Braid groups and Curvature
Talk 1: The Basics

Jon McCammond

UC Santa Barbara

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Braid groups

Dual braids and orthoschemes

It has long been conjectured that the braid groups are non-positively curved in the sense that they have a geometric action on some complete $\mathrm{CAT}(0)$ space. In fact, a promising candidate has been known for some time.

In 2001 Tom Brady constructed a contractible n-dimensional simplicial complex with a free, cocompact, vertex-transitive n-strand braid group action and in 2010 Tom and I added a specific piecewise-euclidean metric to this complex.

I call this the dual braid n-complex with the orthoscheme metric.

Braid groups and CAT(0)

Tom and I conjectured that the dual braid n-complex with the orthoscheme metric is a CAT(0) space for every positive integer n and this conjecture has been established when n is very small. In these talks I outline a recent proof of the full conjecture (joint work with M. Dougherty and S. Witzel).

Theorem (Braid groups are CAT(0))

For every integer $n > 0$, the dual braid n-complex with the orthoscheme metric is a CAT(0) *space and, as a consequence, the n-strand braid group is a* CAT(0) *group.*

The lectures will introduce the complexes, metrics and groups under consideration, and outline the proof.

I like this theorem and its proof because:

- it gives a uniform explanation for many properties of braids,
- the structures we introduce lead to many new questions,
- the proof itself is very pretty (in my opinion),
- it is a new class of examples of CAT(0) spaces, and
- I've been trying to prove this (on and off) for 15 years.

The Revised Plan

The revised plan for these lectures is as follows.

- Talk 1: The Basics
- Talk 2: The Pieces
- Talk 3: The Proof

More precisely,

- Talk 1: curvature conditions and the braid groups
- Talk 2: dual braid complex and orthoschemes
- Talk 3: assemble the pieces and sketch the proof

Curvature conditions

Definition (Triangles)

A geodesic triangle Δ is a triple of points $x, y, z \in X$ and a triple of geodesics $[x, y]$, $[y, z]$ and $[z, x]$ called vertices and sides. A comparison triangle Δ' in $\mathbb{E}$ is a triple of points x', y' and z' so that the corresponding side lengths are equal.

Definition (CAT(0) spaces)

For every point p in a side of Δ in X there is a corresponding p' in a side of Δ' in $\mathbb{E}$ that is the same distance from its vertices. The triangle Δ satisfies the CAT(0) inequality if for all p and q in sides of Δ, $d_X(p, q) \leq d_{\mathbb{E}}(p', q')$. A space X is CAT(0) if all geodesic triangles in X satisfy the CAT(0) inequality.

Convex and Complete

Example ($\mathbb{R}^n$)

n-dimensional Euclidean space is a $\mathrm{CAT}(0)$ space.

It is an easy consequence of the $\mathrm{CAT}(0)$ inequality that geodesics in $\mathrm{CAT}(0)$ spaces are unique.

Definition (Convex and Complete)

A subspace $U \subset X$ is convex if for all $x, y \in U$, the unique geodesic from x to y is in U. A $\mathrm{CAT}(0)$ space is complete if it is complete as a metric space.

There are several ways to construct new $\mathrm{CAT}(0)$ spaces from existing $\mathrm{CAT}(0)$ spaces.

Easy Constructions

Lemma (Convex subspaces)

If X is a $\mathrm{CAT}(0)$ space and $U \subset X$ is a convex subspace then U is a $\mathrm{CAT}(0)$ space.

Lemma (Fixed sets)

If $f \colon X \to X$ is an isometry of a $\mathrm{CAT}(0)$ space X, then the set of points fixed by f is a convex $\mathrm{CAT}(0)$ subspace.

Lemma (Products)

If $X = U \times V$ is a direct product of metric spaces, then X is a $\mathrm{CAT}(0)$ space if and only if both U and V are $\mathrm{CAT}(0)$ spaces.

Gluing

Lemma (Gluing)

Let $X = U \cup V$ be a metric space. If U, V and $U \cap V$ are non-empty complete $\mathrm{CAT}(0)$ spaces, then X is a complete $\mathrm{CAT}(0)$ space.

The gluing lemma is really the gluing theorem since its proof is slightly delicate. Once established, a simple induction extends this from 2 subspaces to n subspaces.

Lemma (Gluing n subspaces)

Let $X = X_1 \cup \cdots \cup X_n$ be a metric space. If for each $\emptyset \neq B \subset [n]$, the corresponding intersection $X_B = \cap_{i \in B} X_i$ is a non-empty complete $\mathrm{CAT}(0)$ space, then X is a complete $\mathrm{CAT}(0)$ space.

The notation $[n]$ means $\{1, 2, \ldots, n\}$.

Non-positive curvature

Non-positively curved means locally $\mathrm{CAT}(0)$.

Definition (Non-positively curved)

Let X be a geodeisc metric space. If every point in X has a neighborhood that is a $\mathrm{CAT}(0)$ space, then X is said to be non-positively curved.

The Cartan-Hadamard Theorem shows that the difference between the local and the global version is purely topological.

Theorem (Cartan-Hadamard)

Let X be a complete connected metric space. If X is non-positively curved then its universal cover is $\mathrm{CAT}(0)$.

Euclidean cell complexes

Definition (Euclidean cell complexes)

Roughly speaking, a euclidean cell complex X is a space constructed by gluing together a collection of convex euclidean polytopes along isometric subpolytopes. The shapes in X are the equivalence classes of these polytopes up to isometry.

We say that X has finitely many shapes when it has only finitely many isometry types of cells. Bridson proved that the local polytope metrics combine to define a well-behaved global metric when the complex has finitely many shapes.

Theorem (Shapes)

If X is connected euclidean cell complex with finitely many shapes, then X is a complete geodesic metric space.

Gromov's criterion

When testing whether a euclidean cell complex is non-positively curved it is sufficient to check whether it is $\mathrm{CAT}(0)$ in the neighborhood of each vertex.

Theorem (Gromov's criterion)

If X is a eucldiean cell complex with finitely many shapes, then X is non-positively curved if and only if every vertex has a neighborhood that is $\mathrm{CAT}(0)$.

Gromov's criterion follows from the observation that if v is a vertex of the polytopal cell containing $x \in X$, then each neighborhood of v contains an isometric copy of a sufficiently small neighborhood of x. In particular, if this neighborhood of v is $\mathrm{CAT}(0)$ then so is the small neighborhood of x.

Group actions

Gromov's criterion can be simplified using group actions.

Definition (Isometric actions)

The action of a group G on a metric space X is by isometries when the action of G preserves the metric on X: for all $g \in G$ and $x, y \in X$, $d_X(g.x, g.y) = d_X(x, y)$.

Combining the group action, Gromov's criterion and the Cartan-Hadamard theorem produces a local $\mathrm{CAT}(0)$ test.

Theorem (Local criterion)

Let G be a group acting vertex-transitively by isometries on a connected and simply-connected euclidean cell complex X with finitely many shapes. If X contains a $\mathrm{CAT}(0)$ subcomplex that contains a neighborhood of a vertex, then X is a $\mathrm{CAT}(0)$ space.

Proof of the Local criterion

Proof.

Let Y be the subcomplex and let v be the vertex. Since Y is CAT(0), it is non-positively curved. By hypothesis and by Gromov's criterion we can find a neighborhood $N(v)$ of a vertex v in X such that $N(v) \subset Y$ is CAT(0). Because the action of G is vertex-transitive, for every vertex $v' \in X$ there is a $g \in G$ such that $g.v = v'$ and since the action is by isometries $g.N(v)$ is a CAT(0) neighborhood of v'. Thus every vertex in X has a CAT(0) neighborhood. By Gromov's criterion X is non-positively curved, by Bridson's theorem X is complete and by hypothesis X connected and simply-connected. Thus by the Cartan-Hadamard Theorem X is a CAT(0) space. □

Geometric actions

We now shift our attention from spaces to groups. Let G be a group acting on a metric space X.

Definition (Group actions)

The action is free if the identity in G is the only element that fixes a point in X. The action is proper if for every point $x \in X$, there is a neighborhood $N(x)$ of x such that the set $\{g \in G \mid g.N(x) \cap N(x) \neq \varnothing\}$ is finite. And the action is cocompact if there is a compact subset $K \subset X$ whose orbit under the G-action is all of X: $G.K = X$.

Definition (Geometric actions)

When the action of G on X is proper, cocompact and by isometries, it is called a geometric action.

CAT(0) groups and NPC groups

Definition (CAT(0) groups and NPC groups)

A group is CAT(0) if it admits a geometric action on some complete CAT(0) space and it is non-positively curved if it is the fundamental group of a compact non-positively curved space.

These concepts are equivalent when the group is torsion-free.

Proposition (Torsion and curvature)

If G is a group with a geometric action on a CAT(0) *space X, then the action of G on X is free if and only if G is torsion-free. As a consequence a group is non-positively curved if and only if it is* CAT(0) *and torsion-free.*

And note that the braid groups are torsion-free.

Braid groups

Labeled configuration spaces

The braid groups have many different equivalent definitions. One of the main ones is as the fundamental group of a configuration space of n unlabeled points in the plane. Let X be a topological space and let X^n be the space of all n-tuples $\vec{x} = (x_1, x_2, \ldots, x_n)$ of elements $x_i \in X$.

Definition (Labeled configuration spaces)

The configuration space of n labeled points in X is the subspace $\textsc{Conf}_n(X)$ of X^n of n-tuples with distinct entries. The thick diagonal of X^n is the subspace

$$\Delta = \{(x_1, \ldots, x_n) \mid x_i = x_j \text{ for some } i \neq j\}$$

where this condition fails. Thus $\textsc{Conf}_n(X) = X^n - \Delta$.

Unlabeled configuration spaces

Definition (Unlabeled configuration spaces)

The symmetric group acts on X^n by permuting coordinates and this action restricts to a free action on $\textsc{Conf}_n(X)$. The configuration space of n unlabeled points in X is the quotient space $\textsc{UConf}_n(X) = (X^n - \Delta)/\textsc{Sym}_n$.

Remark (The map SET)

Since the quotient map sends the n-tuple $(x_1, \ldots, x_n)$ to n-element set $\{x_1, \ldots, x_n\}$, we write

$$\textsc{set}\colon \textsc{Conf}_n(X) \to \textsc{UConf}_n(X)$$

for this natural quotient map.

First Examples

Example (Configuration spaces)

When X is the unit circle and $n = 2$, the space X^2 is a torus, Δ is a $(1,1)$-curve on the torus, its complement $\textsc{Conf}_2(X)$ is homeomorphic to the interior of an annulus and the quotient $\textsc{UConf}_2(X)$ is homeomorphic to the interior of a Möbius band.

Example (Braid arrangement)

Let $\mathbb{C}$ be the complex numbers with its usual topology and let $\vec{z} = (z_1, z_2, \ldots, z_n)$ denote a point in $\mathbb{C}^n$. The thick diagonal of $\mathbb{C}^n$ is a union of hyperplanes called the braid arrangement and the hyperplanes in the arrangement are defined by the equations $z_i - z_j = 0$ for all $i \neq j \in [n]$.

Braids in $\mathbb{C}$

Definition (Braids in $\mathbb{C}$)

The configuration space $\textsc{Conf}_n(\mathbb{C})$ is the complement of the braid arrangement and its fundamental group is called the *n*-strand pure braid group. The *n*-strand braid group is the fundamental group of the quotient configuration space $\textsc{UConf}_n(\mathbb{C}) = \textsc{Conf}_n(\mathbb{C})/\textsc{Sym}_n$ of n unlabeled points.

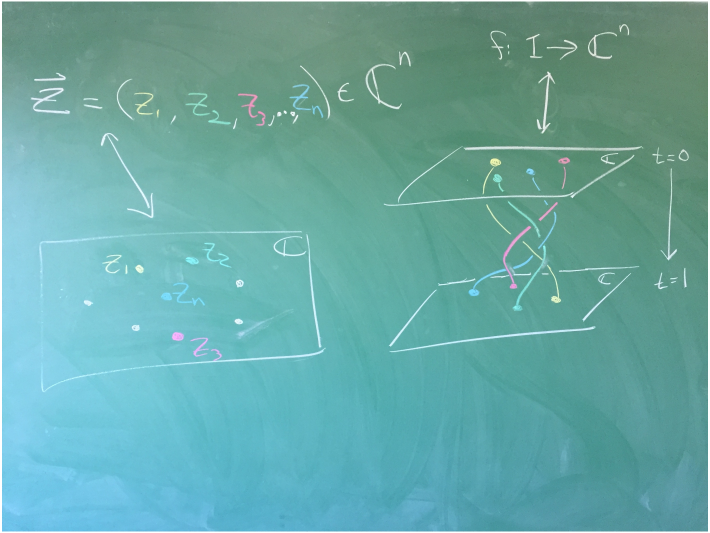
In symbols we have

$$\textsc{PBraid}_n = \pi_1(\textsc{Conf}_n(\mathbb{C}), \vec{z})$$

$$\textsc{Braid}_n = \pi_1(\textsc{UConf}_n(\mathbb{C}), Z)$$

where $\vec{z}$ is some specified basepoint in $\textsc{Conf}_n(\mathbb{C})$ and $Z = \textsc{set}(\vec{z})$ is the corresponding basepoint in $\textsc{UConf}_n(\mathbb{C})$.

The Braid Arrangement



Short exact sequence

Remark (Short exact sequence)

The quotient map $\textsc{set}$ is a covering map, so the induced map

$$\textsc{set}_*\colon \text{PB}\textsc{raid}_n \to \text{B}\textsc{raid}_n$$

on fundamental groups is injective. Since $\text{C}\textsc{onf}_n(\mathbb{C})$ is a regular cover of $\text{UC}\textsc{onf}_n(\mathbb{C})$, the image of $\text{PB}\textsc{raid}_n$ in $\text{B}\textsc{raid}_n$ is a normal subgroup and the quotient is the group $\text{S}\textsc{ym}_n$ of covering transformations. We have a short exact sequence.

$$\text{PB}\textsc{raid}_n \overset{\textsc{set}_*}{\hookrightarrow} \text{B}\textsc{raid}_n \overset{\textsc{perm}}{\twoheadrightarrow} \text{S}\textsc{ym}_n.$$

The second map is called $\textsc{perm}$ because each braid is sent to the induced permutation of the basepoint, an n-element set.

Very few strands

Example ($n = 1$)

The spaces $\text{UC}{\small\text{ONF}}_1(\mathbb{C})$, $\text{C}{\small\text{ONF}}_1(\mathbb{C})$ and $\mathbb{C}$ are equal and contractible, and all three groups in the short exact sequence are trivial.

Example ($n = 2$)

The space $\text{C}{\small\text{ONF}}_2(\mathbb{C})$ is $\mathbb{C}^2 - \mathbb{C}^1$, which retracts to $\mathbb{C}^1 - \mathbb{C}^0$ and then to the unit circle $\mathbb{S}^1 \subset \mathbb{C}$. The quotient space $\text{UC}{\small\text{ONF}}_2(\mathbb{C})$ also deformation retracts to $\mathbb{S}^1$ and the map from $\text{C}{\small\text{ONF}}_2(\mathbb{C})$ to $\text{UC}{\small\text{ONF}}_2(\mathbb{C})$ corresponds to the map from $\mathbb{S}^1$ to itself sending z to z^2. In particular $\text{PB}{\small\text{RAID}}_2 \cong \text{B}{\small\text{RAID}}_2 \cong \mathbb{Z}$, the map SET_* multiplies by 2 and the quotient is $\mathbb{Z}/2\mathbb{Z} \cong \text{S}{\small\text{YM}}_2$.

We assume $n > 2$ from now on.

Braids in $\mathbb{D}$

Let $\mathbb{D} \subset \mathbb{C}$ be the closed unit disk centered at the origin. Restricting to configurations of points that remain in $\mathbb{D}$ does not change the fundamental group of the configuration space.

Proposition (Braids in $\mathbb{D}$)

The configuration space $\text{UCONF}_n(\mathbb{C})$ *deformation retracts to the subspace* $\text{UCONF}_n(\mathbb{D})$*, so for any choice of basepoint* Z *in the subspace,*

$$\pi_1(\text{UCONF}_n(\mathbb{D}), Z) = \pi_1(\text{UCONF}_n(\mathbb{C}), Z) = \text{BRAID}_n.$$

Since the topology of a configuration space only depends on the topology of the original space, we can replace $\mathbb{D}$ with any space P homeomorphic to $\mathbb{D}$.

Braids in P

Corollary (Braids in P)

A homeomorphism $\mathbb{D} \to P$ induces a homeomorphism $h\colon \textsc{UConf}_n(\mathbb{D}) \to \textsc{UConf}_n(P)$. In particular, for any choice of basepoint Z in $\textsc{UConf}_n(\mathbb{D})$, there is an induced isomorphism

$$\pi_1(\textsc{UConf}_n(\mathbb{D}), Z) \cong \pi_1(\textsc{UConf}_n(P), h(Z)) = \textsc{Braid}_n.$$

Remark (Points in ∂P)

When $\textsc{Braid}_n$ is viewed as the mapping class group of a punctured disk, the punctures cannot move into the boundary since this would alter the topological type of the space. When $\textsc{Braid}_n$ is viewed as the fundamental group of a configuration space of unlabeled points, they can move into the boundary.

The extra flexibility is surprisingly useful.

Standard Basepoints and Disks

Definition (Roots of unity)

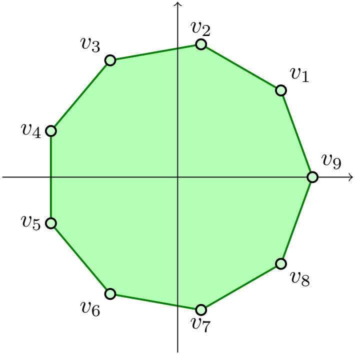
Let $\zeta = e^{2\pi i/n} \in \mathbb{C}$ be a primitive n-th root of unity and let v_i be the point ζ^i for all $i \in \mathbb{Z}$. Since $\zeta^n = 1$, the subscript i should be interpreted as an integer representing $i + n\mathbb{Z} \in \mathbb{Z}/n\mathbb{Z}$.

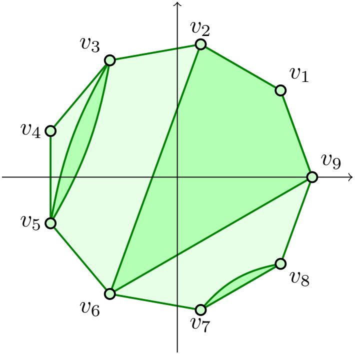
Definition (Standard basepoints and disks)

The standard basepoint for PBRAID_n is $\vec{v} = (v_1, v_2, \ldots, v_n)$ and the standard basepoint for BRAID_n is $V = \{v_1, v_2, \ldots, v_n\}$. Let P be the convex hull of the points in $V = \text{SET}(\vec{v})$. Our standing assumption of $n > 2$ means that P is homeomorphic to the disk $\mathbb{D}$. We call P the standard disk for BRAID_n.

For us the notation BRAID_n means $\pi_1(\text{UCONF}(P), V)$.

Standard Disks and Subdisks





Standard Subdisks

Definition (Subsets of vertices)

For each non-empty $A \subset [n]$ of size k, let $V_A = \{v_i \mid i \in A\} \subset V$.

Definition (Subdisks $k > 2$)

For $k > 2$, let P_A be the convex hull of the points in V_A and note that P_A is a k-gon homeomorphic to $\mathbb{D}$. We call this the standard subdisk for $A \subset [n]$.

Definition (Subdisks $k = 2$)

For $k = 2$ and $A = \{i, j\}$, we define P_A so that it is a topological disk. Take two copies of the path along the straight line segment e_{ij} connecting v_i and v_j and then bend one or both of these copies so that they become injective paths from v_i to v_j with disjoint interiors which together bound a bigon inside P.

Representatives

Definition (Representatives)

Each braid $\alpha \in \text{BRAID}_n$ is a basepoint-preserving homotopy class of a path $f\colon[0,1] \to \text{UCONF}_n(P, V)$ that describes a loop based at the standard basepoint V. We write $\alpha = [f]$ and say that the loop *f represents α*. Greek letters - α, β, δ - are braids and Roman letters - f, g, h - are their representatives.

Vertical drawings of braids in $\mathbb{R}^3$ typically have the $t = 0$ start at the top and the $t = 1$ end at the bottom. As a mnemonic, we use superscripts for information about the start of a braid or a path and subscripts for information about its end.

Strands

Let f be a representative of a braid. A strand of f is a path in P.

Definition (Strand that starts at v_i)

The strand that starts at v_i is the path $f^i \colon [0,1] \to P$ defined by the composition $f^i = \text{PROJ}_i \circ \widetilde{f^{\vec{v}}}$.

Definition (Strand that ends at v_j)

The strand that ends at v_j is the path $f_j \colon [0,1] \to P$ defined by the composition $f_j = \text{PROJ}_j \circ \widetilde{f_{\vec{v}}}$.

When the strand of f that starts at v_i and ends at v_j the path f^i is the same as the path f_j. We write f^i, f_j or f^i_j for this path and we call it the $(i,\cdot)$-strand, the $(\cdot,j)$-strand or the (i,j)-strand of f.

Drawings

A drawing of a braid representative f is the union of the graphs of its strands inside the polygonal prism $[0,1] \times P$.

Definition (Drawings)

To embed this prism into $\mathbb{R}^3$ the complex plane containing P is identified with either the first two coordinates of $\mathbb{R}^3$ and the third coordinate indicates the value $t \in [0,1]$ arranged so that the $t = 0$ start of f is at the top and the $t = 1$ end of f is at the bottom.

Definition (Multiplication)

Let α_1 and α_2 be braids with representatives f_1 and f_2. The product $\alpha_1 \cdot \alpha_2$ is $[f_1.f_2]$ where $f_1.f_2$ is the concatenation of f_1 and f_2. In the drawing of $f_1.f_2$ the drawing of f_1 is above and the drawing of f_2 is before.

Rotations

Definition (Rotations of subdisks)

For $A \subset [n]$ of size $k = |A| > 1$ we define an element $\delta_A \in \textsc{Braid}_n$ that rotates the vertices in V_A. It is the braid represented by the path in $\textsc{UConf}_n(P)$ that fixes the vertices in $V - V_A$ and where every vertex $v_i \in V_A$ travels in a counter-clockwise direction in ∂P_A to the next vertex of P_A.

Definition (Rotations of edges)

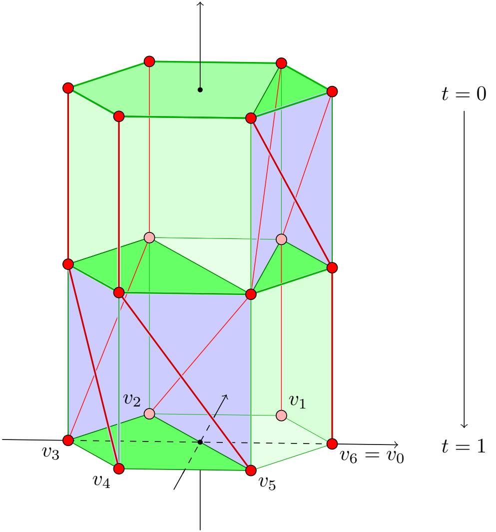
If $A = \{i, j\}$ and $e = e_{ij}$ is the edge connecting v_i and v_j, then we sometimes write δ_e to mean δ_A, the rotation of v_i and v_j around the boundary of the bigon P_A.

The next slide shows that product $\alpha = \alpha_1 \cdot \alpha_2$ where $\alpha = \delta_{\{1,2,3,4,5,6\}}$, $\alpha_1 = \delta_{\{1,5,6\}}$ and $\alpha_2 = \delta_{\{2,3,4,5\}}$. The map $\textsc{Perm}$ sends this product to $(1, 2, 3, 4, 5, 6) = (1, 5, 6) \cdot (2, 3, 4, 5)$.

Product of two rotations

$\alpha_1 = \delta_{\{1,5,6\}}$

$\alpha_2 = \delta_{\{2,3,4,5\}}$



Dual Parabolic Subgroups

For each $A \subset [n]$ of size k, let $B = [n] - A$ and $P^B = P - V_B$.

Lemma (Isomorphic groups)

The inclusion map $P_A \hookrightarrow P^B$ extends to an inclusion map $h\colon \mathrm{UC{\scriptsize ONF}}_k(P_A) \hookrightarrow \mathrm{UC{\scriptsize ONF}}_k(P^B)$ and it induces an isomorphism $h_*\colon \pi_1(\mathrm{UC{\scriptsize ONF}}_k(P_A), V_A) \to \pi_1(\mathrm{UC{\scriptsize ONF}}_k(P^B), V_A)$.

For $k > 1$, P_A is a disk, $\pi_1(\mathrm{UC{\scriptsize ONF}}_k(P_A), V_A)$ is isomorphic to $\mathrm{B{\scriptsize RAID}}_k$ and by the lemma so is $\pi_1(\mathrm{UC{\scriptsize ONF}}_k(P^B), V_A)$.

Definition (Dual Parabolic Subgroups)

For each A of size k, $\mathrm{B{\scriptsize RAID}}_A$ is $g_*(\pi_1(\mathrm{UC{\scriptsize ONF}}_k(P^B), V_A))$ where $g\colon \mathrm{UC{\scriptsize ONF}}_k(P^B) \hookrightarrow \mathrm{UC{\scriptsize ONF}}_n(P)$ is the map that sends $U \in \mathrm{UC{\scriptsize ONF}}_k(P^B)$ to $g(U) = U \cup V_B \in \mathrm{UC{\scriptsize ONF}}_n(P)$.

Note that $g(V_A) = V$. $\mathrm{B{\scriptsize RAID}}_A$ is a dual parabolic subgroup.

Fixing Vertices

Definition (Fixing Vertices)

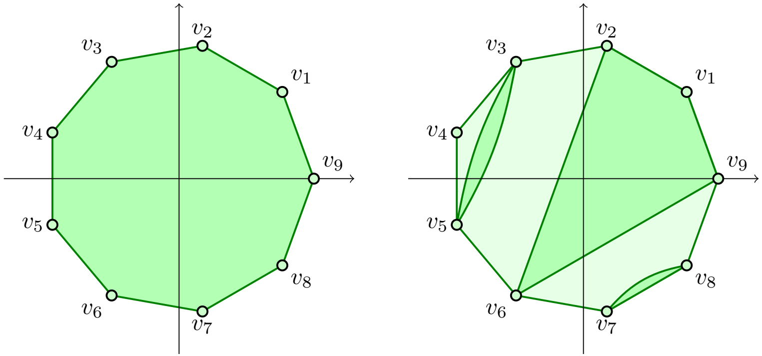
Let $\alpha = [f]$ be a braid in $\textsc{Braid}_n$. We say that f fixes $v_i \in V$ if the strand that starts at v_i is a constant path, f fixes $V_B \subset V$ if it fixes each $v_i \in V_B$ and α fixes V_B if it has some representative f that fixes V_B. Let $\textsc{Fix}(B) = \{\alpha \in \textsc{Braid}_n \mid \alpha \text{ fixes } V_B\}$.

Special representatives can be concatenated and inverted while remaining special, so $\textsc{Fix}(B)$ is a subgroup of $\textsc{Braid}_n$.

Lemma ($\textsc{Fix}(B) = \textsc{Braid}_A$)

If A and B are sets that partition $[n]$, then the fixed subgroup $\textsc{Fix}(B)$ is equal to the parabolic subgroup $\textsc{Braid}_A$.

Disks and Subdisks revisited



Dual Parabolic Intersections

It is straight-forward to show that the collection of of irreducible dual parabolic subgroups is closed under intersection and extremely well-behaved.

Proposition (Dual Parabolic Intersections)

For all $n > 0$ and for every non-empty $B \subset [n]$,

$$\textsc{Fix}_n(B) = \bigcap_{i \in B} \textsc{Fix}_n(\{i\})$$

and, as a consequence, for all non-empty $C, D \subset B$,

$$\textsc{Fix}_n(C \cup D) = \textsc{Fix}_n(C) \cap \textsc{Fix}_n(D).$$