University of Wrocław 29.08.2019

Simulations and algorithmic applications of Markov chains

Syllabus

Paweł Lorek

Course description and objectives

A Markov chain is a sequence of stochastic events (based on probabilities, not on certainties) where the next state of a variable (or a system) depends only on its current state. It can be surprising that these chains (named after A. Markov) are widely present in our daily life, the classical examples include: card shuffling, movements of stock/share prices, modeling the action in a game of Monopoly, Google's PageRank algorithm (Google's secret to becoming the most successful search engine on the web), RC4 (cryptography system), Ising model (a common example from statistical physics), error correcting codes (Viterbi Algorithm, used on almost all mobile phones, invented by A. Viterbi - co-founder of Qualcomm), just to mention a few.

The course is devoted to discrete time Markov chains with finite state space. We gently start with fundamentals (stationary distributions, transition-matrix based simulations, reversibility) and go through monte carlo Markov chain methods (MCMC, a class of algorithms providing one of the currently most popular methods for simulating complicated stochastic systems); rate of convergence methods ("how many times should we shuffle a deck of cards?") - we will study coupling methods, strong stationary times, strong stationary duality, inequalities (Cheeger and Poincaré) for bounding the second-largest eigenvalue of a transition matrix; coupling from the past (CFTP) algorithm (improvement of standard MCMC, allows to obtain an unbiased sample from given distribution on huge state space, e.g., Ising model); estimating winning probabilities in gambler ruin-like problems (first step analysis and Siegmund duality); simulated annealing (a widely used randomized algorithm for various optimization problems); basics of hidden markov models (HMM, a popular machine learning algorithm, e.g., for speech recognition and error corrections); randomized polynomial time approximation schemes (MCMC-originated algorithm for approximating "the answer" to NP-hard related problem, e.g., number of graph colourings).

Teaching methods

Lecture, exercises, discussing students' solutions.

Requirements

Courses

- Algebra
- Probability theory

Course content

- Markov chains on finite state space.
- Simulations of Markov chains. Pseudorandom number generators (and their imperfections).
- Stationary distribution.
- Random walks on graphs. Reversible chains.
- Rate of convergence to stationarity: Coupling methods, bounding second largest eigenvalue of a transition matrix (Poincaré and Cheeger inequalities), Strong Stationary Times.
- Monte Carlo Markov Chains. Metropolis-Hastings algorithm. Gibbs sampler. Ising model. Hard-core model.
- Approximate counting; graph q-colouring.
- Perfect simulation: Propp-Wilson algorithm (Coupling From The Past).
- Strong Stationary Times and Strong Stationary Dual chains. Siegmund duality. Solving gambler ruin-like problems using dualities.
- Hidden Markov models (HMM) with discrete and continuous observations, the Baum-Welch algorithm, the Viterbi algorithm. Applications to error corrections and to classification of time-series.

Learning outcomes

Knowledge

- Student understands the notion of Markov chains and random walks
- Student understands construction of sampling models, including Gibbs sampler and Metropolis-Hasting algorithms
- Student knows how to obtain unbiased sample from some distributions using Coupling From The Past algorithm

- Student knows the constructions and algorithms used in Hidden Markov Models
- Student understands the issue of rate of convergence, knows several techniques to study it

Skills

- Student is be able to analyze the behaviour of a random walk
- Student is be able to assess the performance of some randomized algorithms
- Student knows how to efficiently (and approximately) sample from some distributions on a large state space (knows how to design and implement Metroplolis-Hasting and Gibbs sampler)
- Student is able to use and implement algorithms related to Hidden Markov Models.

Verification methods

written exam, two mid-terms, presentation of problem's solution

Rules and conditions

- *Exercises*: There will be points for 2 midterms and for presenting solutions of problems. To pass, the minimum number of points must be collected.
- *Written exam*: There will be several problems to solve, student must get the required minimum number of them.

Student workload

- Classes with the teacher:
  - lectures - 30 hours
  - exercises - 30 hours
  - written exam - 3 hours
- Student's own work:
  - preparing for classes - 35 hours
  - reading additional material - 22 hours
  - preparing for midterms and an exam - 30 hours

Bibliography

Main book:

[1] Olle Häggström. *Finite Markov Chains and Algorithmic Applications*, Cambridge University Press, 2002.

Suggested book:

[2] Levin, Peres, Wilmer. *Markov Chains and Mixing Times*, American Mathematical Society, 2017.