} \prod_{s=j+1}^{n-1} \left(\frac{q(s)}{p(s)}\right)} = \frac{\sum_{n=j+1}^{i} \prod_{s=j+1}^{n-1} r(s)}{\sum_{n=j+1}^{k} \prod_{s=j+1}^{n-1} r(s)}$$

for $k < i$ (but still $k > j$). Note that in such a case we may have $\rho_{j:i:k}$, thus this has no interpretation in terms of probability anymore.

For given integers n, m, k such that $n \leq m, k \in \{0, \lfloor (m-n+1)/2 \rfloor\}$ define

$$\mathbf{j}_k^{n,m} = \big\{\{j_1, j_2, \ldots, j_k\} : j_1 \geq n+1, j_k \leq m, j_i \leq j_{i+1} - 2 \text{ for } 1 \leq i \leq k-1\big\}. \tag{2.2}$$

For given $\mathbf{p}, \mathbf{q}$ and $\mathbf{j} \in \mathbf{j}_k^{n,m}$ define

$$\delta_{\mathbf{j}}^{n,m} = (-1)^k \prod_{s \in j} r(s) \prod_{s \in \{n, \ldots, m\} \setminus \mathbf{j} \cup (\mathbf{j}-1)} 1 + r(s), \tag{2.3}$$

where $\{n, \ldots, m\}$ is an empty set for $n > m$ and $\mathbf{j} - 1 = \{j_1 - 1, j_2 - 1, \ldots, j_k - 1\}$ for $\mathbf{j} = \{j_1, j_2, \ldots, j_k\}$. Finally, let

$$\xi_k^{n,m} = \sum_{\mathbf{j} \in \mathbf{j}_k^{n,m}} \delta_{\mathbf{j}}^{n,m}. \tag{2.4}$$

Now we are ready to state our main theorem.

Theorem 2.3. *Consider the gambler's ruin problem on $\mathbb{E} = \{0, 1, \ldots, N\}$ described above. We have*

$$EW_{0:i:N} = EW_{0:1:N} - EW_{0:1:i}, \quad \textit{where} \tag{2.5}$$

$$EW_{0:1:i} = \sum_{n=1}^{i-1} \frac{\rho_{0:n:i}}{p(n)} \sum_{s=0}^{\lfloor (i-1-n)/2 \rfloor} \xi_s^{n+1,i-1}. \tag{2.6}$$

Moreover, we have

$$EB_{0:i:N} = EW'_{0:N-i:N}, \tag{2.7}$$

where $W'_{0:N-i:N}$ is defined for a gambler's ruin problem with rates $p'(i) = q(N-i)$ and $q'(i) = p(N-i)$ for $i \in \mathbb{E}$.

The proof of Theorem 2.3 is postponed to Section 5.1.2.

2.1. Constant $r(n) = r = \frac{q(n)}{p(n)}$

In this section we will apply Theorems 2.1 and 2.3 to a gambler's ruin problem with constant $r = \frac{q(i)}{p(i)}$. The winning probabilities $\rho_{0:i:N}$ are known (they are the same as in the classical formulation of the problem), we will focus on a game duration. We have

Corollary 2.4. *Consider the gambler's ruin problem on $\mathbb{E} = \{0, \dots, N\}$ with constant $r = \frac{q(i)}{p(i)}$. We have*

$$
\begin{aligned}
r = 1: \quad ET_{j:i:k} &= \frac{i-j}{k-j} \sum_{n=j+1}^{k-1} \sum_{s=j+1}^{n} \frac{1}{p(s)} - \sum_{n=j+1}^{i-1} \sum_{s=j+1}^{n} \frac{1}{p(s)}, \\
ET_{0:i:N} &= \frac{i}{N} \sum_{n=1}^{N-1} \sum_{s=1}^{n} \frac{1}{p(s)} - \sum_{n=1}^{i-1} \sum_{s=1}^{n} \frac{1}{p(s)}, \\
r \neq 1: \quad ET_{j:i:k} &= \frac{r^j - r^i}{r^j - r^k} \sum_{n=j+1}^{k-1} \left[r^n \sum_{s=j+1}^{n} \frac{r^{-s}}{p(s)} \right] - \sum_{n=j+1}^{i-1} \left[r^n \sum_{s=j+1}^{n} \frac{r^{-s}}{p(s)} \right], \\
ET_{0:i:N} &= \frac{1-r^i}{1-r^N} \sum_{n=1}^{N-1} \left[r^n \sum_{s=1}^{n} \frac{r^{-s}}{p(s)} \right] - \sum_{n=1}^{i-1} \left[r^n \sum_{s=1}^{n} \frac{r^{-s}}{p(s)} \right].
\end{aligned}
$$

Proof. We have $d_k = \prod_{j=1}^{k} r = r^k$. Simple recalculations of (2.1) yield the result. □

For constant r we have that $\delta_{\mathbf{j}}^{n,m}$ (given in (2.3)) for all $i \in \{1, \dots, N-1\}$ depends on $\mathbf{j}$ only through k, thus

$$\xi_k^{n,m} = \sum_{\mathbf{j} \in \mathbf{j}_k^{n,m}} \delta_{\mathbf{j}}^{n,m} = C_k^{n,m} (-r)^k (1+r)^{m+1-n-2k}, \tag{2.8}$$

where $C_k^{n,m} = |\mathbf{j}_k^{n,m}|$. Moreover, we have $|\mathbf{j}_k^{n,m}| = T(m+1-n, k)$, where $T(n,k) = \binom{n-k}{k}$ is the number of subsets of $\{1, 2, ..., n-1\}$ of size k containing no consecutive integers [1].

The proof of the next corollary requires the following lemma.

Lemma 2.5. *Let $n \in \mathbb{N}$ and $r \geq 0$. We have*

$$\sum_{k=0}^{n} \binom{n-k}{k} \left(-\frac{r}{(1+r)^2} \right)^k = \begin{cases} \dfrac{1-r^{n+1}}{(1+r)^n (1-r)} & \text{if } r \neq 1, \\[2ex] \dfrac{n+1}{2^n} & \text{if } r = 1. \end{cases} \tag{2.9}$$

The proof of Lemma 2.5 is given in Section 5.1.2.

Remark 2.6. Note that the assertion of Lemma 2.5 can be stated in the following form (simply substituting $c = \frac{r}{(1+r)^2}$): for $n \in \mathbb{N}$ and $c \in (0, 1/4]$ we have

$$\sum_{k=0}^{n} \binom{n-k}{k} (-c)^k = \begin{cases} \dfrac{1-\gamma^{n+1}}{(1+\gamma)^n (1-\gamma)}, & \text{where } \gamma = \dfrac{1-2c+\sqrt{1-4c}}{2c}, \text{ if } c \in (0, 1/4), \\[2ex] \dfrac{n+1}{2^n} & \text{if } c = 1/4. \end{cases}$$

[1] `http://oeis.org/A011973`

These sums for $c \in \{-1,1\}$ were known ($F(n)$ is the n-th Fibonacci number):

$$\sum_{k=0}^{n} \binom{n-k}{k} = F(n+1),$$

$$\sum_{k=0}^{n} \binom{n-k}{k} (-1)^k = \begin{cases} 1 & \text{if } n \bmod 6 \in \{0,1\}, \\ 0 & \text{if } n \bmod 6 \in \{2,5\}, \\ -1 & \text{if } n \bmod 6 \in \{3,4\}. \end{cases}$$

We will give formulas for $EW_{0:1:i}$ for several cases ($EW_{0:i:N}$ can be calculated via (2.5)).

Corollary 2.7. *Consider the gambler's ruin problem on $\mathbb{E} = \{0,\ldots,N\}$ with constant $r = \frac{q(i)}{p(i)}$. We have:*

$$r = 1: \quad EW_{0:1:i} = \sum_{n=1}^{i-1} \frac{\rho_{0:n:i}}{p(n)} \sum_{s=0}^{\lfloor (i-1-n)/2 \rfloor} \xi_s^{n+1,i-1} = \sum_{n=1}^{i-1} \frac{n/i}{p(n)}(i-n).$$

$$r \neq 1: \quad EW_{0:1:i} = \sum_{n=1}^{i-1} \frac{\rho_{0:n:i}}{p(n)} \sum_{s=0}^{\lfloor (i-1-n)/2 \rfloor} \xi_s^{n+1,i-1} = \sum_{n=1}^{i-1} \frac{\frac{1-r^n}{1-r^i}(1-r^{i-n})}{p(n)(1-r)}. \tag{2.10}$$

Additionally, if $p(n) = p$ is constant (so is $q(n)$ then, since $r(n)$ is constant) we have

$$r = 1: \quad EW_{0:1:i} = \frac{1}{p} \sum_{n=1}^{i-1} \frac{n}{i}(i-n) = \frac{(i-1)(i+1)}{6p}, \tag{2.11}$$

$$\begin{aligned} r \neq 1: \quad EW_{0:1:i} &= \frac{1}{p} \sum_{n=1}^{i-1} \frac{\frac{1-r^n}{1-r^i}(1-r^{i-n})}{1-r} = \frac{1}{p(1-r^i)(1-r)} \sum_{n=1}^{i-1} (1-r^n)(1-r^{i-n}) \\ &= \frac{i(1+r^i) - (1+r)\frac{1-r^i}{1-r}}{p(1-r^i)(1-r)} = \frac{1}{p(1-r)} \left(i\frac{1+r^i}{1-r^i} - \frac{1+r}{1-r} \right). \end{aligned}$$

Proof. We will only show case $r = 1$, general $p(n)$ (the proof for $r \neq 1$ is very similar). Let us calculate $\xi_s^{n+1,i-1}$ first. From (2.8) and form of $C_k^{n,m}$ for $r = 1$ we have

$$\xi_s^{n+1,i-1} = C_s^{n+1,i-1}(-1)^s 2^{i-n-1-2s} = 2^{i-n-1} \binom{i-n-1-s}{s} \left(-\frac{1}{4}\right)^s.$$

From Theorem 2.3 (eq. (2.6)) and the fact that $\rho_{0:n:i} = n/i$ (since $r = 1$) we have

$$\begin{aligned} EW_{0:1:i} &= \sum_{n=1}^{i-1} \frac{n/i}{p(n)} \sum_{s=0}^{\lfloor (i-1-n)/2 \rfloor} \xi_s^{n+1,i-1} \\ &= \sum_{n=1}^{i-1} \frac{n/i}{p(n)} 2^{i-n-1} \sum_{s=0}^{\lfloor (i-1-n)/2 \rfloor} \binom{i-n-1-s}{s} \left(-\frac{1}{4}\right)^s \\ &\overset{\text{Lemma 2.5}}{=} \sum_{n=1}^{i-1} \frac{n/i}{p(n)} 2^{i-n-1} \frac{i-n-1+1}{2^{i-n-1}} = \sum_{n=1}^{i-1} \frac{n/i}{p(n)}(i-n), \end{aligned}$$

what finishes the proof. □

In 1977 Beyer and Waterman [BW77] showed that for a classical case *i.e.,* for constant birth $p(n) = p$ and death $q(n) = q$ rates such that $p + q = 1$, the distribution of $W_{0:i:N}$ is symmetric in p and q (*i.e.,* it has the same distribution for birth rate $p' = q$ and death rate $q' = p$). In 2009 Lengyel [Len09] showed that this holds also for the classical case with ties allowed, *i.e.,* $p+q < 1$. In the following theorem we show that $EW_{0:i:N}$ is symmetric in $\mathbf{p}$ and $\mathbf{q}$ (*i.e.,* it is the same for case with birth deaths $p'(n) = q(n)$ and death rates $q'(n) = p(n)$) as long as $r(n) = \frac{q(n)}{p(n)}$ is constant.

Theorem 2.8. *Consider the gambler's ruin problem on $\mathbb{E} = \{0, \ldots, N\}$ with constant $r = \frac{q(i)}{p(i)}$. We have*

$$EW_{0:i:N} = EW_{0:i:N}(\mathbf{p} \leftrightarrow \mathbf{q}),$$

(i.e., $EW_{0:i:N}$ is symmetric in $\mathbf{p}$ and $\mathbf{q}$).

Proof. By (2.5) it is enough to show that $EW_{0:1:i} = EW_{0:1:i}(\mathbf{p} \leftrightarrow \mathbf{q})$.

Let $W_{0:1:i}$ be defined for rates $\mathbf{p}$ and $\mathbf{q}$, whereas $W'_{0:1:i}$ be defined for rates $\mathbf{p}' = \mathbf{q}$ and $\mathbf{q}' = \mathbf{p}$, thus $r' = 1/r$. Since $r = \frac{q(n)}{p(n)}$, we have $p'(n) = q(n) = rp(n)$.

$$\begin{aligned} EW'_{0:1:i} &= \sum_{n=1}^{i-1} \frac{1}{p'(n)} \frac{(1 - \frac{1}{r^n})}{(1 - \frac{1}{r^i})} \frac{(1 - \frac{1}{r^{i-n}})}{(1 - \frac{1}{r})} = \sum_{n=1}^{i-1} \frac{1}{rp(n)} \frac{r^i(1 - r^n)}{r^n(1 - r^i)} \frac{r(1 - r^{i-n})}{r^{i-n}(1 - r)} \\ &= \sum_{n=1}^{i-1} \frac{1}{p(n)} \frac{(1 - r^n)}{(1 - r^i)} \frac{(1 - r^{i-n})}{(1 - r)}, \end{aligned}$$

what is equal to (2.10). □

It is natural to state the following conjecture.

Conjecture 2.9. *Consider the gambler's ruin problem on $\mathbb{E} = \{0, \ldots, N\}$ with constant $r = \frac{q(i)}{p(i)}$. Then, the distribution of $W_{0:i:N}$ is symmetric in $\mathbf{p}$ and $\mathbf{q}$.*

2.2. The spatially non-homogeneous case

In this Section we consider gambler's ruin problem with birth rates $p(n) = \frac{p}{2cn+1}$ and death rates $q(n) = \frac{q}{2cn+1}$, where c is a non-negative constant. This is often called the spatially non-homogeneous gambler's ruin problem. We will thus still consider case with constant $r(n)$, but with specific rates. As far as we are aware, all results in this section, except the one for $p(n) = q(n) = 1/2$, are new.

Corollary 2.10. *Consider the spatially non-homogeneous gambler's ruin problem. We have*

$$\begin{aligned} r = 1 : ET_{0:i:N} &= \frac{1}{2p} \left(iN \left(1 + \frac{2c}{3} N \right) - i^2 \left(1 + \frac{2c}{3} i \right) \right), \\ r \neq 1 : ET_{0:i:N} &= \frac{1}{p(r-1)} \left(\frac{1 - r^i}{1 - r^N} \left(-cN^2 - N\frac{(cr + c)}{r - 1} - N \right) + ci^2 + i\frac{(cr + c)}{r - 1} + i \right). \end{aligned}$$

Proof. Applying Corollary 2.4 we have:

- Case $r = 1$

$$
\begin{aligned}
ET_{0:i:N} &= \frac{i}{N}\sum_{n=1}^{N-1}\sum_{s=1}^{n}\frac{1}{p(s)} - \sum_{n=1}^{i-1}\sum_{s=1}^{n}\frac{1}{p(s)} = \frac{i}{N}\sum_{n=1}^{N-1}\sum_{s=1}^{n}\frac{2cn+1}{p} - \sum_{n=1}^{i-1}\sum_{s=1}^{n}\frac{2cn+1}{p} \\
&= \frac{1}{p}\left(\frac{i}{N}\sum_{n=1}^{N-1} n(cn+c+1) - \sum_{n=1}^{i-1} n(cn+c+1)\right) \\
&= \frac{1}{p}\left(\frac{i}{N}\frac{1}{6}(N-1)(N(2c(N+1)+3) - \frac{1}{6}(i-1)(i(2c(i+1)+3)\right) \\
&= \frac{1}{2p}\left(iN\left(1+\frac{2c}{3}N\right) - i^2\left(1+\frac{2c}{3}i\right)\right).
\end{aligned}
$$

- Case $r \neq 1$

$$
\begin{aligned}
ET_{0:i:N} &= \frac{1-r^i}{1-r^N}\sum_{n=1}^{N-1}\left[r^n\sum_{s=1}^{n}\frac{r^{-s}}{p(s)}\right] - \sum_{n=1}^{i-1}\left[r^n\sum_{s=1}^{n}\frac{r^{-s}}{p(s)}\right] \\
&= \frac{1}{p}\left(\frac{1-r^i}{1-r^N}\sum_{n=1}^{N-1}\left[r^n\sum_{s=1}^{n}r^{-s}(2cs+1)\right] - \sum_{n=1}^{i-1}\left[r^n\sum_{s=1}^{n}r^{-s}(2cs+1)\right]\right).
\end{aligned}
$$

We have

$$
\sum_{s=1}^{n} r^{-s}(2cs+1) = \frac{r^{-n}}{(r-1)^2}\left(2cr^{n+1} - 2cnr + 2cn - 2cr + r^{n+1} - r^n - r + 1\right)
$$

and

$$
\begin{aligned}
&\sum_{n=1}^{k-1}\left[r^n\frac{r^{-n}}{(r-1)^2}\left(2cr^{n+1} - 2cnr + 2cn - 2cr + r^{n+1} - r^n - r + 1\right)\right] \\
&= \frac{1}{(r-1)^2}\left(-\tfrac{2cr(r-r^k)}{r-1} - c(k-1)kr + c(k-1)k - 2cr(k-1)\right. \\
&\qquad \left. -\frac{r(r-r^k)}{r-1} + \frac{r-r^k}{r-1} + r - kr + k - 1\right) \\
&= \frac{1}{(r-1)^2}\left(-ck^2(r-1) + \frac{(2cr+r-1)(r^k-1)}{r-1} - k(cr+c+r-1)\right).
\end{aligned}
$$

Thus,

$$
\begin{aligned}
ET_{0:i:N} &= \frac{1}{p(r-1)^2}\Big\{ \quad \frac{1-r^i}{1-r^N}\Big(\quad -cN^2(r-1) + \tfrac{(2cr+r-1)(r^N-1)}{r-1} - N(cr+c+r-1)\Big) \\
&\qquad - \Big(-ci^2(r-1) + \tfrac{(2cr+r-1)(r^i-1)}{r-1} - i(cr+c+r-1)\Big)\Big\} \\
&= \frac{1}{p(r-1)^2}\Big\{ \quad \frac{1-r^i}{1-r^N}\Big(\quad -cN^2(r-1) - N(cr+c+r-1)\Big) \\
&\qquad +ci^2(r-1) + i(cr+c+r-1)\Big\} \\
&= \frac{1}{p(r-1)}\left(\frac{1-r^i}{1-r^N}\left(-cN^2 - N\frac{(cr+c)}{r-1} - N\right) + ci^2 + i\frac{(cr+c)}{r-1} + i\right),
\end{aligned}
$$

what was to be shown.

□

Remark 2.11. Note that for $p(n) = q(n) = 1/2$ we have $ET_{0:i:N} = iN\left(1 + \frac{2c}{3}N\right) - i^2\left(1 + \frac{2c}{3}i\right)$, *i.e.*, we obtained Proposition 2.1 from [Lef08].

Concerning the conditional game duration (because of (2.7) it is enough to provide formula only for $EW_{0:i:N}$) we have

Corollary 2.12. *Consider the spatially non-homogeneous gambler's ruin problem. We have*

$$
\begin{aligned}
r = 1: \quad EW_{0:i:N} &= \frac{(N^2-1)(cN+1)}{6p} - \frac{(i^2-1)(ci+1)}{6p}, \\
r \neq 1: \quad EW_{0:i:N} &= \frac{cN+1}{p(1-r)}\left(\frac{r+1}{r-1} - N\frac{r^N+1}{r^N-1}\right) - \frac{ci+1}{p(1-r)}\left(\frac{r+1}{r-1} - i\frac{r^i+1}{r^i-1}\right).
\end{aligned}
$$

Proof. Applying Corollary 2.7 we have:

- $r = 1$

$$
EW_{0:1:i} = \sum_{n=1}^{i-1} \frac{n/i}{p(n)}(i-n) = \frac{1}{p}\sum_{n=1}^{i-1}\frac{n}{i}(i-n)(2cn+1) = \frac{(i-1)(i+1)(ci+1)}{6p}.
$$

- $r \neq 1$

$$
\begin{aligned}
EW_{0:1:i} &= \sum_{n=1}^{i-1}\frac{(1-r^n)(1-r^{i-n})}{p(n)(1-r^i)(1-r)} = \frac{1}{p}\sum_{n=1}^{i-1}\frac{(1-r^n)(1-r^{i-n})}{(1-r^i)(1-r)}(2cn+1) \\
&= \frac{(ci+1)((r+1)(r^i-1) - i(r-1)(r^i+1))}{p(1-r^i)(1-r)^2} \\
&= \frac{ci+1}{p(1-r)}\left(\frac{r+1}{r-1} - i\frac{r^i+1}{r^i-1}\right).
\end{aligned}
$$

Applying (2.5), *i.e.*, $EW_{0:i:N} = EW_{0:1:N} - EW_{0:1:i}$, completes the proof. □

2.3. The classical case.

For constant winning/losing probabilities we recover known results (all given in Sarkar [Sar06]). We state them here for completeness and will indicate how they can be derived from our more general results.

Corollary 2.13. *Consider the gambler's ruin problem on $\mathbb{E} = \{0, 1, \ldots, N\}$ with constant winning/losing probabilities $p(i) = p, q(i) = q, i = 1, \ldots, N-1, p+q = 1$. We have*

$$\rho_{0:i:N} = \begin{cases} \frac{1-r^i}{1-r^N} & \text{if r} = 1, \\ \frac{i}{N} & \text{if r} \neq 1, \end{cases}$$

$$ET_{0:i:N} = \begin{cases} i(N-i) & \text{if r} = 1, \\ \frac{r+1}{r-1}\left(i - N\frac{r^i-1}{r^N-1}\right) & \text{if r} \neq 1, \end{cases}$$

$$EW_{0:i:N} = \begin{cases} \frac{1}{3}(N-i)(N+i) & \text{if r} = 1, \\ \frac{r+1}{r-1}\left[N\frac{r^N+1}{r^N-1} - i\frac{r^i+1}{r^i-1}\right] & \text{if r} \neq 1, \end{cases}$$

$$EB_{0:i:N} = \begin{cases} \frac{1}{3}i(2N-i) & \text{if r} = 1, \\ \frac{r+1}{r-1}\left[N\frac{r^N+1}{r^N-1} - (N-i)\frac{r^{N-i}+1}{r^{N-i}-1}\right] & \text{if r} \neq 1, \end{cases}$$

Results for $ET_{0:i:N}$ follows from Corollary 2.4 (case $r = 1$); $EW_{0:i:N}$ from Corollary 2.7 eq. (2.11) followed by (2.5); $EB_{0:i:N}$ follows from results on $EW_{0:i:N}$ and Theorem 2.3 (eq. (2.7)).

2.4. Example

Fix an integer N and some $p, q > 0$. Consider a gambler's ruin problem with rates

$$p(i) = \frac{p(1+\alpha_1 i)}{2ci+1}, \quad q(i) = \frac{q(1+\alpha_2 i)}{2ci+1},$$

with fixed $\alpha_1, \alpha_2, c \geq 0$ such that $p(i), q(i) > 0, p(i)+q(i) \leq 1, i \in \{1, ..., N\}$. We want to calculate $EW_{0:1:N}$.

2.4.1. $N = 3$

We have

$$\mathbf{p} = \left(0, \frac{p(1+\alpha_1)}{2c+1}, \frac{p(1+2\alpha_1)}{2c+1}, 0\right), \quad \mathbf{q} = \left(0, \frac{q(1+\alpha_2)}{2c+1}, \frac{q(1+2\alpha_2)}{2c+1}, 0\right).$$

Note that in general (for $\alpha_1 \neq \alpha_2$) $r(n) = \frac{q(n)}{p(n)} = \frac{q}{p}\frac{(1+\alpha_2 n)}{(1+\alpha_1 n)}$ is non-constant, thus we will apply Theorem 2.3. Eq. (2.6) takes form

$$EW_{0:1:3} = \sum_{n=1}^{2} \frac{\rho_{0:n:3}}{p(n)} \sum_{s=0}^{\lfloor (2-n)/2 \rfloor} \xi_s^{n+1,2} = \frac{\rho_{0:1:3}}{p(1)}\xi_0^{2,2} + \frac{\rho_{0:2:3}}{p(2)}\xi_0^{3,2}.$$

We need winning probabilities $\rho_{0:1:3}$ and $\rho_{0:2:3}$, which can be calculated from Theorem 2.1:

$$\rho_{0:i:3} = \frac{\sum_{n=1}^{i}\prod_{s=1}^{n-1} r(s)}{\sum_{n=1}^{3}\prod_{s=1}^{n-1} r(s)} = \frac{1+(i-1)r(1)}{1+r(1)+r(1)r(2)} = \frac{1+(i-1)\frac{q}{p}\frac{1+\alpha_2}{1+\alpha_1}}{1+\frac{q}{p}\frac{1+\alpha_2}{1+\alpha_1}+\frac{q^2}{p^2}\frac{(1+\alpha_2)(1+2\alpha_2)}{(1+\alpha_1)(1+2\alpha_1)}} =: \frac{1+(i-1)\frac{q}{p}\frac{1+\alpha_2}{1+\alpha_1}}{\gamma(p,q,\alpha_1,\alpha_2)}.$$

We also need $\xi_0^{2,2}$ and $\xi_0^{3,2}$. We have $\mathbf{j}_0^{2,2} = \mathbf{j}_0^{3,2} = \{\emptyset\}$, thus

$$\xi_0^{2,2} = \delta_{\mathbf{j}}^{2,2} = 1 + r(2) = 1 + \frac{q}{p}\frac{1+2\alpha_2}{1+2\alpha_1}, \quad \xi_0^{3,2} = \delta_{\mathbf{j}}^{3,2} = 1$$

(in the latter the second product was 1, since $\{3,\ldots,2\} \equiv \emptyset$).

Finally,

$$EW_{0:1:3} = \frac{1}{p\gamma(p,q,\alpha_1,\alpha_2)}\left[\frac{2c+1}{1+\alpha_1}\left(1+\frac{q}{p}\frac{(1+2\alpha_2)}{(1+2\alpha_1)}\right) + \left(1+\frac{q}{p}\frac{(1+\alpha_2)}{(1+\alpha_1)}\right)\frac{4c+1}{1+2\alpha_1}\right]. \tag{2.12}$$

Special cases:

- $\alpha_1 = \alpha_2 = \alpha$. Then (2.12) reduces to

$$EW_{0:1:3} = \frac{1+\frac{q}{p}}{p\left(1+\frac{q}{p}+\frac{q^2}{p^2}\right)}\left(\frac{2c+1}{1+\alpha}+\frac{4c+1}{1+2\alpha}\right). \tag{2.13}$$

  Note that in this case $r(n) = \frac{q}{p}$ is constant, thus (2.13) could be derived in an easier way using Corollary 2.7:

$$r = 1: \quad EW_{0:1:3} = \sum_{n=1}^{2}\frac{n}{3}\frac{2cn+1}{p(1+\alpha_1 n)}(3-n) = \frac{2}{3p}\left(\frac{2c+1}{1+\alpha}+\frac{4c+1}{1+2\alpha}\right),$$

$$r \neq 1: \quad EW_{0:1:3} = \sum_{n=1}^{2}\frac{\frac{1-r^n}{1-r^3}(1-r^{3-n})}{(1-r)}\frac{2cn+1}{p(1+\alpha_1 n)} = \frac{1-r^2}{p(1-r^3)}\left(\frac{2c+1}{1+\alpha}+\frac{4c+1}{1+2\alpha}\right),$$

  what is equivalent to (2.13) in both cases. *Note* also that this is not a spatially non-homogeneous case as long as $\alpha > 0$.

- $\alpha_1 = \alpha_2 = 0$. Then (2.12) (and thus (2.13)) reduces to

$$EW_{0:1:3} = \frac{2\left(1+\frac{q}{p}\right)}{p\left(1+\frac{q}{p}+\frac{q^2}{p^2}\right)}\left(\frac{3c+1}{1+\alpha}\right). \tag{2.14}$$

  Note that this is a spatially non-homogeneous case, thus (2.14) could be derived from Corollary 2.12 (we skip the calculations).

- $\alpha_1 = \alpha_2 = 0$ and $c = 0$, then (2.14) reduces to

$$EW_{0:1:3} = \frac{2\left(1+\frac{q}{p}\right)}{p\left(1+\frac{q}{p}+\frac{q^2}{p^2}\right)}.$$

  This situation corresponds to a gambler's ruin problem with constant birth and death rates. In particular, for $p = q = 1/2$ we have $EW_{0:1:3} = \frac{8}{3}$ what agrees with Example 1 in [Len09].

2.4.2. General $N \geq 3, p = q$ and $\alpha_2 = \alpha_1 = 1$

We thus have $p(i) = \frac{p(1+i)}{2ci+1}, q(i) = \frac{q(1+i)}{2ci+1}$. This is constant $r(n) = \frac{q(n)}{p(n)} = \frac{q}{p} = 1$ case, which is however not spatially non-homogeneous. We skip the lengthy calculations, but we can obtain $EW_{0:1:N}$ from Corollary 2.7 (H_N is the N-th harmonic number):

$$\begin{aligned} EW_{0:1:N} &= \sum_{n=1}^{N-1} \frac{n(N-n)(2cn+1)}{pN(1+n)} \\ &= \tfrac{1}{p}\left(\tfrac{c}{3}(N-5)(N+2) + \tfrac{1}{2}(3+N)\right) + \tfrac{1}{Np}(2c-1)(1+N)H_N = \tfrac{c}{3p}N^2 + \Theta(N), \end{aligned}$$

which for $p(i) = p(1+i), q(i) = q(1+i)$ (*i.e.*, for $c = 0$) simplifies to

$$EW_{0:1:N} = \frac{N+3}{2p} - \frac{1}{Np}(N+1)H_N = \frac{N}{2p} + \Theta(log(N)).$$

2.4.3. General $N \geq 3, p = q$ and $\alpha_2 = \alpha_1 = \alpha$

$$\begin{aligned} EW_{0:1:i} &= \sum_{n=1}^{i-1} \frac{n(i-n)(2cn+1)}{pi(1+\alpha n)} \\ &= \frac{1}{6\alpha^4 pi}\left(\alpha(i-1)(\alpha^2 i(2c(i+1)+3) + \alpha(6-6ci) - 12c) + \right. \\ &\qquad \left. 6(\alpha - 2c)(\alpha i + 1)\left[\psi\left(1+\tfrac{1}{\alpha}\right) - \psi\left(i + \tfrac{1}{\alpha}\right)\right]\right), \end{aligned}$$

where ψ is a digamma function. It is known that $\psi(m) = H_{m-1} - \gamma$, where $\gamma = 0.5772156...$ is a known Euler–Mascheroni constant. Let us assume that $\alpha = \frac{1}{m}$ and m is an integer. Then $\psi\left(1 + \frac{1}{\alpha}\right) - \psi\left(i + \frac{1}{\alpha}\right) = H_m - H_{i+m-1}$.

3. Random walk on a polygon

Fix an integer $m \geq 2$. Let

$$\mathbf{p} = (p(0), p(1), \ldots, p(m)), \quad \mathbf{q} = (q(0), q(1), \ldots, q(m)),$$

where $p(i), q(i) > 0, p(i) + q(i) \leq 1$ for $i \in \{0 \ldots, m\}$. Consider the following random walk $\mathbf{X} \equiv \{X_t\}_{t \in \mathbb{N}}$ on $\mathbb{E} = Z_{m+1}$. Being in state i we move to the state $i+1$ with probability $p(i)$, we move to the state $i-1$ with probability $q(i)$, and we do nothing with the remaining probability. Throughout the paper, in the context of a random walk on a polygon, all additions and substractions are performed modulo $m+1$. We will refer to this walk as to a *random walk on a polygon*. The notation intentionally resembles that of gambler's ruin problem. Throughout the section we consider fixed $\mathbf{p}, \mathbf{q}$ and $m \geq 2$ (and omit subscripts $\mathbf{p}, \mathbf{q}$ in random variables below). We are interested in:

$$\begin{aligned} A_i &= \{X : X_0 = i, X_n = i, \forall_{0<t<n} X_t \neq i, \forall_{k \in \mathbb{E}} \exists_{0 \leq t \leq n} X_t = k\} \\ L_{i,j} &= \{X : X_0 = i, X_n = j, \forall_{0<t<n} X_t \neq j, \forall_{k \in \mathbb{E}} \exists_{0 \leq t \leq n} X_t = k\} \\ V_{i,j} &= \inf\{n \geq 1 : X_0 = i, X_n = j, \forall_{k \in \mathbb{E}} \exists_{0 \leq t \leq n} X_t = k\} \\ V_i &= \inf\{n \geq 1 : X_0 = i, \forall_{k \in \mathbb{E}} \exists_{0 \leq t \leq n} X_t = k\} \\ R_i &= \inf\{n_2 \geq 1 : X_0 = i, X_{n_1+n_2} = i, n_1 = \inf\{n \geq 1 : \forall_{k \in \mathbb{E}} \exists_{0 \leq t \leq n} X_t = k\}\} \end{aligned}$$

In other words: A_i is the event that the process starting at i will return for the first time to i after all other vertices are visited; $L_{i,j}$ is the event that the process starting at i will reach for the first time state j after visiting all other vertices; $V_{i,j}$ is the number of steps of the process starting at i to reach for the first time state j after visiting all other vertices; V_i is the number of steps of the process starting at i needed to visit all vertices; R_i is the number of additional steps for the process starting at i needed to reach i after visiting all the vertices.

For $j \preceq i \preceq k$, where $\preceq$ is a cyclic order, *i.e.,* $j \leq i \leq k$ or $i \leq k \leq j$ or $k \leq j \leq i$, let $G(\mathbf{p}, \mathbf{q}, j, i, k)$ denote a gambler's ruin game with i being a starting state, j being a losing state and k being a winning state. Note that independently of j, i, k, winning and losing probabilities $\mathbf{p}, \mathbf{q}$ are fixed.

Notation. In contrast to a usual notation neither $\sum_{k=s}^{t} a_k = 0$ nor $\prod_{k=s}^{t} a_k = 1$ for $t < s - 1$. Since we are considering operations in Z_{m+1}, we define

$$\text{For } t < s \leq m, s - t > 1: \quad \sum_{k=s}^{t} a_k := a_s + a_{s+1} + \ldots + a_m + a_0 + \ldots + a_t,$$

$$\prod_{k=s}^{t} a_k := a_s \cdot a_{s+1} \cdot \ldots \cdot a_m \cdot a_0 \cdot \ldots \cdot a_t,$$

$$\text{For } s = t + 1 \bmod m + 1: \quad \sum_{k=s}^{t} a_k = 0 \qquad \prod_{k=s}^{t} a_k := 1.$$

In all other cases we use usual sums and products. Using this notation, we are ready to state our results.

Theorem 3.1. *Consider the random walk on a polygon described above. We have*

$$P(A_i) = \frac{1}{1 + r(i)} \left(\frac{1}{\sum_{n=i+1}^{i-1} \prod_{s=i+1}^{n-1} r(s)} + \frac{1}{\sum_{n=i+2}^{i} \prod_{s=n}^{i} \left(\frac{1}{r(s)}\right)} \right) \tag{3.1}$$

$$P(L_{i,j}) = \frac{1}{\sum_{n=j+2}^{j-1} \prod_{s=j+2}^{n-1} r(s)} \left(\frac{\sum_{n=i+1}^{j-1} \prod_{s=j+2}^{n-1} r(s)}{\sum_{n=j+1}^{j-1} \prod_{s=j+1}^{n-1} r(s)} + \frac{\sum_{n=j+2}^{i} \prod_{s=j+2}^{n-1} r(s)}{\sum_{n=j+2}^{j} \prod_{s=n}^{j-1} \frac{1}{r(s)}} \right) \tag{3.2}$$

$$\begin{aligned} EV_{i,j} = \; & \rho_{j+1:i:j-1} \left(EW_{j+1:i:j-1} + EB_{j+1:j-1:j} + ET_{j:j+1:j} \right) \\ & + (1 - \rho_{j+1:i:j-1}) \left(EB_{j+1:i:j-1} + EW_{j:j+1:j-1} + ET_{j:j-1:j} \right) \end{aligned} \tag{3.3}$$

$$EV_i = \sum_{j=i+1}^{i-1} P(L_{i,j}) EV_{i,j} \tag{3.4}$$

$$ER_i = \sum_{k=i+1}^{i-1} P(L_{i,k}) ET_{i:k:i} \tag{3.5}$$

The proof of Theorem 3.1 is postponed to Section 5.2.1.

Constant $r(n) = r = \frac{q(n)}{p(n)}$..

In this case the starting point does not matter, we consider $i = 0$. Note that $P(A_i)$ and $P(L_{i,j})$ depend on $p(n)$ and $q(n)$ only through $r(n)$, thus they must reduce to known results for constant birth $p(n) = p$ and death $q(n) = q$ rates (see (3.1) and (3.3) in [Sar06]). Indeed, substituting $r(n) = r$ to (3.1) and (3.2) yields

Corollary 3.2. *Consider the random walk on polygon with constant* $r(n) = \frac{q(n)}{p(n)}$*, then we have*

$$P(A_0) = \begin{cases} \frac{1}{m} & \text{if } r = 1, \\ \frac{r-1}{r+1}\frac{r^m+1}{r^m-1} & \text{if } r \neq 1, \end{cases}$$

$$P(L_{0,j}) = \begin{cases} \frac{1}{m} & \text{if } r = 1, \\ \frac{r^{m-j}(r-1)}{r^m-1} & \text{if } r \neq 1. \end{cases}$$

We skip the formulas for $EV_{0,j}$, EV_0 and ER_0 in this case, noting that they can be derived from Corollaries 2.4 and 2.7.

Constant $q(n) = q, p(n) = p$.
First, let us recall formulas for EV_0, ER_0 for the case $p + q = 1$.

Corollary 3.3. *[Sar06] Consider the random walk on a polygon with constant* $q(n) = q, p(n) = p, p+q = 1$*. We have*

$$EV_0 = \begin{cases} \frac{m(m+1)}{2} & \text{if } r = 1, \\ \frac{r+1}{r-1}\left[m - \frac{1}{r-1} - \frac{m^2}{r^m-1} + \frac{(m+1)^2}{r^{m+1}-1}\right] & \text{if } r \neq 1, \end{cases}$$

$$ER_0 = \begin{cases} \frac{1}{6}(m+1)(m+2) & \text{if } r = 1, \\ \frac{r+1}{r-1}\left[\frac{r}{r-1} - \frac{m(m+2)}{r^m-1} + \frac{(m+1)^2}{r^{m+1}-1}\right] & \text{if } r \neq 1, \end{cases}$$

In the case $p+q \leq 1$ note that $EB_{j:i:k} = \frac{1}{p(1+r)}EB^1_{j:i:k}$, $EW_{j:i:k} = \frac{1}{p(1+r)}EW^1_{j:i:k}$, $ET_{j:i:k} = \frac{1}{p(1+r)}ET^1_{j:i:k}$, where superscript 1 denotes the case $p + q = 1$. Thus Theorem 3.1 implies $EV_0 = EV_0^1, ER_0 = ER_0^1$, *i.e.*, we have

Corollary 3.4. *Consider the random walk on a polygon with constant* $q(n) = q, p(n) = p$*. We have*

$$EV_0 = \begin{cases} \frac{m(m+1)}{4p} & \text{if } r = 1, \\ \frac{1}{p(r-1)}\left[m - \frac{1}{r-1} - \frac{m^2}{r^m-1} + \frac{(m+1)^2}{r^{m+1}-1}\right] & \text{if } r \neq 1, \end{cases}$$

$$ER_0 = \begin{cases} \frac{1}{12p}(m+1)(m+2) & \text{if } r = 1, \\ \frac{1}{p(r-1)}\left[\frac{r}{r-1} - \frac{m(m+2)}{r^m-1} + \frac{(m+1)^2}{r^{m+1}-1}\right] & \text{if } r \neq 1, \end{cases}$$

4. Fastest Strong Stationary Time for a symmetric random walk on a circle

Consider an ergodic Markov chain $\mathbf{X} = \{X_k\}_{k\geq 0} \sim (\nu, \mathbf{P}_X)$ on a finite state space $\mathbb{E} = \{\mathbf{e}_1, \ldots, \mathbf{e}_M\}$ with a stationary distribution π, initial distribution ν and a transition matrix $\mathbf{P}_X$. We are interested in measuring nonstationarity of $\mathbf{X}_k$ via **separation "distance"**

$$sep(\nu\mathbf{P}_X^k, \pi) = \max_{\mathbf{e}\in\mathbb{E}} \left(1 - \frac{\nu\mathbf{P}_X^k(\mathbf{e})}{\pi(\mathbf{e})}\right).$$

Note that it is not symmetric, that is why it is not an actual distance, however it is an upper bound on a **total variation distance** $d_{TV}(\nu \mathbf{P}^k_X, \pi) = 1/2 \sum_{\mathbf{e}\in\mathbb{E}} |P(X_k = \mathbf{e}) - \pi(\mathbf{e})|$.
A random variable T is a **strong stationary time** (SST) T for $\mathbf{X}$ if it is a randomized stopping time for $\mathbf{X}$ such that

$$\forall(\mathbf{e} \in \mathbb{E}) \ P(X_k = \mathbf{e}|T = k) = \pi(\mathbf{e}).$$

The notion of separation distance fits perfectly into a notion of SST, in [AD87] it is shown that for an SST T we have

$$sep(\nu \mathbf{P}^k_X, \pi) \le P(T > k).$$

We say that T is a **fastest strong stationary time** (FSST) if $sep(\nu \mathbf{P}^k_X, \pi) = P(T > k)$.

In this section we consider a symmetric random walk on a polygon with constant rates $p(i) = q(i) = p$ on d points (*i.e.,* $m = d-1$). Moreover, we will refer to the random walk as to a *symmetric random walk on a circle* (to be consistent with [DF90], we will compare our result to a result from this article) on $\mathbb{Z}_d$, *i.e.,* $\{0, \ldots, d-1\}$. We will show a construction of a fastest strong stationary time for this symmetric random walk on a circle, moreover we have

Lemma 4.1. *For the fastest strong stationary time T for a symmetric random walk on a circle with $d = 2N$ we have*

$$ET = \begin{cases} \dfrac{2N^2+1}{12p} & \text{for } p \in (0, 1/3] \text{ and } N > 1, \\[2mm] \dfrac{1}{4p} & \text{for } p \in (0, 1/4] \text{ and } N = 1, \\[2mm] \dfrac{1}{2(1-2p)} & \text{for } p \in (1/4, 1/2) \text{ and } N = 1. \end{cases}$$

Remark 4.2. *A construction of a strong stationary time for a symmetric random walk on a circle with $p = 1/3$ is presented in [DF90]. For $d = 2^a, a > 1$ their construction yields an SST T_0 such that*

$$ET_0 = \frac{3}{2} 2^{2a} \left(2^{-4} + 2^{-6} + \cdots + 2^{-2(a-1)} + 2 \times 2^{-2a}\right) = \frac{1}{8} d^2 + 1$$

(see the bottom of the page 1484 in [DF90]), whereas Lemma 4.1 states that a fastest strong stationary time T fulfills ($N = d/2$)

$$ET = \frac{1}{8} d^2 + \frac{1}{4},$$

what means that a construction from [DF90] does not yield a fastest strong stationary time (authors mention this fact in their Example 3.1). Note that ET and ET_0 differ by $\frac{3}{4}$ (independently of d).

Strong stationary duality. For a general description of a strong stationary duality see [DF90] (total ordering and set-valued chains) and [LS12], [Lor18] (general partial ordering). Here we will describe this duality for chains on the same state space. Let both $\mathbf{X} \sim (\nu, \mathbf{P}_X)$ and $\mathbf{X}^* \sim (\nu^*, \mathbf{P}^*_X)$ be chains on $\mathbb{E} = \{\mathbf{e}_1, \ldots, \mathbf{e}_M\}$, chain $\mathbf{X}$ is ergodic with a stationary distribution π, whereas $\mathbf{X}^*$ is an absorbing chain with a unique absorbing state $\mathbf{e}_M$. We say that a stochastic matrix of size $d \times d$ is a **link** if $\Lambda(\mathbf{e}_M, \mathbf{e}) = \pi(\mathbf{e})$ for all $\mathbf{e} \in \mathbb{E}$. We say that $\mathbf{X}^*$ is a **strong stationary dual** of $\mathbf{X}$ with the link Λ if

$$\nu = \nu^* \Lambda \quad \text{and} \quad \Lambda \mathbf{P}_X = \mathbf{P}^*_X \Lambda. \tag{4.1}$$

Diaconis and Fill [DF90] proved that the absorption time T^* of $\mathbf{X}^*$ is an SST for $\mathbf{X}$. If the corresponding T^* is an FSST for $\mathbf{X}$, then the chain $\mathbf{X}^*$ is called a **sharp SSD**.

Fix some partial ordering $\preceq$ on $\mathbb{E}$, such that $\mathbf{e}_1$ is the minimum and $\mathbf{e}_M$ is the maximum. Let $\mathbf{C}(\mathbf{e}_i, \mathbf{e}_j) = \mathbf{1}(\mathbf{e}_i \preceq \mathbf{e}_j)$ be the corresponding *ordering matrix* (always invertible, the inverse $\mathbf{C}^{-1}$ is called the Möbius matrix). Assume that $\nu(\mathbf{e}_1) = 1$ (*i.e.,* chain $\mathbf{X}$ starts in $\mathbf{e}_1$), then (4.1) implies that also $\nu^*(\mathbf{e}_1) = 1$. Let $\overleftarrow{\mathbf{P}}_X$ be a transition matrix of a time reversed chain, *i.e.,* $\overleftarrow{\mathbf{P}}_X(\mathbf{e}_i, \mathbf{e}_j) = \frac{\pi(\mathbf{e}_j)}{\pi(\mathbf{e}_i)} \mathbf{P}_X(\mathbf{e}_j, \mathbf{e}_i)$. We have

Theorem 4.3 (Theorem 2 in [LS12], Remark 2.2 in [LS16], simplified version). *Let $\mathbf{X} \sim (\nu, \mathbf{P}_X)$ be an ergodic Markov chain on a finite state space $\mathbb{E} = \{\mathbf{e}_1, \ldots, \mathbf{e}_M\}$ starting at $\mathbf{e}_1$ (i.e., $\nu(\mathbf{e}_1) = 1$), with a stationary distribution π, partially ordered by $\preceq$ (with ordering matrix $\mathbf{C}$), with $\mathbf{e}_1$ being the minimum and $\mathbf{e}_M$ being the maximum. Assume that $\mathbf{C}^{-1}\overleftarrow{\mathbf{P}}_X\mathbf{C}$ is a non-negative matrix. Then there exists a sharp SSD $\mathbf{X}^* \sim (\nu^*, \mathbf{P}^*_X)$ on $\mathbb{E}$ with $\nu^*(\mathbf{e}_1) = 1$ and transitions*

$$\mathbf{P}^*_X(\mathbf{e}_i, \mathbf{e}_j) = \frac{H(\mathbf{e}_j)}{H(\mathbf{e}_i)} \left(\mathbf{C}^{-1}\overleftarrow{\mathbf{P}}_X\mathbf{C}\right)(\mathbf{e}_j, \mathbf{e}_i) \tag{4.2}$$

with a unique absorbing state $\mathbf{e}_M$, where $H(\mathbf{e}) = \sum_{\mathbf{e}' \preceq \mathbf{e}} \pi(\mathbf{e})$.

Remark 4.4. *The condition that $\mathbf{C}^{-1}\overleftarrow{\mathbf{P}}_X\mathbf{C}$ is a non-negative matrix was called ${}^{\downarrow}$-Möbius monotonicity in [LS12].*

Proof of Lemma 4.1. First, we will construct a sharp SSD for this symmetric random walk on a circle using Theorem 4.3. It will be more convienient to work with states numerated as $1^*, 2^*, \ldots, d^* = (2N)^*$ (instead of $0, 1, \ldots, 2N-1$). Our walk $\mathbf{X}$ moves either right or left, or it does not move, i.e., it has the transition matrix:

$$\mathbf{P}_X(i^*, j^*) = \begin{cases} 1-2p & \text{if } \; j^* = i^*, \\ p & \text{if } \; (j^* = (i+1)^*, i^* \neq (2N)^*) \;\vee\; (j^* = (i-1)^*, i^* \neq 1^*) \\ & \quad \vee\, (j^* = 1, i^* = (2N)^*) \vee (j^* = (2N)^*, i^* = 1). \end{cases}$$

It will be even more convienient to work with another enumeration of states. Consider a set of states $\{1, \ldots, 2N\}$ and let us define a bijection between this set and the set $\{1^*, \ldots, (2N)^*\}$ in the following way:

$$\sigma(i^*) = \begin{cases} 2i-1 & \text{if } \; i \leq N, \\ 2(2N-i+1) & \text{if } \; i > N. \end{cases}, \qquad \sigma^{-1}(i) = \begin{cases} \left(\frac{i+1}{2}\right)^* & \text{if } \quad \text{is odd,} \\ \left(2N - \frac{i}{2} + 1\right)^* & \text{if } \quad \text{is even.} \end{cases}$$

The bijection for $d = 2N = 8$ is following

$$\begin{aligned} \sigma((1^*, 2^*, 3^*, 4^*, 5^*, 6^*, 7^*, 8^*)) &= (1, 3, 5, 7, 8, 6, 4, 2), \\ \sigma^{-1}((1, 2, 3, 4, 5, 6, 7, 8)) &= (1^*, 8^*, 2^*, 7^*, 3^*, 6^*, 4^*, 5^*), \end{aligned}$$

it is depicted in Fig. 1 (left). The transition matrix of the chain $\mathbf{X}$ can be rewritten as:

$$\mathbf{P}_X(i, j) = \begin{cases} 1-2p & \text{if } \; i = j, \\ p & \text{if } \; |i-j| = 2 \;\vee\; (i=1, j=2) \;\vee\; (i=2, j=1) \;\vee \\ & \quad (i = 2N-1, j = 2N) \;\vee\; (i = 2N, j = 2N-1). \end{cases}$$

Continuing our example $d = 2N = 8$ we have (using enumeration of states $1, 2, \ldots, 2N$))

$$\mathbf{P}_X = \begin{bmatrix} 1-2p & p & p & 0 & 0 & 0 & 0 & 0 \\ p & 1-2p & 0 & p & 0 & 0 & 0 & 0 \\ p & 0 & 1-2p & 0 & p & 0 & 0 & 0 \\ 0 & p & 0 & 1-2p & 0 & p & 0 & 0 \\ 0 & 0 & p & 0 & 1-2p & 0 & p & 0 \\ 0 & 0 & 0 & p & 0 & 1-2p & 0 & p \\ 0 & 0 & 0 & 0 & p & 0 & 1-2p & p \\ 0 & 0 & 0 & 0 & 0 & p & p & 1-2p \end{bmatrix}$$

We will now compute an SSD chain using total ordering $1 < 2 < \ldots < 2N$. Mapping the total ordering $1 < 2 < \ldots 2N$ into the ordering on original states $1^*, 2^*, \ldots, (2N)^*$, we have

$$i^* \prec j^* \Leftrightarrow \sigma(i^*) < \sigma(j^*),$$

Note that $\prec$ is also a total ordering, we have

$$1^* \prec (2N)^* \prec 2^* \prec (2N-1)^* \prec \cdots \prec (N+1)^*.$$

We will thus work with total ordering $1 < 2 \ldots < 2N$ – which is equivalent (with easier notation) to working with $1^* \prec (2N)^* \prec \ldots \prec (N+1)^*$.

The ordering matrix for total ordering is $\mathbf{C}(i,j) = \mathbf{1}(i \leq j)$, the Möbius matrix (*i.e.,* the inverse of $\mathbf{C}$) is then following:

$$\mathbf{C}^{-1}(i,j) = \begin{cases} 1 & \text{if} \quad i = j, \\ -1 & \text{if} \quad i = j-1, i < 2N. \end{cases}$$

We have

$$H(i) = \sum_{j \leq i} \pi(j) = \sum_{j \leq i} \frac{1}{2N} = \frac{i}{2N}. \tag{4.3}$$

Using above derivations and the fact that the chain is reversible ($\overleftarrow{\mathbf{P}}_X = \mathbf{P}_X$), for $i < 2N$ we have:

$$\begin{aligned} \left(\mathbf{C}^{-1}\overleftarrow{\mathbf{P}}_X\mathbf{C}\right)(i,j) &= \left(\mathbf{C}^{-1}\mathbf{P}_X\mathbf{C}\right)(i,j) = \sum_l \mathbf{C}^{-1}(i,l) \sum_{k \leq j} \mathbf{P}_X(l,k) \\ &= \sum_{k \leq j} \mathbf{P}_X(i,k) - \mathbf{P}_X(i+1,k) \\ &= \mathbf{P}_X(i,j) + \left(\sum_{k<j} \mathbf{P}_X(i,k) - \mathbf{P}_X(i+1,k+1) \right) - \mathbf{P}_X(i+1,1), \end{aligned}$$

whereas for $i = 2N$ we have

$$\left(\mathbf{C}^{-1}\overleftarrow{\mathbf{P}}_X\mathbf{C}\right)(i,j) = \sum_{k \leq j} \mathbf{P}_X(2N,k) = \begin{cases} 0 & \text{if} \quad j < 2N-2, \\ p & \text{if} \quad j = 2N-2, \\ 2p & \text{if} \quad j = 2N-1, \\ 1 & \text{if} \quad j = 2N, \end{cases}$$

We also have:

$$\mathbf{P}_X(i,k) - \mathbf{P}_X(i+1,k+1) = \begin{cases} p & \text{if} \quad (i=1, j=2) \ \vee \ (i=2, j=1), \\ -p & \text{if} \quad (i=2N-1, j=2N-2) \ \vee \ (i=2N-2, j=2N-1). \end{cases}$$

Using above derivations we can easily calculate all the cases:

$$\left(\mathbf{C}^{-1}\overleftarrow{\mathbf{P}}_X\mathbf{C}\right)(i,j) = \begin{cases} 1-2p & \text{if} \quad 1 < i = j < 2N-1, \\ 1-3p & \text{if} \quad i = j = 1 \ \vee \ i = j = 2N-1, \\ 1 & \text{if} \quad i = j = 2N, \\ p & \text{if} \quad |i-j| = 2, \ j \neq 2N, \\ 2p & \text{if} \quad i = 2N, \ j = 2N-1. \end{cases} \tag{4.4}$$

Continuing our example $d = 2N = 8$ we have (again, using enumeration $1, 2, \ldots, 8$)

$$\mathbf{C}^{-1}\overleftarrow{\mathbf{P}}_X\mathbf{C} = \begin{bmatrix} 1-3p & 0 & p & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-2p & 0 & p & 0 & 0 & 0 & 0 \\ p & 0 & 1-2p & 0 & p & 0 & 0 & 0 \\ 0 & p & 0 & 1-2p & 0 & p & 0 & 0 \\ 0 & 0 & p & 0 & 1-2p & 0 & p & 0 \\ 0 & 0 & 0 & p & 0 & 1-2p & 0 & 0 \\ 0 & 0 & 0 & 0 & p & 0 & 1-3p & 0 \\ 0 & 0 & 0 & 0 & 0 & p & 2p & 1 \end{bmatrix}.$$

Combining (4.4) with (4.3) and noting that $\frac{H(i)}{H(j)} = \frac{i}{j}$, Theorem 4.3 and yields the following transitions of a sharp SSD chain $\mathbf{X}^*$ (written using the original enumeration of states)

$$\mathbf{P}^*_X(i^*, j^*) = \frac{H(j^*)}{H(i^*)}\left(\mathbf{C}^{-1}\overleftarrow{\mathbf{P}}_X\mathbf{C}\right)(j^*, i^*) = \frac{H(\sigma(j^*))}{H(\sigma(i^*))}\left(\mathbf{C}^{-1}\overleftarrow{\mathbf{P}}_X\mathbf{C}\right)(\sigma(j^*), \sigma(i^*)),$$

thus

$$\mathbf{P}^*_X(i^*, j^*) = \begin{cases} 1-2p & \text{if} \quad 1 < \sigma(j^*) = \sigma(i^*) < 2N-1, \\ 1-3p & \text{if} \quad \sigma(j^*) = \sigma(i^*) = 1 \vee \sigma(j^*) = \sigma(i^*) = 2N-1, \\ 1 & \text{if} \quad \sigma(j^*) = \sigma(i^*) = 2N, \\ p\frac{\sigma(j^*)}{\sigma(i^*)} & \text{if} \quad |\sigma(j^*) - \sigma(i^*)| = 2, \sigma(i^*) \neq 2N, \\ 2p\frac{\sigma(j^*)}{\sigma(i^*)} & \text{if} \quad \sigma(j^*) = 2N, \sigma(i^*) = 2N-1. \end{cases}$$

We leave it to the reader to check that the condition $|\sigma(j^*) - \sigma(i^*)| = 2$ for $j, i \leq N$ or $j, i > N$ is equivalent to $|j - i| = 1$, whereas for $j \leq N, i > N$ or for $i \leq N, j > N$ the condition is never met. Thus, the transition matrix of $\mathbf{X}^*$ can rewritten in the following way, using ordering $\prec$:

$$\mathbf{P}^*_X(i^*, j^*) = \begin{cases} 1-2p & \text{if} \quad j^* = i^*, 2^* \preceq i^* \prec N \text{ or } (N+1)^* \prec i^* \preceq (2N)^*, \\ 1-3p & \text{if} \quad j^* = i^*, i^* \in \{1^*, N^*\}, \\ 1 & \text{if} \quad j^* = i^* = (N+1)^*, \\ \frac{(2i+1)p}{2i-1} & \text{if} \quad j = i+1, 1^* \preceq i^* \prec N^*, \\ \frac{(2i-3)p}{2i-1} & \text{if} \quad j = i-1, 1^* \prec i^* \preceq N^*, \\ \frac{(2N-i)p}{2N-i+1} & \text{if} \quad j = i+1, (N+2)^* \preceq i^* \prec (2N)^*, \\ \frac{(2N-i+2)p}{2N-i+1} & \text{if} \quad j = i-1, (N+2)^* \preceq i^* \preceq (2N)^*, \\ \frac{4Np}{2N-1} & \text{if} \quad i^* = N^*, j^* = (N+1)^*. \end{cases}$$

- First, let us consider case $p \in (0, 1/3]$ and $N > 1$.

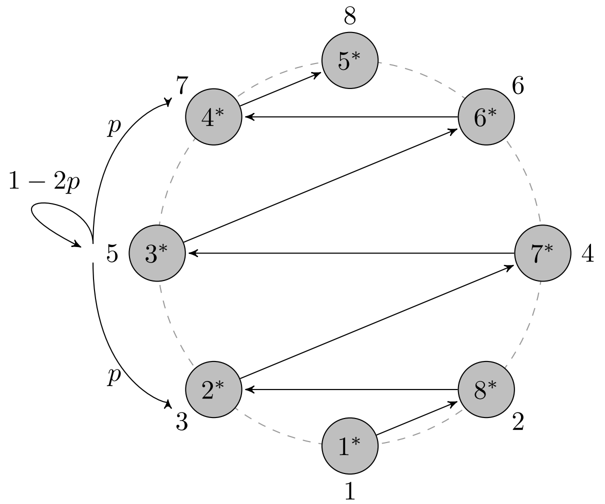
  Note that the assumption $p \in (0, 1/3]$ implies that $\mathbf{P}^*_X$ is a transition matrix. Continuing the example $d = 2N = 8$, we have (using the enumeration $1^*, 2^*, \ldots, (2N)^*$)

$$\mathbf{P}^*_X = \begin{bmatrix} 1-3p & 3p & 0 & 0 & 0 & 0 & 0 & 0 \\ p/3 & 1-2p & 5/3p & 0 & 0 & 0 & 0 & 0 \\ 0 & 3/5p & 1-2p & 7/5p & 0 & 0 & 0 & 0 \\ 0 & 0 & 5/7p & 1-3p & \frac{16p}{7} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4/3p & 1-2p & 2/3p & 0 \\ 0 & 0 & 0 & 0 & 0 & 3/2p & 1-2p & p/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2p & 1-2p \end{bmatrix}.$$

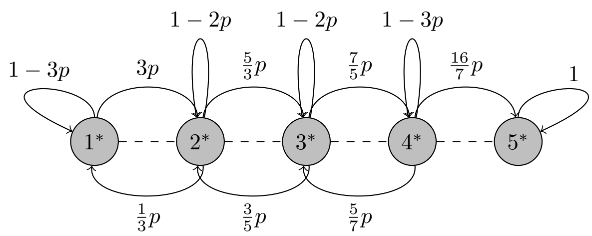
Note that the resulting chain (recall, it starts at 1^*) will never reach states $(N+2)^*, \ldots, (2N)^*$. Denote the resulting chain on $\{1^*, \ldots, (N+1)^*\}$ by $\mathbf{Y}^*$. This is a birth and death chain with a unique absorbing state $(N+1)^*$, let us write down the relevant transitions only

$$\mathbf{P}^*_Y(i^*, j^*) = \begin{cases} 1-3p & \text{if} \quad j = i, i \in \{1, N\}, \\ 1-2p & \text{if} \quad j = i, 2 \le i < N, \\ 1 & \text{if} \quad j = i = N+1, \\ \frac{(2i+1)p}{2i-1} & \text{if} \quad j = i+1, 1 \le i < N, \\ \frac{4Np}{2N-1} & \text{if} \quad i = N, j = N+1, \\ \frac{(2i-3)p}{2i-1} & \text{if} \quad j = i-1, 1 < i \le N. \end{cases}$$

For $d = 2N = 8$ the transitions are depicted in Fig. 1 (right).



X and the "Zig-zag" ordering



A sharp SSD $\mathbf{X}^*$

Figure 1: Case $d = 2N = 8$: "zig-zag" ordering and state space of $\mathbf{X}$ (left), the corresponding sharp SSD $\mathbf{Y}^*$ (right)

Since there is no confusion (in the chain $\mathbf{Y}^*$), we will identify a state i^* simply with i. Let $T \equiv T_{1:1:N+1}$ denote the absorption time (in $N+1$) of $\mathbf{Y}^*$ (starting at 1). Using Theorem 2.2 we have:

$$ET_{1:1:N+1} = \sum_{n=1}^{N} \left[d_n \sum_{s=1}^{n} \frac{1}{p(s) d_s} \right]. \tag{4.5}$$

Let us write a formula for $p(s)$ explicitly:

$$p(s) = \begin{cases} \frac{(2s+1)p}{2s-1} & \text{if} \quad i < N, \\ \frac{4Np}{2N-1} & \text{if} \quad i = N. \end{cases} \tag{4.6}$$

We need to compute $d(s)$. For $1 \le s < N$ we have

$$d_s = \prod_{i=2}^{s} \frac{q(i)}{p(i)} = \prod_{i=2}^{s} \frac{\frac{2i-3}{2i-1}}{\frac{2i+1}{2i-1}} = \prod_{i=2}^{s} \frac{2i-3}{2i+1} = \frac{3}{(2s-1)(2s+1)}$$

and for $s = N$ we have

$$d_N = \prod_{i=2}^{N} \frac{q(i)}{p(i)} = d_{N-1}\frac{q(N)}{p(N)} = \frac{3}{(2N-3)(2N-1)}\frac{\frac{2N-3}{2N-1}}{\frac{4N}{2N-1}} = \frac{3}{4N(2N-1)}.$$

Plugging above formulas for $p(s), d_s$ in (4.5) (and using a formula $\sum_{s=1}^{n}(2s-1)^2 = \frac{n(2n-1)(2n+1)}{3}$) we obtain for $N > 1$: $ET_{1:1:N+1} =$

$$\begin{aligned}
& \sum_{n=1}^{N}\left[d_n\sum_{s=1}^{n}\frac{1}{p(s)d_s}\right] \\
= & \sum_{n=1}^{N-1}\left[d_n\sum_{s=1}^{n}\frac{1}{p(s)d_s}\right] + d_N\sum_{s=1}^{N}\frac{1}{p(s)d_s} \\
= & \sum_{n=1}^{N-1}\left[d_n\sum_{s=1}^{n}\frac{1}{p(s)d_s}\right] + d_N\sum_{s=1}^{N-1}\frac{1}{p(s)d_s} + \frac{d_N}{p(N)d_N} \\
= & \sum_{n=1}^{N-1}\left[\frac{3}{(2n-1)(2n+1)}\sum_{s=1}^{n}\frac{1}{p\frac{2s+1}{2s-1}\frac{3}{(2s-1)(2s+1)}}\right] + \frac{3}{4N(2N-1)}\sum_{s=1}^{N-1}\frac{1}{p\frac{2s+1}{2s-1}\frac{3}{(2s-1)(2s+1)}} + \frac{1}{p\frac{4N}{2N-1}} \\
= & \sum_{n=1}^{N-1}\left[\frac{1}{p(2n-1)(2n+1)}\sum_{s=1}^{n}(2s-1)^2\right] + \frac{1}{p4N(2N-1)}\sum_{s=1}^{N-1}(2s-1)^2 + \frac{2N-1}{p4N} \\
= & \sum_{n=1}^{N-1}\left[\frac{1}{p(2n-1)(2n+1)}\frac{n(2n-1)(2n+1)}{3}\right] + \frac{1}{p4N(2N-1)}\frac{(N-1)(2N-3)(2N-1)}{3} + \frac{2N-1}{p4N} \\
= & \frac{1}{3p}\sum_{n=1}^{N-1} n + \frac{(N-1)(2N-3)}{p12N} + \frac{2N-1}{p4N} \\
= & \frac{4N}{12pN}\frac{N(N-1)}{2} + \frac{(N-1)(2N-3)}{p12N} + \frac{3(2N-1)}{p12N} \\
= & \frac{2N^2(N-1)+(N-1)(2N-3)+3(2N-1)}{12pN} \\
= & \frac{2N^3-2N^2+2N^2-5N+3+6N-3}{12pN} = \frac{2N^3+N}{12pN} = \frac{2N^2+1}{12p}.
\end{aligned}$$

.

- Now consider case $N = 1$.

  We can directly compute a separation distance $sep(\nu\mathbf{P}_X^k, \pi)$ for $\mathbf{X}$ starting at 1 (i.e., $\nu = (1, 0)$). We have

$$sep(\nu\mathbf{P}_X^k, \pi) = \max_{i\in\{1,2\}}\left(1 - \frac{\mathbf{P}_X^k(1,i)}{\frac{1}{2}}\right) = 1 - 2\min_{i\in\{1,2\}}\mathbf{P}_X^k(1,i). \tag{4.7}$$

  Spectral decomposition yields

$$\mathbf{P}_X^k = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & (1-4p)^k \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1+(1-4p)^k & 1-(1-4p)^k \\ 1-(1-4p)^k & 1+(1-4p)^k \end{pmatrix} \tag{4.8}$$

and thus

$$sep(\nu \mathbf{P}_X^k, \pi) = 1 - \min\{1 + (1-4p)^k, 1-(1-4p)^k\} = \begin{cases} (1-4p)^k & \text{if} \quad p \in (0, 1/4), \\ (4p-1)^k & \text{if} \quad p \in (1/4, 1/2). \end{cases}$$

On the other hand we know that there always exists a fastest strong stationary time T (see Proposition 1.10 (b) in [DF90]), i.e., $sep(\nu \mathbf{P}_X^k, \pi) = P(T > k)$. For $p \in (0, 1/4)$ we have that T has distribution $Geo(4p)$, whereas for $p \in (1/4, 1/2)$ we have $P(T > k) = (4p-1)^k = (1 - 2(1-2p))^k$, thus T has distribution $Geo(2(1-2p))$. It implies that

$$ET = \begin{cases} \dfrac{1}{4p} & \text{if} \quad p \in (0, 1/4), \\ \dfrac{1}{2(1-2p)} & \text{if} \quad p \in (1/4, 1/2). \end{cases}$$

□

Remark 4.5. For a case $N = 1$ and $p \leq 1/4$ we can have a duality-based proof, similar to the one we had for $N > 1$. From equation (4.6) we have $p(1) = p(N) = 4p$, using Theorem 2.2 we directly have

$$ET_{1:1:2} = d_1 \frac{1}{p(1) d_1} = \frac{1}{p(1)} = \frac{1}{4p}.$$

Let us have a closer look at this case. Note that both, a random walk on a circle and a resulting strong stationary dual, are the chains on two points. The ordering matrix is given by $\mathbf{C} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and we directly have

$$\mathbf{P}_X = \begin{pmatrix} 1-2p & 2p \\ 2p & 1-2p \end{pmatrix}, \quad \mathbf{C}^{-1} \overleftarrow{\mathbf{P}}_X \mathbf{C} = \begin{pmatrix} 1-4p & 2p \\ 0 & 1 \end{pmatrix}.$$

From (4.2) we obtain (with $\pi(1) = \pi(2) = 1/2$)

$$\mathbf{P}_X^* = \begin{pmatrix} 1-4p & 4p \\ 0 & 1 \end{pmatrix}.$$

The transitions of $\mathbf{X}$ and $\mathbf{X}^*$ are depicted in Figure 2.

Random walk $\mathbf{X}^*$ A sharp SSD $\mathbf{X}^*$

Figure 2: Case $d = 2N = 2$: Original random walk on a circle $\mathbf{X}$ (left), the corresponding sharp SSD $\mathbf{X}^*$ (right)

Of course, time to absorption in $\mathbf{X}^*$ has $Geo(4p)$ distribution, thus $ET = \frac{1}{4p}$.

Remark 4.6. *Note that the assumptions on p in Lemma 4.1 (i.e., $p \leq 1/3$ for $N > 1$ and casee $p \leq 1/4, p \in (1/4, 1/2)$ for $N = 1$) are equivalent to non-negativity of the resulting matrix $\mathbf{P}_X^*$. In other words the assumption implies that $\mathbf{X}$ is $^{\uparrow}$-Möbius monotne (it is if and only if condition).*

5. Proofs

5.1. Gambler's ruin problem, absorbing birth and death chain

5.1.1. Proof of Theorems 2.1 and 2.2

Proof of Theorem 2.1. Consider the birth and death chain with j and k ($j < k$) as recurrent absorbing states ($p(j) = q(j) = p(k) = q(k) = 0$). First step analysis yields (for $j < i < k$)

$$ET_{j:i:k} = p(i)(1 + ET_{j:i+1:k}) + q(i)(1 + ET_{j:i-1:k}) + (1 - q(i) - p(i))(1 + ET_{j:i:k}), \tag{5.1}$$

thus

$$ET_{j:i+1:k} = ET_{j:i:k} + \frac{q(i)}{p(i)}\left(ET_{j:i:k} - ET_{j:i-1:k} - \frac{1}{q(i)}\right). \tag{5.2}$$

Since $ET_{j:j:k} = 0$, we have:

$$ET_{j:j+2:k} = ET_{j:j+1:k}\left(1 + \frac{q(j+1)}{p(j+1)}\right) - \frac{q(j+1)}{p(j+1)}\frac{1}{q(j+1)}.$$

Recall that $d_s = \prod_{i=j+1}^{s} \frac{q(i)}{p(i)}$ (where $d_j = 1$), iterating the above equations yields:

$$ET_{j:i:k} = ET_{j:j+1:k}\sum_{s=j}^{i-1} d_s - \sum_{s=j+1}^{i-1}\left[d_s \sum_{m=j+1}^{s}\frac{1}{p(m)d_m}\right], \tag{5.3}$$

what can be checked by induction. Plugging (5.3) into (5.2) we have:

$ET_{j:i+1:k} =$

$$\begin{aligned}
& ET_{j:j+1:k}\sum_{n=j}^{i-1} d_n - \sum_{n=j+1}^{i-1}\left[d_n \sum_{s=j+1}^{n}\frac{1}{p(s)d_s}\right] \\
& +\frac{q(i)}{p(i)}\quad\Bigg(ET_{j:j+1:k}\sum_{n=j}^{i-1} d_n - \sum_{n=j+1}^{i-1}\left[d_n \sum_{s=j+1}^{n}\frac{1}{p(s)d_s}\right] \\
& \qquad -ET_{j:j+1:k}\sum_{n=j}^{i-2} d_n - \sum_{n=j+1}^{i-2}[d_n \sum_{m=j+1}^{n}\frac{1}{p(s)d_s}] - \frac{1}{q(i)}\Bigg) \\
= \; & ET_{j:j+1:k}\sum_{n=j}^{i-1} d_n - \sum_{n=j+1}^{i-1}\left[d_n \sum_{s=j+1}^{n}\frac{1}{p(s)d_s}\right] \\
& + \frac{q(i)}{p(i)}\left(ET_{j:j+1:k}d_{i-1} - d_{i-1}\sum_{s=j+1}^{i-1}\frac{1}{p(s)d_s} - d_i\frac{1}{d_i q(i)}\right) \\
= \; & ET_{j:j+1:k}\sum_{n=j}^{i-1} d_n - \sum_{n=j+1}^{i-1}\left[d_n \sum_{s=j+1}^{n}\frac{1}{p(s)d_s}\right] + ET_{j:j+1:k}d_i - d_i\sum_{s=j+1}^{i-1}\frac{1}{p(s)d_s} - d_i\frac{1}{d_i p(i)} \\
= \; & ET_{j:j+1:k}\sum_{n=j}^{i} d_n - \sum_{n=j+1}^{i-1}\left[d_n \sum_{s=j+1}^{n}\frac{1}{p(s)d_s}\right] - d_i\sum_{s=j+1}^{i}\frac{1}{p(s)d_s} \\
= \; & ET_{j:j+1:k}\sum_{n=j}^{i} d_n - \sum_{n=j+1}^{i}\left[d_n \sum_{s=j+1}^{n}\frac{1}{p(s)d_s}\right]
\end{aligned}$$

.

Since $ET_{j:k:k} = 0$, we have:

$$0 = ET_{j:j+1:k}\sum_{n=j}^{k-1} d_n - \sum_{n=j+1}^{k-1}\left[d_n \sum_{s=j+1}^{n}\frac{1}{p(s)d_s}\right] \Rightarrow ET_{j:j+1:k} = \frac{\displaystyle\sum_{n=j+1}^{k-1}\left[d_n \sum_{s=j+1}^{n}\frac{1}{p(s)d_s}\right]}{\displaystyle\sum_{n=j}^{k-1} d_n},$$

thus

$$ET_{j:i:k} = \frac{\sum_{n=j+1}^{k-1}\left[d_n \sum_{s=j+1}^{n}\frac{1}{p(s)d_s}\right]}{\sum_{n=j}^{k-1} d_n}\sum_{n=j}^{i-1} d_n - \sum_{n=j+1}^{i-1}\left[d_n \sum_{s=j+1}^{n}\frac{1}{p(s)d_s}\right],$$

what was to be shown.

□

Proof of Theorem 2.2. Similarly as to proof of the Theorem 2.1 we consider birth and death chain on $\{j, \ldots, k\}$ $(j < k)$, however now only k is absorbing (*i.e.*, $p(k) = q(k) = q(j)0$, but $p(j) > 0$). For $i : j < i < k$ we can rewrite Eq. (5.1):

$$ET_{j:i:k} = p(i)(1 + ET_{j:i+1:k}) + q(i)(1 + ET_{j:i-1:k}) + (1 - q(i) - p(i))(1 + ET_{j:i:k}),$$

we have

$$ET_{j:i:k} = ET_{j:i+1:k} - \frac{q(i)}{p(i)}\left(ET_{j:i:k} - ET_{j:i-1:k} - \frac{1}{q(i)}\right). \tag{5.4}$$

However, for $i = j$ we have

$$ET_{j:j:k} = (1 - p(j))(1 + ET_{j:j:k}) + p(j)(1 + ET_{j:j+1:k}),$$

i.e.,

$$ET_{j:j:k} = \frac{1}{p(j)} + ET_{j:j+1:k}.$$

Recall that $d_s = \prod_{i=j+1}^{s}\frac{q(i)}{p(i)}$ (where $d_j = j$), iterating the above equations yields:

$$ET_{j:i:k} = ET_{j:i+1:k} + \sum_{s=j}^{i}\frac{d_i}{p(s)d_s}, \tag{5.5}$$

what can be checked by induction. Plugging (5.5) (for $i := i - 1$) into (5.4) we have:

$$\begin{aligned} ET_{j:i:k} &= ET_{j:i+1:k} - \frac{q(i)}{p(i)}\left(ET_{j:i:k} - \left(ET_{j:i:k} + \sum_{s=j}^{i-1}\frac{d_{i-1}}{p(s)d_s}\right) - \frac{1}{q(i)}\right) \\ &= ET_{j:i+1:k} + \frac{d_i}{d_{i-1}}\left(\left(\sum_{s=j}^{i-1}\frac{d_{i-1}}{p(s)d_s}\right) + \frac{d_{i-1}}{p(i)d_i}\right) \\ &= ET_{j:i+1:k} + \sum_{s=j}^{i}\frac{d_i}{p(s)d_s}. \end{aligned}$$

Since $ET_{j:k:k} = 0$, we have:

$$ET_{j:k-1:k} = \sum_{s=1}^{k-1}\frac{d_{k-1}}{p(s)d_s}.$$

Iterating the above equations yields:

$$ET_{j:i:k} = \sum_{n=i}^{k-1} \left[d_n \sum_{s=1}^{n} \frac{1}{p(s)d_s} \right],$$

what was to be shown.

□

5.1.2. Proof of Lemma 2.5 and Theorem 2.3

Proof of Lemma 2.5. Denote by $f(n)$ lhs of (2.9) and by $h(n)$ its rhs. We will show that generating functions of f and h are equal. Let us start with $\mathfrak{g}_f(x)$, the generating function of f at x:

$\mathfrak{g}_f(x) =$

$$\begin{aligned}
\sum_{n=0}^{\infty} f(n)x^n &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{n-k}{k}\left(-\frac{r}{(1+r)^2}\right)^k x^n = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\binom{n-k}{k}\left(-\frac{r}{(1+r)^2}\right)^k x^n \\
&= \sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\binom{n-k}{k}\left(-\frac{r}{(1+r)^2}\right)^k x^n \\
&= \sum_{k=0}^{\infty}\sum_{n=0}^{\infty}\binom{n}{k}\left(-\frac{r}{(1+r)^2}\right)^k x^{n+k} = \sum_{k=0}^{\infty}\left(-\frac{r}{(1+r)^2}\right)^k x^k \sum_{n=0}^{\infty}\binom{n}{k}x^n
\end{aligned}$$

Applying $\sum_{n=0}^{\infty}\binom{n}{k}x^n = \frac{x^k}{(1-x)^{k+1}}$ we have

$$\begin{aligned}
\mathfrak{g}_f(x) &= \sum_{k=0}^{\infty}\left(-\frac{r}{(1+r)^2}\right)^k x^k \frac{x^k}{(1-x)^{k+1}} = \frac{1}{1-x}\sum_{k=0}^{\infty}\left(\frac{-rx^2}{(1+r)^2(1-x)}\right)^k \\
&= \frac{1}{(1-x)}\frac{(1+r)^2(1-x)}{(1+r)^2(1-x)+rx^2} = \frac{(1+r)^2}{(1+r)^2(1-x)+rx^2}.
\end{aligned}$$

On the other hand, the generating function of h is following:

$$\begin{aligned}
\mathfrak{g}_h(x) &= \sum_{n=0}^{\infty} h(n)x^n = \sum_{n=0}^{\infty}\frac{1-r^{n+1}}{(1+r)^n(1-r)}x^n = \frac{1}{(1-r)}\left(\sum_{n=0}^{\infty}\frac{1}{(1+r)^n}x^n - \sum_{n=0}^{\infty}\frac{r^n}{(1+r)^n}x^n\right) \\
&= \frac{1}{(1-r)}\left(\sum_{n=0}^{\infty}\frac{1}{(1+r)^n}x^n - \sum_{n=0}^{\infty}\frac{r^n}{(1+r)^n}x^n\right) = \frac{1}{(1-r)}\left(\frac{1+r}{1+r-x} - r\frac{1+r}{1+r-xr}\right) \\
&= \frac{1+r}{(1-r)}\frac{1+r-xr-r-r^2-xr}{(1+r-x)(1+r-xr)} = \frac{1+r}{(1-r)}\frac{(1+r)(1-r)}{(1+r)^2-(1+r)(x+xr)+x^2r} \\
&= \frac{(1+r)^2}{(1+r)^2(1-x)+rx^2},
\end{aligned}$$

thus $\mathfrak{g}_h(x) = \mathfrak{g}_f(x)$, what finishes the proof.

□

The following lemma will be needed in the proof of Theorem 2.3.

Lemma 5.1. *Consider the gambler's ruin problem with general rates* $\mathbf{p}, \mathbf{q}$. *Define*

$$\begin{aligned}
a_i &= -\frac{\rho_{0:i:i+1}}{p(i)}, \\
b_i &= \frac{(p(i)+q(i))\rho_{0:i:i+1}}{p(i)}, \\
c_i &= -\frac{q(i)}{p(i)}\rho_{0:i-1:i+1}.
\end{aligned}$$

Then, for all $N \geq 1$ *we have*

$$\prod_{j=2}^{N}\begin{pmatrix} b_j & c_j & a_j \\ 1 & 0 & 0 \\ 0 & 0 & 1\end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & a_1 \\ 1 & 0 & 0 \\ 0 & 0 & 1\end{pmatrix} = \begin{pmatrix} 1 & 0 & A_N \\ 1 & 0 & A_{N-1} \\ 0 & 0 & 1\end{pmatrix},$$

where

$$A_M = -\sum_{n=1}^{M} \frac{1}{p(n)} \rho_{0:n:M+1} \sum_{k=0}^{\lfloor (M-n)/2 \rfloor} \xi_k^{n+1,M},$$

$\xi_k^{n+1,M}$ *was defined in (2.4).*

Proof. Recall that $\mathbf{j}_k^{n,m}$ was defined in (2.2) as

$$\mathbf{j}_k^{n,m} = \big\{\{j_1, j_2, \ldots, j_k\} : j_1 \geq n+1, j_k \leq m, j_i \leq j_{i+1} - 2 \text{ for } i \in \{1, k-1\}\big\}.$$

For given $\mathbf{p}, \mathbf{q}$, b_n, c_n and $\mathbf{j} \in \mathbf{j}_k^{n,m}$ define

$$D_{\mathbf{j}}^{n,m} = b_n b_{n+1} \ldots b_{j_1-2} c_{j_1} b_{j_1+1} b_{j_1+2} \ldots b_{j_2-2} c_{j_2} \ldots b_{j_{k-1}+1} b_{j_{k-1}+2} \ldots b_{j_k-2} c_{j_k} b_{j_k+1} b_{j_k+2} \ldots b_m$$

and let

$$S_k^{n,m} = \sum_{\mathbf{j} \in \mathbf{j}_k^{n,m}} D_{\mathbf{j}}^{n,m}.$$

Let

$$\begin{aligned}
\alpha_i &= -\frac{1}{p(i)}, \\
\beta_i &= \frac{(p(i) + q(i))}{p(i)} = 1 + r(i), \\
\gamma_i &= -\frac{q(i)}{p(i)} = -r(i).
\end{aligned}$$

$D_{\mathbf{j}}^{n,m}$ can be rewritten as

$$\begin{aligned}
D_{\mathbf{j}}^{n,m} &= \rho_{0:n:m+1} \beta_n \beta_{n+1} \cdots \beta_{j_1-2} \gamma_{j_1} \beta_{j_1+1} \beta_{j_1+2} \cdots \beta_{j_2-2} \gamma_{j_2} \cdots \\
&\quad \cdot \beta_{j_{k-1}+1} \beta_{j_{k-1}+2} \ldots \beta_{j_k-2} \gamma_{j_k} \beta_{j_k+1} \beta_{j_k+2} \cdots \beta_m \\
&= (-1)^k \prod_{s \in \mathbf{j}} r(s) \prod_{s \in \{n,\ldots,m\} \setminus \mathbf{j} \cup \mathbf{j}-1} 1 + r(s) = \rho_{0:n:m+1} \delta_{\mathbf{j}}^{n,m}.
\end{aligned}$$

Thus $S_k^{n,m} = \sum_{\mathbf{j} \in \mathbf{j}_k^{n,m}} D_{\mathbf{j}}^{n,m} = \rho_{0:n:m+1} \sum_{\mathbf{j} \in \mathbf{j}_k^{n,m}} \delta_{\mathbf{j}}^{n,m} =: \rho_{0:n:m+1} \xi_k^{n,m}$ and A_M can be rewritten as

$$A_M = \sum_{n=1}^{M} a_n \sum_{k=0}^{\lfloor (M-n)/2 \rfloor} S_k^{n+1,M}.$$

We will show this by induction.

- For $M = 1$ we have

$$A_1 = \sum_{n=1}^{1} a_n \sum_{k=0}^{\lfloor (1-n)/2 \rfloor} S_k^{n+1,1} = a_1 \sum_{k=0}^{\lfloor 0/2 \rfloor} S_k^{2,1} = a_1 S_0^{2,1} = a_1.$$

- For $N \geq M \geq 2$ assuming $A_M = \sum_{n=1}^{M} a_n \sum_{k=0}^{\lfloor (M-n)/2 \rfloor} S_k^{n+1,M}$ we shall prove that $A_{N+1} = b_{N+1} A_N + c_{N+1} A_{N-1} + a_{N+1}$. We have

$$
\begin{aligned}
&b_{N+1}A_N + c_{N+1}A_{N-1} + a_{N+1} = \\
&= b_{N+1}\sum_{n=1}^{N} a_n \sum_{k=0}^{\lfloor (N-n)/2\rfloor} S_k^{n+1,N} + c_{N+1}\sum_{n=1}^{N-1} a_n \sum_{k=0}^{\lfloor (N-n-1)/2\rfloor} S_k^{n+1,N-1} + a_{N+1}\\
&= \sum_{n=1}^{N} a_n \sum_{k=0}^{\lfloor (N-n)/2\rfloor} b_{N+1}\sum_{\mathbf{j}_k^{n+1,N}} D^{n+1,N}_{\mathbf{j}_k^{n+1,N}} + \sum_{n=1}^{N-1} a_n \sum_{k=0}^{\lfloor (N-n-1)/2\rfloor} c_{N+1}\sum_{\mathbf{j}_k^{n+1,N-1}} D^{n+1,N-1}_{\mathbf{j}_k^{n+1,N-1}} + a_{N+1}\\
&= \sum_{n=1}^{N} a_n \sum_{k=0}^{\lfloor (N+1-n)/2\rfloor} \sum_{\mathbf{j}_k^{n+1,N+1}:j_k\neq N+1} D^{n+1,N+1}_{\mathbf{j}_k^{n+1,N+1}}\\
&\quad + \sum_{n=1}^{N} a_n \sum_{k=0}^{\lfloor (N+1-n)/2\rfloor} \sum_{\mathbf{j}_k^{n+1,N+1}:j_k= N+1} D^{n+1,N+1}_{\mathbf{j}_k^{n+1,N+1}} + a_{N+1}\\
&= \sum_{n=1}^{N+1} a_n \sum_{k=0}^{\lfloor (N+1-n)/2\rfloor} \sum_{\mathbf{j}_k^{n+1,N+1}} D^{n+1,N+1}_{\mathbf{j}_k^{n+1,N+1}} = \sum_{n=1}^{N+1} a_n \sum_{k=0}^{\lfloor (N+1-n)/2\rfloor} S_k^{n+1,N+1} = A_{N+1}
\end{aligned}
$$

what finishes the proof.

□

Proof of Theorem 2.3. First step analysis yields (for $N > i > 1$):

$$
\begin{aligned}
EW_{0:i:N} &= (1+EW_{0:i-1:N})P(X_1 = i-1|X_0 = i, X_T = N)\\
&\quad +(1+EW_{0:i:N})P(X_1 = i|X_0 = i, X_T = N)\\
&\quad +(1+EW_{0:i+1:N})P(X_1 = i+1|X_0 = i, X_T = N).
\end{aligned}
$$

We have $EW_{0:N:N} = 0$ and for simplicity we also set $EW_{0:0:N} = 0$. We have

$$
\begin{aligned}
P(X_1 = i-1|X_0 = i, X_T = N) &= \tfrac{P(X_1=i-1|X_0=i)P(X_T=N|X_1=i-1)}{P(X_T=N|X_0=i)} = \frac{q(i)\rho_{0:i-1:N}}{\rho_{0:i:N}} = q(i)\rho_{0:i-1:i},\\
P(X_1 = i|X_0 = i, X_T = N) &= \frac{(1-p(i)-q(i))\rho_{0:i:N}}{\rho_{0:i:N}} = 1-p(i)-q(i),\\
P(X_1 = i+1|X_0 = i, X_T = N) &= \frac{p(i)\rho_{0:i+1:N}}{\rho_{0:i:N}} = p(i)\rho_{0:i+1:i}.
\end{aligned}
$$

For $i = 1$ we have

$$EW_{0:1:N} = [1+EW_{0:1:N}](1-p(1)-q(1)) + [1+EW_{0:2:N}]p(1)\rho_{0:2:1},$$

thus

$$EW_{0:2:N} = \frac{(p(1)+q(1)-1)\rho_{0:1:2}}{p(1)} - 1 + \frac{(p(1)+q(1))\rho_{0:1:2}}{p(1)}EW_{0:1:N}.$$

For $1 \le i \le N$ we have

$$EW_{0:i:N} = (1+EW_{0:i-1:N})q(i)\rho_{0:i-1:i} + (1+EW_{0:i:N})(1-p(i)-q(i)) + (1+EW_{0:i+1:N})p(i)\rho_{0:i+1:i} \quad (5.6)$$

and

$$\begin{aligned}
EW_{0:i+1:N} &= \frac{(p(i)+q(i))\rho_{0:i:i+1}}{p(i)} - \frac{q(i)}{p(i)}\rho_{0:i-1:i+1} - 1 - \frac{\rho_{0:i:i+1}}{p(i)} \\
&\quad + \frac{(p(i)+q(i))\rho_{0:i:i+1}}{p(i)} EW_{0:i:N} - \frac{q(i)}{p(i)}\rho_{0:i-1:i+1} EW_{0:i-1:N}, \\
&= b_i + c_i - 1 + a_i + b_i EW_{0:i:N} + c_i EW_{0:i-1:N} \\
&\stackrel{(*)}{=} a_i + b_i EW_{0:i:N} + c_i EW_{0:i-1:N},
\end{aligned} \tag{5.7}$$

where a_i, b_i, c_i were defined in Lemma 5.1 and in $(*)$ we used the fact that

$$\begin{aligned}
b_i + c_i &= \frac{(p(i)+q(i))\rho_{0:i:i+1}}{p(i)} - \frac{q(i)}{p(i)}\rho_{0:i:i+1} \\
&= \frac{p(i)+q(i)}{p(i)} \frac{\sum_{n=1}^{i}\prod_{k=1}^{n-1}\left(\frac{q(k)}{p(k)}\right)}{\sum_{n=1}^{i+1}\prod_{k=1}^{n-1}\left(\frac{q(k)}{p(k)}\right)} - \frac{q(i)}{p(i)} \frac{\sum_{n=1}^{i-1}\prod_{k=1}^{n-1}\left(\frac{q(k)}{p(k)}\right)}{\sum_{n=1}^{i+1}\prod_{k=1}^{n-1}\left(\frac{q(k)}{p(k)}\right)} \\
&= \frac{\sum_{n=1}^{i}\prod_{k=1}^{n-1}\left(\frac{q(k)}{p(k)}\right) + \frac{q(i)}{p(i)}\sum_{n=1}^{i}\prod_{k=1}^{n-1}\left(\frac{q(k)}{p(k)}\right) - \frac{q(i)}{p(i)}\sum_{n=1}^{i-1}\prod_{k=1}^{n-1}\left(\frac{q(k)}{p(k)}\right)}{\sum_{n=1}^{i+1}\prod_{k=1}^{n-1}\left(\frac{q(k)}{p(k)}\right)} \\
&= \frac{\sum_{n=1}^{i}\prod_{k=1}^{n-1}\left(\frac{q(k)}{p(k)}\right) + \frac{q(i)}{p(i)}\sum_{n=i}^{i}\prod_{k=1}^{n-1}\left(\frac{q(k)}{p(k)}\right)}{\sum_{n=1}^{i+1}\prod_{k=1}^{n-1}\left(\frac{q(k)}{p(k)}\right)} \\
&= \frac{\sum_{n=1}^{i}\prod_{k=1}^{n-1}\left(\frac{q(k)}{p(k)}\right) + \prod_{k=1}^{i}\left(\frac{q(k)}{p(k)}\right)}{\sum_{n=1}^{i+1}\prod_{k=1}^{n-1}\left(\frac{q(k)}{p(k)}\right)} = \frac{\sum_{n=1}^{i+1}\prod_{k=1}^{n-1}\left(\frac{q(k)}{p(k)}\right)}{\sum_{n=1}^{i+1}\prod_{k=1}^{n-1}\left(\frac{q(k)}{p(k)}\right)} = 1.
\end{aligned}$$

Equations (5.6) and (5.7) can be written in a matrix form:

$$\begin{pmatrix} EW_{0:i+1:N} \\ EW_{0:i:N} \\ 1 \end{pmatrix} = \begin{pmatrix} b_i & c_i & a_i \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} EW_{0:i:N} \\ EW_{0:i-1:N} \\ 1 \end{pmatrix}. \tag{5.8}$$

Note that $c_1 = -\frac{q_1}{p_1} W_0^2 = -\frac{q_1}{p_1} 0 = 0$ and

$$b_1 = \frac{(p(1)+q(1))\rho_{0:1:2}}{p(1)} = \frac{p(1)+q(1)}{p(1)} \frac{\sum_{n=1}^{1}\prod_{k=1}^{n-1}\left(\frac{q(k)}{p(k)}\right)}{\sum_{n=1}^{2}\prod_{k=1}^{n-1}\left(\frac{q(k)}{p(k)}\right)} = \frac{1+\frac{q(1)}{p(1)}}{1} \frac{1}{1+\frac{q(1)}{p(1)}} = 1,$$

thus using (5.8) recursively we obtain

$$\begin{aligned}
\begin{pmatrix} 0 \\ EW_{0:N-1:N} \\ 1 \end{pmatrix} = \begin{pmatrix} EW_{0:N:N} \\ EW_{0:N-1:N} \\ 1 \end{pmatrix} &= \prod_{j=2}^{N-1} \begin{pmatrix} b_j & c_j & a_j \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & a_1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} EW_{0:1:N} \\ EW_{0:0:N} \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & A_{N-1} \\ 1 & 0 & A_{N-2} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} EW_{0:1:N} \\ 0 \\ 1 \end{pmatrix},
\end{aligned}$$

where A_N is given in Lemma 5.1, what implies

$$EW_{0:1:N} = -A_{N-1}$$

and thus proves (2.6). Equation (2.5) follows from the fact that $W_{0:1:N} \stackrel{(distr)}{=} W_{0:1:i} + W_{0:i:N}$ (Markov property, $W_{0:1:i}$ and $W_{0:i:N}$ are independent).

□

5.2. Random walk on a polygon

5.2.1. Proof of Theorem 3.1

Proof of eq. (3.1) . Let F_i denote the event that at the first time we leave state i (recall, ties are allowed) we move clockwise. Similarly, let F_i^c denotes the event that at the first time we leave state i we move counterclockwise. We have

$$\begin{aligned} P(F_i) &= \frac{p(i)}{p(i)+q(i)} = \frac{1}{1+r(i)}, \\ P(F_i^c) &= \frac{q(i)}{p(i)+q(i)} = \frac{r(i)}{1+r(i)} \end{aligned}$$

and

$$P(A_i) = P(F_i)P(A_i|F_i) + P(F_i^c)P(A_i|F_i^c) = \frac{1}{1+r(i)}P(A_i|F_i) + \frac{r(i)}{1+r(i)}P(A_i|F_i^c).$$

- For $P(A_i|F_i)$ we have: we start at $i+1$ and we have to reach $i-1$ before reaching i. This is the probability of winning in the game $G(\mathbf{p}, \mathbf{q}, i, i+1, i-1)$. We thus have

$$P(A_i|F_i) = \rho_{i:i+1:i-1} = \frac{1}{\sum_{n=i+1}^{i-1} \prod_{s=i+1}^{n-1} r(s)}.$$

- Similarly for $P(A_i|F_i^c)$ we have: we start at $i-1$, and we have to reach $i+1$ before reaching i which corresponds to losing in the game $G(\mathbf{p}, \mathbf{q}, i+1, i-1, i)$. We thus have

$$P(A_i|F_i^c) = 1 - \rho_{i+1:i-1:i} = 1 - \frac{\sum_{n=i+2}^{i-1} \prod_{s=i+2}^{n-1} r(s)}{\sum_{n=i+2}^{i} \prod_{s=i+2}^{n-1} r(s)} = \frac{\prod_{s=i+2}^{i-1} r(s)}{\sum_{n=i+2}^{i} \prod_{s=i+2}^{n-1} r(s)} = \frac{1}{\sum_{n=i+2}^{i} \prod_{s=n}^{i-1} \left(\frac{1}{r(s)}\right)}.$$

Finally

$$\begin{aligned} P(A_i) &= \frac{1}{(1+r(i)) \sum_{n=i+1}^{i-1} \prod_{s=i+1}^{n-1} r(s)} + \frac{r(i)}{(1+r(i)) \sum_{n=i+2}^{i} \prod_{s=n}^{i-1} \left(\frac{1}{r(s)}\right)} \\ &= \frac{1}{(1+r(i)) \sum_{n=i+1}^{i-1} \prod_{s=i+1}^{n-1} r(s)} + \frac{1}{(1+r(i)) \sum_{n=i+2}^{i} \prod_{s=n}^{i} \left(\frac{1}{r(s)}\right)}. \end{aligned}$$

□

Proof of eq. (3.2). Let us define $T_1 = \inf\{t : X_t = j-1 \vee X_t = j+1 | X_0 = i\}$ and consider separately two cases when at T_1 we are at $j-1$ or $j+1$. The first one corresponds to winning, whereas the second one corresponds to losing in the game $G(\mathbf{p}, \mathbf{q}, j+1, i, j-1)$. The winning probability is

$$\rho_{j+1:i:j-1}.$$

In the first case (when we get to the $j-1$ before $j+1$) vertex j will be the last one if we reach $j+1$ earlier - this can be interpreted as losing in the game $G(\mathbf{p}, \mathbf{q}, j+1, j-1, j)$, what happens with probability:

$$1 - \rho_{j+1:j-1:j} \quad = \quad 1 - \frac{\sum_{n=j+2}^{j-1} \prod_{s=j+2}^{n-1} r(s)}{\sum_{n=j+2}^{j} \prod_{s=j+2}^{n-1} r(s)} = \frac{\prod_{s=j+2}^{j-1} r(s)}{\sum_{n=j+2}^{j} \prod_{s=j+2}^{n-1} r(s)}.$$

In the second case (when we get to the $j+1$ before $j-1$) vertex j will be the last one if we reach $j-1$ earlier - this can be interpreted as winning in the game $G(\mathbf{p}, \mathbf{q}, j, j+1, j-1)$, what happens with probability:

$$\rho_{j:j+1:j-1} \quad = \quad \frac{1}{\sum_{n=j+1}^{j-1} \prod_{s=j+1}^{n-1} r(s)}.$$

Finally:

$$\begin{aligned}
P(L_{i,j}) &= (1 - \rho_{j+1:i:j-1})\rho_{j:j+1:j-1} + \rho_{j+1:i:j-1}(1 - \rho_{j+1:j-1:j}) \\
&= \left(1 - \frac{\sum_{n=j+2}^{i} \prod_{s=j+2}^{n-1} r(s)}{\sum_{n=j+2}^{j-1} \prod_{s=j+2}^{n-1} r(s)}\right) \frac{1}{\sum_{n=j+1}^{j-1} \prod_{s=j+1}^{n-1} r(s)} + \frac{\sum_{n=j+2}^{i} \prod_{s=j+2}^{n-1} r(s)}{\sum_{n=j+2}^{j-1} \prod_{s=j+2}^{n-1} r(s)} \frac{\prod_{s=j+2}^{j-1} r(s)}{\sum_{n=j+2}^{j} \prod_{s=j+2}^{n-1} r(s)} \\
&= \frac{\sum_{n=i+1}^{j-1} \prod_{s=j+2}^{n-1} r(s)}{\sum_{n=j+2}^{j-1} \prod_{s=j+2}^{n-1} r(s)} \frac{1}{\sum_{n=j+1}^{j-1} \prod_{s=j+1}^{n-1} r(s)} + \frac{\sum_{n=j+2}^{i} \prod_{s=j+2}^{n-1} r(s)}{\sum_{n=j+2}^{j-1} \prod_{s=j+2}^{n-1} r(s)} \frac{\prod_{s=j+2}^{j-1} r(s)}{\sum_{n=j+2}^{j} \prod_{s=j+2}^{n-1} r(s)} \\
&= \frac{1}{\sum_{n=j+2}^{j-1} \prod_{s=j+2}^{n-1} r(s)} \left(\frac{\sum_{n=i+1}^{j-1} \prod_{s=j+2}^{n-1} r(s)}{\sum_{n=j+1}^{j-1} \prod_{s=j+1}^{n-1} r(s)} + \frac{\left(\sum_{n=j+2}^{i} \prod_{s=j+2}^{n-1} r(s)\right)\left(\prod_{s=j+2}^{j-1} r(s)\right)}{\sum_{n=j+2}^{j} \prod_{s=j+2}^{n-1} r(s)} \right) \\
&= \frac{1}{\sum_{n=j+2}^{j-1} \prod_{s=j+2}^{n-1} r(s)} \left(\frac{\sum_{n=i+1}^{j-1} \prod_{s=j+2}^{n-1} r(s)}{\sum_{n=j+1}^{j-1} \prod_{s=j+1}^{n-1} r(s)} + \frac{\sum_{n=j+2}^{i} \prod_{s=j+2}^{n-1} r(s)}{\sum_{n=j+2}^{j} \prod_{s=n}^{j-1} \frac{1}{r(s)}} \right).
\end{aligned}$$

□

Proof of eqs. (3.3), (3.4) and (3.5) . Let us start with the expectation of $V_{i,j}$ – number of steps to visit all vertices starting at i when j is the last visited vertex. As earlier, let $T_1 = \inf\{t : X_t = j-1 \vee X_t = j+1\}$. We have two cases:

- If $X_{T_1} = j-1$ (and j was the last visited vertex) then the expected game time consists of: expected time to win in $G(\mathbf{p}, \mathbf{q}, j+1, i, j-1)$, expected time to lose in $G(\mathbf{p}, \mathbf{q}, j+1, j-1, j)$ and expected duration of the game $G(\mathbf{p}, \mathbf{q}, j, j+1, j)$. That is:

$$EW_{j+1:i:j-1} + EB_{j+1:j-1:j} + ET_{j:j+1:j}$$

- If $X_{T_1} = j+1$ (and j was last visited vertex) then the expected game time consists of: expected time to lose in $G(\mathbf{p}, \mathbf{q}, j+1, i, j-1)$, expected time to win in $G(\mathbf{p}, \mathbf{q}, j, j+1, j-1)$ and expected duration of the game $G(\mathbf{p}, \mathbf{q}, j, j-1, j)$. That is:

$$EB_{j+1:i:j-1} + EW_{j:j+1:j-1} + ET_{j:j-1:j}$$

Now, conditioning on X_{T_1}, we obtain:

$$\begin{aligned} EV_{i,j} = \rho_{j+1:i:j-1}\left(EW_{j+1:i:j-1} + EB_{j+1:j-1:j} + ET_{j:j+1:j}\right) \\ +(1-\rho_{j+1:i:j-1})\left(EB_{j+1:i:j-1} + EW_{j:j+1:j-1} + ET_{j:j-1:j}\right). \end{aligned}$$

Equations (3.4) and (3.5) are simply obtained by conditioning on the states.

□

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