ENTROPY AND QUASIMORPHISMS

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Abstract. Let S be a compact oriented surface. We construct homogeneous quasimorphisms on $\mathrm{Diff}(S, \mathrm{area})$, on $\mathrm{Diff}_0(S, \mathrm{area})$ and on $\mathrm{Ham}(S)$ generalizing the constructions of Gambaudo-Ghys and Polterovich.

We prove that there are infinitely many linearly independent homogeneous quasimorphisms on $\mathrm{Diff}(S, \mathrm{area})$, on $\mathrm{Diff}_0(S, \mathrm{area})$ and on $\mathrm{Ham}(S)$ whose absolute values bound from below the topological entropy. In case when S has a positive genus, the quasimorphisms we construct on $\mathrm{Ham}(S)$ are C^0-continuous.

We define a bi-invariant metric on these groups, called the entropy metric, and show that it is unbounded. In particular, we reprove the fact that the autonomous metric on $\mathrm{Ham}(S)$ is unbounded.

1. Introduction

Let M be a smooth compact manifold with some fixed Riemannian metric. Let $f: M \to M$ be a continuous function. Recall that the topological entropy of f may be defined as follows. Let $\mathbf{d}$ be the metric on M induced by some

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Riemannian metric. For $p \in \mathbf{N}$ define a new metric $\mathbf{d}_{f,p}$ on M by

$$\mathbf{d}_{f,p}(x,y) = \max_{0 \leq i \leq p} \mathbf{d}(f^i(x), f^i(y)).$$

Let $M_f(p,\epsilon)$ be the minimal number of ϵ-balls in the $\mathbf{d}_{f,p}$-metric that cover M. The topological entropy $h(f)$ is defined by

$$h(f) = \lim_{\epsilon \to 0} \limsup_{p \to \infty} \frac{\log M_f(p,\epsilon)}{p},$$

where the base of log is two. It turns out that $h(f)$ does not depend on the choice of Riemannian metric, see [6, 14].

It is notoriously difficult to compute topological entropy of a given diffeomorphism, even to detect whether entropy of a given diffeomorphism is non zero is a difficult task in most cases.

We consider the case when M is a compact connected oriented surface S endowed with an area form. Let $\mathrm{Diff}(S, \mathrm{area})$ and $\mathrm{Diff}_0(S, \mathrm{area})$ be groups, where the first group is the group of area-preserving diffeomorphisms of S, and the second group is the group of area-preserving diffeomorphisms of S isotopic to the identity. If S has a boundary, then we assume that diffeomorphisms are identity in some fixed neighborhood of the boundary.

In the first part of the paper we revise and extend the construction of quasimorphisms on $\mathrm{Diff}_0(S, \mathrm{area})$ given by Gambaudo-Ghys [17] and Polterovich [23], see also [7, 8]. The main advantage of our approach is that it allows to treat all surfaces in an unified way and to show there are infinitely many linearly independent homogeneous quasimorphisms on $\mathrm{Diff}(S, \mathrm{area})$ whose restrictions on $\mathrm{Diff}_0(S, \mathrm{area})$ are linearly independent.

In the second part of our work we show that there are infinitely many linearly independent homogeneous quasimorphisms on $\mathrm{Diff}(S, \mathrm{area})$ and on $\mathrm{Diff}_0(S, \mathrm{area})$ whose absolute values bound from below the topological entropy. The same holds for the group $\mathrm{Ham}(S)$ of Hamiltonian diffeomorphisms of S. More precisely, we apply the construction described in Section 2 to quasimorphisms on mapping class groups constructed by Bestvina and Fujiwara in [3]. We prove that these quasimorphisms are Lipschitz with respect to the topological entropy. Our work is inspired by the paper of Gambaudo and Pecou [16] who constructed a dynamical cocycle on the group $\mathrm{Diff}(D^2, \mathrm{area})$ which bounds from below the topological entropy.

Recall that a function ψ from a group G to the reals is called a quasimorphism if there exists D such that

$$|\psi(a) - \psi(ab) + \psi(b)| < D$$

for all $a, b \in G$. Minimal such D is called the defect of ψ and denoted by D_ψ. A quasimorphism ψ is homogeneous if $\psi(a^n) = n\psi(a)$ for all $n \in \mathbf{Z}$ and

$a \in G$. Quasimorphism ψ can be homogenized by setting

$$\overline{\psi}(a) := \lim_{p\to\infty} \frac{\psi(a^p)}{p}.$$

The vector space of homogeneous quasimorphisms on G is denoted by $Q(G)$. For more information about quasimorphisms and their connections to different branches of mathematics, see [13]. Throughout the paper we assume that the surface S is always connected. Our main result is the following

Theorem 1. *Let S be a compact oriented Riemannian surface and let the group $G = \mathrm{Diff}(S, \mathrm{area})$ or $G = \mathrm{Diff}_0(S, \mathrm{area})$ or $G = \mathrm{Ham}(S)$. Then there exists an infinite dimensional subspace of $Q(G)$ such that every Ψ in this subspace is Lipschitz with respect to the topological entropy, i.e., there exists a positive constant C_Ψ, which depends only on Ψ, such that for every $f \in G$ we have*

$$|\Psi(f)| \le C_\Psi h(f).$$

Let $\mathrm{Ent}(S) \subset G$ be the set of entropy-zero diffeomorphisms. This set is conjugation invariant and it generates G, see Lemma 5.1. In other words, a diffeomorphism of S is a finite product of entropy-zero diffeomorphisms. One may ask for a minimal decomposition and this question leads to the concept of the entropy norm which we define by

$$\|f\|_{\mathrm{Ent}} := \min\{k \in \mathbf{N} \mid f = h_1 \cdots h_k,\ h_i \in \mathrm{Ent}(S)\}.$$

It is the word norm associated with the generating set $\mathrm{Ent}(S)$. This set is conjugation invariant, so is the entropy norm. The associated bi-invariant metric is denoted by $\mathbf{d}_{\mathrm{Ent}}$. It follows from the work of Burago-Ivanov-Polterovich [12] and Tsuboi [26, 27] that for many manifolds all conjugation invariant norms on $\mathrm{Diff}_0(M)$ are bounded. Hence the entropy norm is bounded in those cases.

We show that the situation is different for G. More precisely, as a corollary of our main result we obtain the following

Theorem 2. *Let S be a compact oriented Riemannian surface and let the group $G = \mathrm{Diff}_0(S, \mathrm{area})$ or $G = \mathrm{Diff}(S, \mathrm{area})$ or $G = \mathrm{Ham}(S)$. Then the diameter of $(G, \mathbf{d}_{\mathrm{Ent}})$ is infinite. Moreover, in case when S is a closed disc, for each $m \in \mathbf{N}$ there exists a bi-Lipschitz embedding*

$$\mathbf{Z}^m \hookrightarrow (G, \mathbf{d}_{\mathrm{Ent}}),$$

where $\mathbf{Z}^m$ is endowed with the l^1-metric.

Remark. There exists another conjugation invariant word norm on $\mathrm{Ham}(S)$, the autonomous norm. It is unbounded in the case when S is a compact oriented surface, see [8, 9, 10, 11, 17]. Theorem 2 together with the fact that every autonomous diffeomorphism of a surface has entropy zero, see [29], gives a new proof of unboundedness of the autonomous norm on $\mathrm{Ham}(S)$.

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2. Quasimorphisms on diffeomorphisms groups of surfaces

Let S be a compact oriented surface endowed with an area form. In this section we define two linear maps

$$\mathcal{G}_{S,n}\colon Q(\mathrm{MCG}(S,n)) \to Q(\mathrm{Diff}(S,\mathrm{area})),$$

$$\mathcal{G}^0_{S,n}\colon Q(\mathrm{K}(S,n)) \to Q(\mathrm{Diff}_0(S,\mathrm{area})).$$

Here $\mathrm{MCG}(S,n)$ is the mapping class group of S with n punctures and $\mathrm{K}(S,n)$ is a certain subgroup of $\mathrm{MCG}(S,n)$. If S is a disk, then the group $\mathrm{K}(S,n)$ is isomorphic to the braid group on n strands and $\mathcal{G}^0_{S,n}$ is the map defined by Gambaudo-Ghys [7, 17]. If S is a closed surface of genus $g \geq 2$, then $\mathrm{K}(S,1)$ is isomorphic to the fundamental group of S and $\mathcal{G}^0_{S,1}$ is the map defined by Polterovich [23].

The main difference in our approach is that we work with mapping class groups, instead of braid groups. Our definition of $\mathcal{G}^0_{S,n}$ is isotopy free, which allows us to define $\mathcal{G}^0_{S,n}$ in the case when $\pi_1(\mathrm{Diff}_0(S,\mathrm{area}))$ is non-trivial. It also allows us to extend the construction to the case of $\mathrm{Diff}(S,\mathrm{area})$.

The map $\mathcal{G}_{S,n}$ is an extension of $\mathcal{G}^0_{S,n}$ in a sense that we have the following commutative diagram

$$\begin{array}{ccc} Q(\mathrm{MCG}(S,n)) & \xrightarrow{\mathcal{G}_{S,n}} & Q(\mathrm{Diff}(S,\mathrm{area})) \\ \downarrow & & \downarrow \\ Q(\mathrm{K}(S,n)) & \xrightarrow{\mathcal{G}^0_{S,n}} & Q(\mathrm{Diff}_0(S,\mathrm{area})) \end{array}$$

The horizontal maps are restrictions to subgroups. Many quasimorphisms we construct on $\mathrm{Diff}(S,\mathrm{area})$ restrict to non-trivial quasimorphisms on $\mathrm{Diff}_0(S,\mathrm{area})$. In particular, they do not arise as a pull back of quasimorphims on $\mathrm{MCG}(S)$ by the quotient map

$$\mathrm{Diff}(S,\mathrm{area}) \to \mathrm{MCG}(S).$$

In the remaining part of this section we define maps $\mathcal{G}_{S,n}$, $\mathcal{G}^0_{S,n}$ and discuss their basic properties.

2.A. **Configuration space.** Let D^2 be an open disc in the Euclidean plane. Let X_n be the configuration configuration space of n points in D^2. We fix $z = (z_1, \dots, z_n) \in X_n$.

Let $ev_z \colon \mathrm{Diff}(D^2) \to X_n$ be defined by $f \to f(z) = (f(z_1), \dots, f(z_n))$.

It is shown in [16, Section 3.2 and Theorem 4], that there is a subset H_n of X_n with the following properties: it is a union of submanifolds of codimension 1, there exists a map $h \colon X_n \backslash H_n \to \mathrm{Diff}(D^2)$ which is a section of ev_z, i.e., $ev_z \circ h$ is the identity on $X_n \backslash H_n$.

Denote by $\Omega_n = X_n \backslash H_n$, and let h be a section described as follows: let $x = (x_1, \dots, x_n) \in \Omega_n$. Let P_i be a geodesic segment connecting z_i to x_i. The fact that $x \notin H_n$ guaranties, that x_i and z_i do not lie on P_j for every i and $j \neq i$ (for details see [16, Section 3.2]). By $N_\epsilon(P_i)$ we denote the ϵ-neighborhood of P_i. By the definition of H_n, we can pick a small $\epsilon(x) \in \mathbf{R}_+$ such that $x_i, z_i \notin N_{\epsilon(x)}(P_j)$ for every i and $j \neq i$. The choice of $\epsilon(x)$ can be made such that the function $\epsilon(x)$ is C^1-continuous on Ω_n.

Let $h_{x_i} \in \mathrm{Diff}(D^2)$ be a map which pushes z_i to x_i along P_i and is supported on $N_{\epsilon(x)}(P_i)$. We set $h_x = h_{x_1} \circ h_{x_2} \circ \ldots \circ h_{x_n}$. By the definition, h_x maps z_i to x_i. This is a C^1-continuous section of ev_z.

2.B. **The cocycle.** Let S be a compact oriented surface with an area form. We take a map $j \colon D^2 \to S$ which is an attachment of an open 2-cell to the 1-skeleton of S. The image of D^2 is of full measure in S. In what follows we always regard D^2 as a subset of S.

The area form on S induces the volume form on $X_n(S)$, which is the configuration space of n-points in S. Spaces $X_n = X_n(D^2)$ and Ω_n are full measure subsets of $X_n(S)$. The group $\mathrm{Diff}(S, \mathrm{area})$ acts on $X_n(S)$ preserving the measure. Let $\mathrm{MCG}(S, n) = \mathrm{MCG}(S, \{j(z_1), \dots, j(z_n)\})$, where $\{z_i\}$ are defined in Section 2.A. We define a cocycle

$$\gamma_{S,n} \colon \mathrm{Diff}(S, \mathrm{area}) \times X_n \to \mathrm{MCG}(S, n)$$

by the formula

$$\gamma_{S,n}(f, x) = [h^{-1}_{f(x)} \circ f \circ h_x].$$

To be fully correct the map $\gamma_{S,n}(f) \colon X_n \to X_n$ is not defined on X_n, but on a full measure subset of X_n which depends on f, namely on $\Omega_{n,f} = \Omega_n \cap f^{-1}(\Omega_n)$. It is easy to show that $\gamma_{S,n}$ is a cocycle, i.e.,

$$\gamma_{S,n}(fg, x) = \gamma_{S,n}(f, g(x))\gamma_{S,n}(g, x).$$

Consider the forgetful map

$$F\colon \operatorname{MCG}(S,n) \to \operatorname{MCG}(S)$$

and denote $\mathrm{K}(S,n) = Ker(F)$. If $f \in \operatorname{Diff}_0(S, \text{area})$ then

$$\gamma_{S,n}(f,x) = [h^{-1}_{f(x)} \circ f \circ h_x]$$

is homotopic to the identity in S, possibly by a homotopy which can move points $j(z_i)$. Thus $F(\gamma_{S,n}(f,x)) = 1$ and $\gamma_{S,n}(f,x) \in \mathrm{K}(S,n)$. It follows that we can restrict $\gamma_{S,n}$ to $\operatorname{Diff}_0(S,\text{area})$ and obtain the cocycle:

$$\gamma^0_{S,n}\colon \operatorname{Diff}_0(S,\text{area}) \times X_n \to \mathrm{K}(S,n).$$

In the same way $\gamma_{S,n}$ restricts to $\operatorname{Ham}(S)$.

2.B.1. *Relation to braids.* Let us recall a construction due to Gambaudo and Ghys [17, Section 5.2]. With an isotopy f_t, $t \in [0,1]$ and a point $x = (x_1, \dots, x_n) \in X_n(S)$, we associate a braid $\gamma'_{S,n}(f,x) \in \mathbf{B}_n(S)$ in the following way: we connect z with x using geodesic segments as in Section 2.A, then we connect x with $f(x)$ by $f_t(x)$, $t \in [0,1]$ and at the end we connect $f(x)$ again with z using geodesic segments.

Let us now describe the relation between $\gamma'_{S,n}(f,x)$ and $\gamma_{S,n}(f,x)$. Recall the Birman map:

$$Push\colon \mathbf{B}_n(S) \to \operatorname{MCG}(S,n),$$

where $\mathbf{B}_n(S) = \pi_1(X_n(S), z)$ is the braid group of S on n strings and the definition of $Push$ is the following: let $\gamma(t)$, $t \in [0,1]$, be a loop in $X_n(S)$ based at z and $\psi_t \in \operatorname{Diff}(S)$ an isotopy such that $\psi_t(z) = \gamma(t)$. Then $Push([\gamma]) = \psi_1$. From this description of $Push$ it is immediate that $Push(\gamma'_{S,n}(f,x)) = \gamma_{S,n}(f,x)$.

2.B.2. *Finitely many mapping classes.* We say that a function γ defined on a probability space X has essentially finite image, if there exists a full measure subset of X on which γ has finite image.

Lemma 2.1. *For given $f \in \operatorname{Diff}(S,\text{area})$, the map*

$$\gamma_{S,n}(f)\colon X_n \to \operatorname{MCG}(S,n)$$

has essentially finite image.

Before the proof we need some preparations. The following Proposition is an immediate consequence of the cocycle condition.

Proposition 2.2. *Let $f_i \in \operatorname{Diff}(S,\text{area})$, $i = 1, \dots, n$. Assume that functions $\gamma_{S,n}(f_i)$ have essentially finite images. Then $f_1 f_2 \dots f_n$ has essentially finite image.*

To prove Lemma 2.1 it is enough to prove it for some generating set of $\operatorname{Diff}(S,\text{area})$. Let us consider the following three types of diffeomorphisms.

- Morse autonomous diffeomorphisms: let H be a Morse function on S. A Morse autonomous diffeomoprhism f is the Hamiltonian diffeomorphism defined by H, i.e. f is the time-one map of the flow f_t given by a vector field X_H, where X_H is defined by the equation $dH = \iota_{X_H}$ area.
- Hamiltonian pushes: let σ be a simple loop in S and let A be a tubular neighborhood of σ. A Hamiltonian push is an element in $\mathrm{Diff}_0(S, \mathrm{area})$ which is the identity on the complement of A and when restricted to $A \cong [0,1] \times S^1$ it is a time-t map for some $t \in \mathbf{R}$ of a Hamiltonian defined by $H(s, \psi) = g(s)$ where g is a monotone function such that $g(\delta) = 0$ and $g(1-\delta) = 1$ for all $\delta < \frac{1}{3}$.
- Area-preserving Dehn twists: the standard Dehn twist of the annulus $[0,1] \times S^1$ is given by $D(s, \psi) = (s, \psi + s)$. Note that D preserves the Lebesgue measure on $[0,1] \times S^1$. Let σ be a simple loop in S and let A be a tubular neighbourhood of σ. An area-preserving Dehn twist associated to σ is a map which is the identity on the complement of A and on A it is the pull-back of D by some area-preserving diffeomorphism between A and $[0,1] \times S^1$.

Lemma 2.3. *Let S be a closed oriented surface. Then Morse autonomous diffeomorphisms, Hamiltonian pushes and area-preserving Dehn twists generate* $\mathrm{Diff}(S, \mathrm{area})$.

Proof. Note that the set of Morse autonomous diffeomorphisms is a conjugacy invariant subset of $\mathrm{Ham}(S)$. If follows from the simplicity of $\mathrm{Ham}(S)$ that this set generates $\mathrm{Ham}(S)$. Now consider the flux homomorphism

$$Flux\colon \mathrm{Diff}_0(S, \mathrm{area}) \to H^1(S, \mathbf{R})/\Gamma.$$

It is known that $Ker(Flux) = \mathrm{Ham}(S)$ and for every $c \in H^1(S, \mathbf{R})/\Gamma$ one can find a product of Hamiltonian pushes p such that $Flux(p) = c$. Thus Morse autonomous diffeomorphisms and Hamiltonian pushes generate $\mathrm{Diff}_0(S, \mathrm{area})$. Recall that

$$\mathrm{MCG}(S) = \mathrm{Diff}(S, \mathrm{area}) / \mathrm{Diff}_0(S, \mathrm{area}).$$

Now the Lemma follows from the fact that $\mathrm{MCG}(S)$ is generated by mapping classes of area-preserving Dehn twists. □

Proof of Lemma 2.1. First we consider the case when S is a closed oriented surface. It follows from Lemma 2.2 that it is enough to prove the statement for Morse autonomous diffeomorphisms, Hamiltonian pushes and area-preserving Dehn twists.

Let f be a Morse autonomous diffeomorphism. There exists a full measure subset $X_n^0(S) < X_n(S)$ where the set of braids associated to f is finite, see [8, Section 2.C]. It means that the set $\{\gamma'_{S,n}(f, x) \mid x \in X_n^0(S)\}$ is finite.

The same analysis as in [8, Section 2.C] is applied to Hamiltonian pushes.

In the case of area-preserving Dehn twists we proceed as follows. Let f be a Dehn twist supported in annulus $A \subset S$. We can assume that $z_i \notin A$ for all $i = 1, \ldots, n$. Thus $[f] \in \mathrm{MCG}(S, n)$ and

$$\gamma_{S,n}(f,x)[f^{-1}] = [h^{-1}_{f(x)} \circ f \circ h_x \circ f^{-1}] \in \mathrm{MCG}(S,n).$$

Since h_x and $h_{f(x)}$ are isotopic to the identity, this implies that

$$\gamma_{S,n}(f,x)[f^{-1}] \in im(Push) = K(S,n).$$

Let $x = (x_1, \ldots, x_n) \in X_n(S)$ and P_{z_i,x_i} an interval connecting z_i with x_i as in Section 2.A. Note that $f \circ h_x \circ f^{-1}$ is a diffeomorphism which pushes z_i to $f(x_i)$ along the curve $f(P_{z_i,x_i})$. Let $\delta_{S,n}(f,x)$ be a braid described as follows: first we connect z_i with $f(x_i)$ by curves $f(P_{z_i,x_i})$ and then we connect $f(x_i)$ with z_i by $P_{z_i,f(x_i)}$. It follows from the definition of $Push$ in Subsection 2.B.1, that $Push(\delta_{S,n}(f,x)) = \gamma_{S,n}(f,x)[f^{-1}]$. Now the same analysis as in [8, Section 2.C] shows that there are finitely many braids of the form $\delta_{S,n}(f,x)$, and the proof follows for closed S.

Assume that S has a boundary. In this case we embed S into a closed surface $\overline{S}$ such that $i\colon \mathrm{MCG}(S,n) \to \mathrm{MCG}(\overline{S},n)$ is an embedding. For example, one can cap each boundary component of S with a torus with one boundary component. We extend the area form from S to $\overline{S}$. It is possible to define the geodesic segments P_i for $\overline{S}$ such that they agree with the geodesic segments defined for S (see Section 2.A). Now for $x \in S$ and $f \in \mathrm{Diff}(S, \mathrm{area})$ we have that $i \circ \gamma_{S,n}(f,x) = \gamma_{\overline{S},n}(\bar{f},x)$, where $\bar{f}$ is the extension of f to $\mathrm{Diff}(\overline{S}, \mathrm{area})$ by the identity. □

2.C. **Definition of $\mathcal{G}_{S,n}$ and $\mathcal{G}^0_{S,n}$.** Let $\psi \in Q(\mathrm{MCG}(S,n))$ and a diffeomorphism f in $\mathrm{Diff}(S, \mathrm{area})$. By Lemma 2.1 the function $x \to \psi \circ \gamma_{S,n}(f,x)$ is integrable. We define

$$\mathcal{G}'_{S,n}(\psi)(f) = \int_{X_n} \psi \circ \gamma_{S,n}(f,x)dx\,.$$

Lemma 2.4. *The function $\mathcal{G}'_{S,n}(\psi)$ is a quasimorphism.*

Proof.

$$\begin{aligned}
\mathcal{G}'_{S,n}(\psi)(fg) &= \int_{X_n} \psi \circ \gamma_{S,n}(fg,x)dx \\
&= \int_{X_n} \psi(\gamma_{S,n}(f,g(x))\gamma_{S,n}(g,x))dx \\
&\leq \int_{X_n} \psi \circ \gamma_{S,n}(f,g(x)) + \psi \circ \gamma_{S,n}(g,x) + D_\psi dx \\
&= \mathcal{G}'_{S,n}(\psi)(f) + \mathcal{G}'_{S,n}(\psi)(g) + \mathrm{Area}(S)D_\psi\,.
\end{aligned}$$

In the last equality we used the fact that g preserves the measure, thus $\mathcal{G}'_{S,n}(\psi)(f) = \int_{X_n} \psi \circ \gamma_{S,n}(f, g(x))dx$. In a similar way one shows that $\mathcal{G}'_{S,n}(\psi)(fg) \geq \mathcal{G}'_{S,n}(f) + \mathcal{G}'_{S,n}(g) - \mathrm{Area}(S)D_\psi$. □

We define $\mathcal{G}_{S,n}(\psi)$ to be the stabilization of $\mathcal{G}'_{S,n}(\psi)$, i.e.,

$$\mathcal{G}_{S,n}(\psi)(f) = \lim_{p\to\infty} \frac{\mathcal{G}'_{S,n}(\psi)}{p},$$

$$\mathcal{G}_{S,n} \colon Q(\mathrm{MCG}(S,n)) \to Q(\mathrm{Diff}(S, \mathrm{area})).$$

The map $\mathcal{G}^0_{S,n}$ is defined in the same way as $\mathcal{G}_{S,n}$, except that now instead of $\gamma_{S,n}$ we use $\gamma^0_{S,n}$. In this situation we obtain a linear map

$$\mathcal{G}^0_{S,n} \colon Q(\mathrm{K}(S,n)) \to Q(\mathrm{Diff}_0(S, \mathrm{area})).$$

It is also defined from $Q(\mathrm{K}(S,n))$ to $Q(\mathrm{Ham}(S))$.

2.D. **Embedding Theorem.** The proof of the following theorem is a variation of the proof of Ishida [19], see also [8, 10]. We present it for the reader convenience.

Theorem 2.5. *$\mathcal{G}_{S,n}$ and $\mathcal{G}^0_{S,n}$ are injective.*

Proof. We give a proof in the case of $\mathcal{G}_{S,n}$. The argument for the injectivity of $G^0_{S,n}$ goes along the same lines. Let $\psi \in Q(\mathrm{MCG}(S,n))$ and let γ in $\mathrm{MCG}(S,n)$ such that $\psi(\gamma) \neq 0$. Dehn twists generate $\mathrm{MCG}(S,n)$, thus we can express γ as a product of Dehn twists along some finite set of simple loops $\mathcal{C}$. We assume that z_i does not lie on any loop in $\mathcal{C}$. Let N be a small tubular neighborhood of loops in $\mathcal{C}$ such that $z_i \notin N$. We choose f such that $[f] = \gamma$ and f is supported in N. The idea of the proof is to show that we can choose N in a way that $\mathcal{G}_{S,n}(f) \neq 0$. In what follows we will split X_n into two pieces: one which has a small volume, and one on which f is the identity. This allows us to control the value of the integral.

Step 1. By definition f is the identity on $D^2\backslash N$. Let X_N be a set of tuples in X_n which have at least one coordinate in N. Then the area of X_N goes to zero when the volume of N goes to zero. Let $\gamma = \gamma_{S,n}$ and

$$C = \sup\{|\psi(\gamma(f,x))| \mid x \in X_n\}.$$

We have that

$$\frac{\psi(\gamma(f^p, x))}{p} = \frac{\psi(\gamma(f, f^{p-1}(x)) \dots \gamma(f,x))}{p} \leq \frac{pC + pD_\psi}{p} = C + D_\psi.$$

Thus

$$A_N = \int_{X_N} \lim_{p\to\infty} \frac{\psi(\gamma(f^p, x))}{p} dx \leq Area(X_N)(C + D_\psi).$$

Step 2. Let $U = D^2\backslash N$ and consider $U^n \subset X_n$. Note that that $X_n = U^n \cup X_N$ and f acts identically on U^n. The set U is open and has finitely many connected components. Denote them by $U_1, \ldots, U_k$. Let $x \in U^n$. By $c_j(x)$ we denote the number of coordinates of x which belong to U_j. The following two claims show that $\psi(\gamma(f,x))$ depends only on the numbers $\{c_j(x)\}_{j=1}^k$.

Claim 1. Let $x, y \in U^n$. Assume that $x_i = y_{\sigma(i)}$ for some permutation $\sigma \in \mathrm{Sym}_n$ and $i = 1, \ldots, n$. Then $\gamma(f,x)$ and $\gamma(f,y)$ are conjugated in $\mathrm{MCG}(S,n)$.

Proof. Consider a map $h_x^{-1}h_y$. This map permutes the points z_i, thus $[h_x^{-1}h_y] \in \mathrm{MCG}(S,n)$. Then $[h_x^{-1}h_y]\gamma(f,y)[h_y^{-1}h_x] = \gamma(f,x)$. □

Claim 2. Let $x, y \in U^n$ such that x_i, y_i belong to the same connected component of U for $i = 1, \ldots, n$. Then $\gamma(f,x)$ and $\gamma(f,y)$ are conjugated in $\mathrm{MCG}(S,n)$.

Proof. Let g be a diffeomorphism of S such that $g(x_i) = y_i$ and g is supported on U. In particular g can be taken to be a map which pushes x_i towards y_i and x_i travels all the time in the same connected component of U. We consider the mapping class $[h_y^{-1} \circ g \circ h_x] \in \mathrm{MCG}(S,n)$. The maps g and f have disjoint supports, hence they commute. Then

$$[h_x^{-1} \circ g^{-1} \circ h_y]\gamma(f,y)[h_y^{-1} \circ g \circ h_x] = [h_x^{-1} \circ g^{-1} \circ f \circ g \circ h_x] = \gamma(f,x).$$

□

The set U^n splits into finitely many connected components of the form $U_s = U_{s(1)} \times \ldots \times U_{s(n)}$, where $s\colon \{1, \cdots, n\} \to \{1, \cdots, k\}$. Let $C = (c_1, \ldots, c_k)$ be a partition such that $c_1 + \ldots + c_k = n$. Consider a component $U_s = U_{s(1)} \times \ldots \times U_{s(n)}$ of U^n for which $c_j = \#s^{-1}(j)$. We say that U_s is associated to C. The function $\psi(\gamma(f))$ is constant on connected components associated to C, and on each component it has the same value. Denote it by $\psi(\gamma(f,C))$. Let $L(C)$ be the number of connected components associated to C. Every connected component associated to C has the same volume $\mathrm{vol}(C) = \mathrm{area}(U_1)^{c_1} \ldots \mathrm{area}(U_k)^{c_k}$.

Recall that $\gamma(f,z) = [f]$ and $\psi([f]) \neq 0$. Let $C_0 = (c_1, \ldots, c_k)$ be the partition corresponding to z, that is c_i is the number of coordinates of z which lie in U_i. Then $\psi(\gamma(f,C_0)) = \psi([f]) \neq 0$. Since f is the identity on U, we have $\gamma(f^p,x) = \gamma(f,x)^p$. Now we compute:

$$\begin{aligned} B_N &= \int_{U^n} \lim_{p\to\infty} \frac{\psi(\gamma(f^p,x))}{p} dx = \int_{U^n} \psi(\gamma(f,x)) dx \\ &= \sum_{C\colon c_1+\ldots+c_k=n} \psi(\gamma(f,C)) L(C)\,\mathrm{vol}(C). \end{aligned}$$

If we treat area(U_i) as a free variable, this integral is a homogeneous polynomial in k variables of degree n. Denote this polynomial by P. The coefficient of the monomial vol(C_0) is $\psi(\gamma(f, C_0))L(C_0)$ and is non-zero. The area(U_i) depends on the neighborhood N. If we start shrinking N such that area(N) converges to zero, then area(U_i) converges to some number V_i. We can assume that $P(V_1, \ldots, V_k) \neq 0$. Indeed, if it is not the case, we can modify a little the loops $\mathcal{C}$ which were chosen to express $[f]$ in terms of Dehn twists. Then the values of V_i change freely, except that we have a constrain $V_1 + \ldots + V_k = \text{area}(S)$. It is easy to see, that a homogeneous polynomial P is non-trivial on every affine non-linear subspace of codimension one, so we can arrange V_i such that $P(V_1, \ldots, V_k) \neq 0$.

Now we shrink N, then area$(N) \to 0$ and area$(U_i) \to V_i$. We have that $A_N \to 0$ and $B_N \to P(V_1, \ldots, V_k) \neq 0$. Since $\mathcal{G}_{S,n}(\psi)(f) = A_N + B_N$, then for some N we have $\mathcal{G}_{S,n}(\psi)(f) \neq 0$. □

3. Curve complex

Let S be a connected oriented surface (possibly with boundary and punctures). A simple closed curve is called essential if it is not isotopic to a boundary curve, not isotopic to a curve going around exactly one puncture, and it is not isotopic to a point.

The curve complex $\mathcal{C}(S)$ of S was first defined by Harvey [18]. This simplicial complex is defined as follows: for vertices we take isotopy classes of essential simple closed curves in S. A collection of $k+1$ vertices $\{\alpha_i\}_{i=1}^{k}$ form a k-simplex whenever this collection can be realized by pairwise disjoint closed curves in S. A celebrated result of Masur-Minsky states that $\mathcal{C}(S)$ is hyperbolic [22]. We write $\mathbf{d}_{\mathcal{C}(S)}$ for the induced combinatorial path-metric on $\mathcal{C}(S)$ which assigns unit length to each edge of $\mathcal{C}(S)$.

The intersection number $\iota_S(\alpha, \beta)$ between two simple closed curves α, β on S is defined to be the minimal number of geometric intersections between α' and β' where α' is isotopic to α and β' is isotopic to β. Recall that a surface S of genus g with k boundary components and n punctures is called non-sporadic if $3g + n + k - 4 > 0$. Proof of the following lemma may be found in [25].

Lemma 3.1. *Let S be a non-sporadic surface. Then for all simple closed curves α, β with $\iota_S(\alpha, \beta) \neq 0$ we have*

$$\mathbf{d}_{\mathcal{C}(S)}(\alpha, \beta) \leq 2 \log \iota_S(\alpha, \beta) + 2.$$

Lemma 3.2. *Let S be a compact oriented surface and $p_1, \ldots, p_n \in S$. Let $S' = S \setminus \{p_1, \ldots, p_n\}$ and assume that S' is non-sporadic. Then for every Riemannian metric on S there exists a constant C such that for each two essential simple closed curves α, β in S' we have $\iota_{S'}(\alpha, \beta) \leq C l(\alpha) l(\beta)$, where $l(\alpha)$ is the Riemannian length of α.*

Proof. An analogous statement is proved in [1], c.f. [21, Lemma 4.2]. The difference is that there one works with homotopy classes of curves on the compact surface S and not on the punctured surface.

We construct a specific metric on S such that we are able to use an argument from [1]. Then, by comparing metrics, the statement of the lemma holds for any Riemannian metric on S. Let D_i be a small disc centered at p_i and let $S_o = S \setminus (D_1 \cup \ldots \cup D_n)$. We fix a hyperbolic metric on S_o such that all boundary loops ∂D_i are totally geodesic and have the same length ϵ. The induced length is denoted by l_{S_o}.

Let S_r be a 2-dimensional round sphere of radius r and let B be a ball in S_r of perimeter ϵ. We consider $S_{r,o} = S_r \setminus B$. By p we denote the point in S_r which is antipodal to the center of the ball B. Let x and y be two different points in ∂B and let $b \subset \partial B$ be an embedded arc which connects x to y. If the radius r of S_r is big compared to ϵ, then the arc b has the following property. Let γ be an arc in $S_{r,o} \setminus p$ which connects x to y. Assume that b and γ are homotopic in $S_{r,o} \setminus \{p\}$ relatively to $\{x, y\}$. Then $l(\gamma) \geq l(b)$, where l is a Riemannian length with respect to the round metric on S_r.

Now we construct a metric on S. We start with the surface S_o. To each boundary component ∂D_i we glue a copy of $S_{r,o}$ along the boundary. We obtain a surface homeomorphic to S. Note that in each copy of $S_{r,o}$ there is one antipodal point p. These antipodal points naturally correspond to points $\{p_i\}_{i=1}^n$. On S we consider the path length l_S induced by the hyperbolic length l_{S_o} on S_o and round metrics on copies of $S_{r,o}$.

Let α be an essential simple closed curve in $S' = S \setminus \{p_1, \ldots, p_n\}$. Since S_o is a deformation retract of S', α is homotopic to a simple closed curve that is contained in the hyperbolic surface S_o. Let γ_α be the unique hyperbolic geodesic contained in S_o which is homotopic to α in S'.

Claim. The loop γ_α has the minimal length among all simple loops homotopic to α in S'.

Proof. Let γ be a simple loop homotopic to α in S'. Assume, that γ is not contained in S_o. Then there exists a boundary loop of S_o, say ∂D_i, which intersects γ in at least two points. Let x and y be two distinct points in $\gamma \cap \partial D_i$ and let a be the arc contained in γ which connects x to y and is disjoint from the interior of S_o. Since a is an embedded arc, it is homotopic in $S_{r,o} \setminus \{p_i\}$ relative to $\{x, y\}$ to one of the arcs in ∂D_i whose end points are x and y. Denote this arc by b (see Figure 3.1). By construction, $l_S(b) \leq l_S(a)$. Hence if we substitute a by b, we obtain a new loop γ', which is homotopic to γ in S' and $l_S(\gamma') \leq l_S(\gamma)$. Repeating this procedure we find a loop γ'' such that $\gamma'' \subset S_o$ and $l_S(\gamma'') \leq l_S(\gamma)$. Then $l_S(\gamma_\alpha) \leq l_S(\gamma'')$ and the claim follows. □

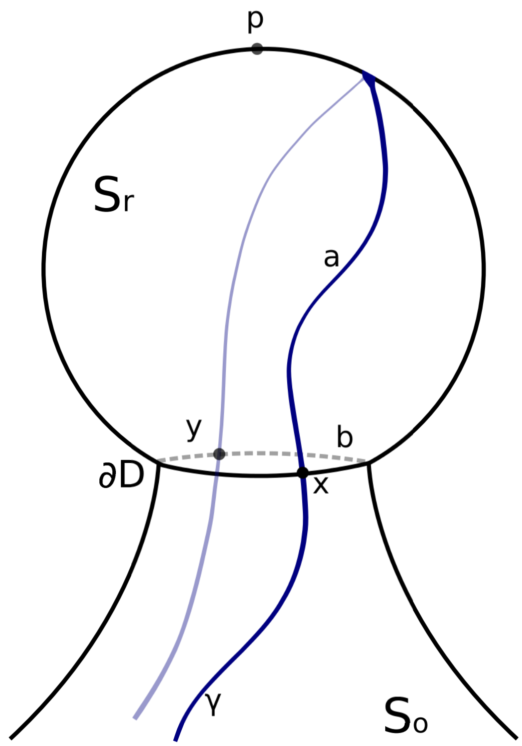

FIGURE 3.1. Loop γ and arcs a and b.

Let α and β be essential simple loops in S'. We prove that there is a constant C such that

$$\iota_{S'}(\alpha, \beta) \leq C l_S(\alpha) l_S(\beta).$$

It follows from the claim, that it is enough to prove that

$$\iota_{S'}(\alpha, \beta) \leq C l_S(\gamma_\alpha) l_S(\gamma_\beta).$$

Let us repeat the argument from [1]. We can assume that $\gamma_\alpha \neq \gamma_\beta$, otherwise this inequality is trivial. Let r_1 be a positive number which is less then the injectivity radius of the exponential map of the surface S_o equipped with the hyperbolic length l_{S_o}. The geodesic γ_α may be covered by fewer then $\frac{l_S(\gamma_\alpha)}{r_1} + 1$ geodesic arcs, each of which is contained in a geodesic disc. The same holds for γ_β. Note that if an arc is close to a boundary of S_o such a disk may contain part of the boundary of S_o, but this does not affect the argument. Now a small arc of γ_α intersects a small arc of γ_β in at most one point. Thus we have

$$\iota_{S'}(\alpha, \beta) \leq \iota_{S'}(\gamma_\alpha, \gamma_\beta) \leq \left(\frac{l_S(\gamma_\alpha)}{r_1} + 1\right)\left(\frac{l_S(\gamma_\beta)}{r_1} + 1\right).$$

Since the length l_S of every essential simple closed curve in S' is greater or equal then r_1, we get

$$\iota_{S'}(\alpha,\beta) \leq \frac{4}{r_1^2} l_S(\gamma_\alpha) l(\gamma_\beta).$$

Now let $\mathbf{g}$ be any Riemannian metric on S. It is easy to see, that the length l induced on S by $\mathbf{g}$ and l_S are comparable. Thus, there exists C such that for every loop α we have $l_S(\alpha) < Cl(\alpha)$. This finishes the proof of the lemma. □

4. Mapping class groups

Mapping class group $\mathrm{MCG}(S)$ of an oriented surface S is defined to be a group of orientation preserving diffeomorphisms of S which fix the boundary pointwise modulo diffeomorphisms which are isotopic to the identity. Since an element in $\mathrm{MCG}(S)$ takes homotopy classes of disjoint essential simple closed curves to homotopy classes of disjoint essential simple closed curves, $\mathrm{MCG}(S)$ acts by isometries on the curve complex $(\mathcal{C}(S), \mathbf{d}_{\mathcal{C}(S)})$.

Let $[f] \in \mathrm{MCG}(S)$ and α an essential simple closed curve in S. Recall that the translation length of $[f]$ is

$$\tau_S([f]) := \lim_{p\to\infty} \frac{\mathbf{d}_{\mathcal{C}(S)}(f^p(\alpha),\alpha)}{p}.$$

Translation length is independent of the choice of α and vanishes on all periodic and reducible elements of $\mathrm{MCG}(S)$.

Proposition 4.1. *Let S be a compact surface and $p_1,\dots,p_n \in S$. Let $S' = S\backslash\{p_1,\dots,p_n\}$ be a non-sporadic surface. Let $\mathbf{g}$ be a Riemannian metric on S such that the length of every essential simple closed curve is greater than one. Then there exists a constant B such that for every $[f] \in \mathrm{MCG}(S')$ we have*

$$\tau_{S'}([f]) \leq 2\log l(f(\alpha)) + B$$

for every essential simple closed curve $\alpha \subset S'$.

Proof. Let $\alpha \subset S'$ be an essential simple closed curve. We have

$$\tau_{S'}([f]) \leq \mathbf{d}_{\mathcal{C}(S')}(f(\alpha),\alpha).$$

Note that by the definition of $\mathcal{C}(S')$ we have $\mathbf{d}_{\mathcal{C}(S')}(\alpha,\beta) \leq 1$ if and only if $\iota_{S'}(\alpha,\beta) = 0$. We take a constant $C_1 := \max\{C, 1\}$, where C is a constant from Lemma 3.2. Now Lemma 3.1 together with Lemma 3.2 gives us the following inequality

$$\begin{aligned}\mathbf{d}_{\mathcal{C}(S')}(f(\alpha),\alpha) &\leq 2\log\left(C_1 l(f(\alpha)) l(\alpha)\right) + 2\\ &= 2\log l(f(\alpha)) + 2(\log\left(C_1 l(\alpha)\right) + 1).\end{aligned}$$

Combining the last two inequalities we obtain

$$\tau_{S'}([f]) \leq 2\log l(f(\alpha)) + B,$$

where $B = 2(\log(C_1 l(\alpha)) + 1)$. □

4.A. **Bestvina-Fujiwara quasimorphisms.** Here we describe a construction of quasimorphisms on mapping class groups due to Bestvina and Fujiwara [3].

Let S be an oriented surface and let ω be a finite oriented path in $\mathcal{C}(S)$. By $|\omega|$ we denote the length of ω. Let σ be a finite path. We set

$$|\sigma|_\omega = \{\text{the maximal number of non} - \text{overlapping copies of } \omega \text{ in } \sigma\}.$$

Let α, β be two vertices in $\mathcal{C}(S)$ and let W be an integer such that $0 < W < |\omega|$. Define

$$c_{\omega,W}(\alpha,\beta) = \mathbf{d}_{\mathcal{C}(S)}(\alpha,\beta) - \inf(|\sigma| - W|\sigma|_\omega),$$

where σ ranges over all paths from α to β.

Let $\alpha \in \mathcal{C}(S)$. We define $\psi_\omega \colon \operatorname{MCG}(S) \to \mathbf{R}$ by

$$\psi_\omega([f]) = c_{\omega,W}(\alpha, f(\alpha)) - c_{\omega^{-1},W}(\alpha, f(\alpha)).$$

Bestvina and Fujiwara proved that ψ_ω is a quasimorphism [3]. The induced homogeneous quasimorphism is denoted by $\overline{\psi}_\omega$. We denote by $Q_{\mathrm{BF}}(\operatorname{MCG}(S))$ the space of homogeneous quasimorphisms on $\operatorname{MCG}(S)$ which is spanned by Bestvina-Fujiwara quasimorphisms. In [3] it is proved that $Q_{\mathrm{BF}}(\operatorname{MCG}(S))$ is infinite dimensional whenever S is a non-sporadic surface.

Let $i^*\colon Q(\operatorname{MCG}(S,n)) \to Q(\mathrm{K}(S,n))$ be a homomorphism induced by the inclusion map $i\colon K(S,n) \to \operatorname{MCG}(S,n)$.

Corollary 4.2. *Let S be a closed oriented surface and $n \in \mathbf{N}$ such that S with n punctures is non-sporadic. Then*

$$\mathcal{G}_{S,n}(Q_{BF}(\operatorname{MCG}(S,n))) < Q(\operatorname{Diff}(S,\text{area}))$$

and

$$\mathcal{G}^0_{S,n} \circ i^*(Q_{BF}(\operatorname{MCG}(S,n))) < Q(\operatorname{Diff}_0(S,\text{area}))$$

are infinite dimensional.

Proof. The maps $\mathcal{G}_{S,n}$ and $\mathcal{G}^0_{S,n}$ are injective by Theorem 2.5. Since the space $Q_{BF}(\operatorname{MCG}(S,n))$ is infinite dimensional [3], it follows that $\mathcal{G}_{S,n}(Q_{BF}(\operatorname{MCG}(S,n)))$ is infinite dimensional.

It is left to prove that $i^*(Q_{BF}(\operatorname{MCG}(S,n)))$ is infinite dimensional. Since $K(S,n)$ is an infinite normal subgroup of $\operatorname{MCG}(S,n)$, it is non reducible by a theorem of Ivanov [20, Corollary 7.13]. In order to prove that the space $i^*(Q_{BF}(\operatorname{MCG}(S,n)))$ is infinite dimensional it is enough to show that $K(S,n)$ is not virtually abelian, see [3, Theorem 12]. There are three cases.

Case 1. The surface S is a sphere and $n > 3$. In this case the mapping class group of S is trivial and thus the group $K(S, n)$ is nothing but $\mathrm{MCG}(S, n)$ which is not virtually abelian.

Case 2. The surface S is a torus and $n > 1$. It follows from the Birman sequence for torus [4] that the group $K(S, n)$ maps onto $\mathbf{B}_n(S)$ modulo center, where $\mathbf{B}_n(S)$ is the torus braid group on n strings. Let $\mathbf{P}_n(S)$ be the pure torus braid group on n strings. By removing $n-2$ strings we get an epimorphism from $\mathbf{P}_n(S)$ to $\mathbf{P}_2(S)$ which is isomorphic to $\mathbf{Z}^2 \times \mathbf{F}_2$. It follows that $\mathbf{B}_n(S)$ modulo center is not virtually abelian, and so is $K(S, n)$.

Case 3. The surface S is hyperbolic. It follows from the Birman exact sequence [4, 5] that the group $K(S, n)$ is isomorphic to $\mathbf{B}_n(S)$ which is the braid group of S on n strings. Let $\mathbf{P}_n(S)$ be the pure braid group of S on n strings. By removing $n-1$ strings we get an epimorphism from $\mathbf{P}_n(S)$ to $\mathbf{P}_1(S)$ which contains $\mathbf{F}_2$. It follows that $\mathbf{B}_n(S)$ is not virtually abelian, and so is $K(S, n)$. □

Lemma 4.3. *Let S be an oriented surface. Then for every quasimorphism $\psi \in Q_{\mathrm{BF}}(\mathrm{MCG}(S))$ there is a positive constant C_ψ such that for every $[f]$ in $\mathrm{MCG}(S)$ we have*

$$|\psi([f])| \leq C_\psi \tau_S([f])$$

Proof. It follows from the definition of $Q_{\mathrm{BF}}(\mathrm{MCG}(S))$ that for each ψ in $Q_{\mathrm{BF}}(\mathrm{MCG}(S))$ there exist $k \in \mathbf{N}$, $a_1, \ldots, a_k \in \mathbf{R}$ and $\omega_1, \ldots, \omega_k$ finite oriented paths in $\mathcal{C}(S)$ such that

$$\psi = \sum_{i=1}^{k} a_i \overline{\psi}_{\omega_i}.$$

Combining the definition of $\overline{\psi}_{\omega_i}$ with triangle inequality we get

$$\overline{\psi}_{\omega_i}([f]) \leq \tau_S([f]).$$

It follows that

$$|\psi([f])| \leq \left(\sum_{i=1}^{k} |a_i| \right) \tau_S([f]).$$

□

5. Proofs

5.A. **Proof of Theorem 1.** Assume that S is a closed surface. We discuss the case of a surface with boundary at the end. Equip S with a metric as in Proposition 4.1. Denote $\mathcal{G} = \mathcal{G}_{S,n}$, $\gamma = \gamma_{S,n}$ and assume that n is such that S with n punctures is non-sporadic. At the end of the proof we discuss the case when $\mathcal{G} = \mathcal{G}^0_{S,n}$. We pick an essential simple closed curve α in the punctured surface $S' = S \setminus \{z_1, \dots, z_n\}$ (see Section 2.A). Let $\psi \in Q_{BF}(\mathrm{MCG}(S, n))$ and $f \in \mathrm{Diff}(S, \mathrm{area})$. Then

$$\begin{aligned} |\mathcal{G}(\psi)(f)| &\le \int_{X_n} \lim_{p\to\infty} \frac{|\psi \circ \gamma(f^p, x))|}{p} dx \\ &\le C_\psi \int_{X_n} \lim_{p\to\infty} \frac{\tau_{S'} \circ \gamma(f^p, x)}{p} dx \\ &\le 2C_\psi \int_{X_n} \lim_{p\to\infty} \frac{\log(l(h^{-1}_{f^p(x)} \circ f^p \circ h_x(\alpha)))}{p} dx\,, \end{aligned}$$

where the second inequality is by Lemma 4.3, the third inequality is by Proposition 4.1.

Let $x \in \Omega_n$ (see Section 2.A) and let U_x be an open set such that $x \in U_x$ and the closure of U_x is in Ω_n. It follows from the Poincaré recurrence theorem that for almost all x, after passing to a subsequence, we can assume that $f^p(x) \in U_x$. Due to C^1-continuity of the function h^{-1}_x on Ω_n, there exists a constant K_x such that

$$\sup_{p\ge 0} \|h^{-1}_{f^p(x)}\|_1 \le K_x,$$

where $\|\cdot\|_1$ is the C^1-norm. Thus for almost every $x \in X_n$ we get

$$\lim_{p\to\infty} \frac{\log(l(h^{-1}_{f^p(x)} \circ f^p \circ h_x(\alpha)))}{p} \le \lim_{p\to\infty} \frac{\log(K_x l(f^p \circ h_x(\alpha)))}{p}.$$

This yields

$$|\mathcal{G}(\psi)(f)| \le 2C_\psi \int_{X_n} \lim_{p\to\infty} \frac{\log(l(f^p \circ h_x(\alpha)))}{p} dx\,.$$

We apply Yomdin result [28, Theorem 1.4] and get that for almost every $x \in X_n$

$$\lim_{p\to\infty} \frac{\log(l(f^p \circ h_x(\alpha)))}{p} \le h(f).$$

Combining last two inequalities we get

$$|\mathcal{G}(\psi)(f)| \le 2C_\psi \,\mathrm{area}(S) h(f).$$

Since this inequality applies to any $\psi \in Q_{BF}(\mathrm{MCG}(S, n))$, then by Corollary 4.2 the space of quasimorphisms which are Lipschitz with respect to the entropy is infinite dimensional.

In case when $\mathcal{G} = \mathcal{G}^0_{S,n}$ the proof is the same. The fact that the space of quasimorphisms on $\mathrm{Diff}_0(S, \mathrm{area})$ bounding entropy from below is infinite dimensional again follows from Corollary 4.2.

Let us discuss the case of $\mathrm{Ham}(S)$. It is a simple group which is isomorphic to the commutator subgroup of $\mathrm{Diff}_0(S, \mathrm{area})$ [2]. Since every quasimorphism in $Q_{BF}(\mathrm{MCG}(S, n))$ vanishes on reducible elements, the space $i^* \circ Q_{BF}(\mathrm{MCG}(S, n))$ contains no non-trivial homomorphisms to the reals.

In addition, for every $\psi \in i^* \circ Q_{BF}(\mathrm{MCG}(S,n))$ the map $\mathcal{G}^0_{S,n}(\psi)$ is not a homomorphism. Note that the the kernel of the restriction homomorphism $Q(\mathrm{Diff}_0(S,\mathrm{area})) \to Q(\mathrm{Ham}(S))$ is the space $\mathrm{Hom}(\mathrm{Diff}_0(S,\mathrm{area}),\mathbf{R})$. Therefore the space

$$\frac{Q(\mathrm{Diff}_0(S,\mathrm{area}))}{\mathrm{Hom}(\mathrm{Diff}_0(S,\mathrm{area}),\mathbf{R})}$$

is isomorphic to a subspace of $Q(\mathrm{Ham}(S))$. It follows that the map

$$\mathcal{G}^0_{S,n} \colon i^* \circ Q_{BF}(\mathrm{MCG}(S,n)) \to Q(\mathrm{Ham}(S))$$

is injective. Now the proof is identical to the case of $\mathrm{Diff}_0(S,\mathrm{area})$.

At last let us comment on the case when S has a boundary. In this case we embed S into a closed surface $\overline{S}$. Then each one of the groups $\mathrm{Diff}(S,\mathrm{area})$, $\mathrm{Diff}_0(S,\mathrm{area})$ and $\mathrm{Ham}(S)$ embed in the usual way into the groups $\mathrm{Diff}(\overline{S},\mathrm{area})$, $\mathrm{Diff}_0(\overline{S},\mathrm{area})$ and $\mathrm{Ham}(\overline{S})$ respectively. It follows from the proof of the embedding theorem that $\mathcal{G}_{\overline{S},n}(\psi)$ is non-trivial on $\mathrm{Diff}(S,\mathrm{area})$ provided ψ is non-trivial. Similarly, $\mathcal{G}^0_{\overline{S},n}(i^* \circ \psi)$ is non-trivial on $\mathrm{Diff}_0(S,\mathrm{area})$ and on $\mathrm{Ham}(S)$ provided $i^* \circ \psi$ is non-trivial. □

5.B. **Proof of Theorem 2.** We start with the following

Lemma 5.1. *Let $G = \mathrm{Diff}(S,\mathrm{area})$, $G = \mathrm{Diff}_0(S,\mathrm{area})$ or $G = \mathrm{Ham}(S)$. Then G is generated by the set $\mathrm{Ent}(S)$ of entropy-zero diffeomorphisms of G.*

Proof. **Case 1:** $G = \mathrm{Ham}(S)$. Denote by $\mathcal{D}$ the set of embedded discs in S of area less than or equal to half of $\mathrm{area}(S)$. Then by fragmentation lemma [2] for every $f \in G$ there exists a finite collection of discs $\{D_i\}_{i=1}^k$ in $\mathcal{D}$ and diffeomorphisms $\{h_i\}_{i=1}^k$ such that each h_i is supported in D_i and $f = h_1 \circ \ldots \circ h_k$. Each $h_i \in \mathrm{Diff}(D_i,\mathrm{area})$ is generated by autonomous diffeomorphisms [9]. An autonomous diffeomorphism is a flow of a vector field, and as such has zero entropy, see [29], and the proof follows.

Case 2: $G = \mathrm{Diff}_0(S,\mathrm{area})$. There is a surjective homomorphism $Flux$ from G to $\frac{\mathrm{H}^1(S,\mathbf{R})}{\Gamma}$, where Γ is the flux group of $Flux$. The kernel of $Flux$ is $\mathrm{Ham}(S)$ [2]. Take $f \in G$. Then there exists an autonomous (and therefore of zero entropy) diffeomorphism h of S such that $Flux(f \circ h) = 0$. Hence $f \circ h \in \mathrm{Ham}(S)$ which is generated by entropy-zero diffeomorphisms and so is G.

Case 3: $G = \mathrm{Diff}(S,\mathrm{area})$. Mapping class group of S is isomorphic to $\frac{\mathrm{Diff}(S,\mathrm{area})}{\mathrm{Diff}_0(S,\mathrm{area})}$. It is generated by Dehn twists. Recall that every Dehn twist has a representative of zero entropy and the group $\mathrm{Diff}_0(S,\mathrm{area})$ is generated by entropy-zero diffeomorphisms. Hence the group G is generated by entropy-zero diffeomorphisms. □

Lemma 5.2. *Let G be a group, $\mathcal{S}$ its generating set and $\mathbf{d}_\mathcal{S}$ the induced word metric on G. Let $\psi\colon G \to \mathbf{R}$ a non-trivial homogeneous quasimorphism which vanishes on $\mathcal{S}$. Then*

$$\operatorname{diam}(G, \mathbf{d}_\mathcal{S}) = \infty.$$

Proof. Let $g \in G$. Let $s_1, \ldots, s_k \in \mathcal{S}$ such that $g = s_1 \circ \ldots \circ s_k$ and $\|g\|_\mathcal{S} = k$, where $\|\cdot\|_\mathcal{S}$ is the induced word norm on G. Then since ψ vanishes on $\mathcal{S}$ we have

$$|\psi(g)| = \left|\psi(g) - \sum_{i=1}^{k} \psi(s_i)\right| \leq D_\psi \|g\|_\mathcal{S}.$$

Take $h \in G$ such that $\psi(h) \neq 0$. Then for every $n \in \mathbf{N}$

$$\|h^n\|_\mathcal{S} \geq n \left(\frac{|\psi(h)|}{D_\psi}\right).$$

□

It follows from Theorem 1 that there are infinitely many linearly independent homogeneous quasimorphisms on $\operatorname{Diff}(S, \text{area})$, on $\operatorname{Diff}_0(S, \text{area})$ and on $\operatorname{Ham}(S)$ which vanish on the set of entropy-zero diffeomorphisms. By Lemma 5.2 we have

$$\operatorname{diam}(\operatorname{Diff}(S, \text{area}), \mathbf{d}_{\text{Ent}}) = \infty, \quad \operatorname{diam}(\operatorname{Diff}_0(S, \text{area}), \mathbf{d}_{\text{Ent}}) = \infty\,,$$
$$\operatorname{diam}(\operatorname{Ham}(S), \mathbf{d}_{\text{Ent}}) = \infty\,.$$

Now we prove the second statement of the theorem. Let $S = D^2$ be the unit disc in the plane centered at zero and $m \in \mathbf{N}$. Note that in this case

$$\operatorname{Diff}(D^2, \text{area}) = \operatorname{Diff}_0(D^2, \text{area}) = \operatorname{Ham}(D^2).$$

Let $r < \frac{1}{m}$. Denote by D_r the disc in the plane of radius r centered at zero. Denote $\mathcal{G} = \mathcal{G}_{D^2,n}$ and $\mathcal{G}_r = \mathcal{G}_{D_r,n}$. The inclusion $D_r \subset D^2$ induces and isomorphism $K(D_r, n) \simeq K(D^2, n)$. Note that $K(D^2, n) = \operatorname{MCG}(D^2, n)$ which is isomorphic via Birman isomorphism to the Artin braid group $\mathbf{B}_n$. From now on for $x \in X_n$ and $f \in \operatorname{Diff}(D^2, \text{area}))$ we regard a mapping class $\gamma(f, x)$ as an element of $\mathbf{B}_n$. We have:

$$\mathcal{G}\colon Q(\mathbf{B}_n) \to Q(\operatorname{Diff}(D^2, \text{area}))$$
$$\mathcal{G}_r\colon Q(\mathbf{B}_n) \to Q(\operatorname{Diff}(D_r, \text{area})).$$

We extend every diffeomorphism in $\operatorname{Diff}(D_r, \text{area})$ by identity on D^2 and get an injective homomorphism

$$i_r\colon \operatorname{Diff}(D_r, \text{area}) \to \operatorname{Diff}(D^2, \text{area}).$$

Lemma 5.3. *Let $n \geq 4$. Then $\mathcal{G}_r = i_r^* \circ \mathcal{G}$ on the linear subspace $Q_{\text{BF}}(\mathbf{B}_n)$ of $Q(\mathbf{B}_n)$.*

Proof. Denote by $X_{n,r}$ the space of all ordered n-tuples of distinct points in D_r. Let $\psi \in Q_{BF}(\mathbf{B}_n)$ and $f \in \mathrm{Diff}(D_r, \mathrm{area})$. We have

$$\mathcal{G}(\psi)(i_r(f)) = \lim_{p\to\infty}\left(\int\limits_{X_{n,r}} \frac{\psi(\gamma(f^p;x))}{p}\,dx + \int\limits_{X_n\setminus X_{n,r}} \frac{\psi(\gamma(f^p;x))}{p}\,dx\right)$$
$$= \mathcal{G}_r(\psi)(f) + \int\limits_{X_n\setminus X_{n,r}} \lim_{p\to\infty}\frac{\psi(\gamma(f^p;x))}{p}\,dx\,.$$

Let $inc\colon \mathbf{B}_{n-1} \to \mathbf{B}_n$ be the standard inclusion of $\mathbf{B}_{n-1}$ into $\mathbf{B}_n$. Recall that by definition $i_r(f)$ is the identity on $D^2 \setminus D_r$. It follows that for each $x \in X_n \setminus X_{n,r}$ the braid

$$\gamma(f^p;x) = \alpha_{1,p,x} \circ \gamma'_{f^p,x} \circ \alpha_{2,p,x},$$

where the braid $\gamma'_{f^p,x} \in inc(\mathbf{B}_{n-1})$ and the word length of the braids $\alpha_{1,p,x}$ and $\alpha_{2,p,x}$ is bounded for all p and x. Hence for each $x \in X_n \setminus X_{n,r}$ we have

$$\lim_{p\to\infty}\frac{\psi(\gamma(f^p;x))}{p} = \lim_{p\to\infty}\frac{\psi(\gamma'_{f^p,x})}{p} = 0,$$

where the last equality follows from the fact that $inc(\mathbf{B}_{n-1})$ is reducible in $\mathbf{B}_n$ and every quasimorphism in $Q_{\mathrm{BF}}(\mathbf{B}_n)$ vanishes on reducible elements. This finishes the proof of the lemma. □

Let us continue the proof. It follows from Theorem 1 that the subspace

$$\mathcal{G}_r(Q_{\mathrm{BF}}(\mathbf{B}_n)) < Q(\mathrm{Diff}(D_r, \mathrm{area}))$$

is infinite dimensional for $n \geq 4$ and that every quasimorphism in this space vanishes on the set of entropy-zero diffeomorphisms. It follows from [9, Lemma 3.10] that there exist $\{\Psi_{i,n,r}\}_{i=1}^m$ in $\mathcal{G}_r(Q_{\mathrm{BF}}(\mathbf{B}_n))$ and $\{f_{i,n,r}\}_{i=1}^m$ in $\mathrm{Diff}(D_r, \mathrm{area})$ such that

$$\Psi_{i,n,r}(f_{j,n,r}) = \delta_{ij},$$

where δ_{ij} is the Kronecker delta.

Set $f_i := i_r(f_{i,n,r})$. It follows from Lemma 5.3 that $\Psi_{i,n}(f_j) = \delta_{ij}$, where $\Psi_{i,n} \in \mathcal{G}(Q_{\mathrm{BF}}(\mathbf{B}_n))$ and it is defined using the same quasimorphism from $Q_{\mathrm{BF}}(\mathbf{B}_n)$ as $\Psi_{i,n,r}$. Each f_j is supported in D_r. Since $r < \frac{1}{m}$ there exists a family of diffeomorphisms $\{h_i\}_{i=1}^m \in \mathrm{Diff}(D^2, \mathrm{area})$ such that $h_i \circ f_i \circ h_i^{-1}$ and $h_j \circ f_j \circ h_j^{-1}$ have disjoint supports for $i \neq j$. Denote by $\hat{f}_i := h_i \circ f_i \circ h_i^{-1}$ and let

$$J\colon \mathbf{Z}^m \to \mathrm{Diff}(D^2, \mathrm{area}),$$

where

$$J(k_1, \ldots, k_m) = \hat{f}_1^{k_1} \ldots \hat{f}_m^{k_m}.$$

It is clear that this map is a monomorphism. We prove it is bi-Lipschitz. Since all $\hat{f}_i$ commute with each other and $\Psi_{i,n}(\hat{f}_j) = \delta_{ij}$, we obtain

$$\|\hat{f}_1^{k_1} \circ \ldots \circ \hat{f}_m^{k_m}\|_{\mathrm{Ent}} \geq \frac{|\Psi_{i,n}(\hat{f}_1^{k_1} \circ \ldots \circ \hat{f}_m^{k_m})|}{D_{\Psi_{i,n}}} = \frac{|k_i|}{D_{\Psi_{i,n}}}\,,$$

where $D_{\Psi_{i,n}}$ is the defect of the quasimorphism $\Psi_{i,n}$. We denote by $\mathfrak{D}_m := \max_i D_{\Psi_{i,n}}$ and obtain the following inequality

$$\|\hat{f}_1^{k_1} \circ \ldots \circ \hat{f}_m^{k_m}\|_{\mathrm{Ent}} \geq (m \cdot \mathfrak{D}_m)^{-1} \sum_{i=1}^{m} |k_i|\,.$$

Denote by $\mathfrak{M}_J := \max_i \|\hat{f}_i\|_{\mathrm{Ent}}$. Now we have the following inequality

$$\|\hat{f}_1^{k_1} \circ \ldots \circ \hat{f}_m^{k_m}\|_{\mathrm{Ent}} \leq \sum_{i=1}^{m} |k_i| \cdot \|\hat{f}_i\|_{\mathrm{Ent}} \leq \mathfrak{M}_J \cdot \sum_{i=1}^{m} |k_i|\,.$$

Last two inequalities conclude the proof of the theorem. □

6. Final remarks

(1) Let S be a compact oriented surface of positive genus and n such that S with n punctures is non-sporadic. Since each quasimorphism in $Q_{BF}(\mathrm{MCG}(S,n))$ vanishes on reducible elements, the induced quasimorphism on $\mathrm{Ham}(S)$ vanishes on diffeomorphisms supported in a disc. Hence every quasimorphism that lies in the image of $\mathcal{G}^0_{S,n}$ is C^0-continuous, see [15, Theorem 1.7].

(2) Let S be a compact oriented surface. One can easily show that there is an infinite family of egg-beater Hamiltonian diffeomorphisms $\{f_i\}_{i=1}^{\infty}$ of S (for definition see [24]), and a family of linearly independent quasimorphisms $\Psi_i \in Q(\mathrm{Diff}(S, \mathrm{area}))$ which are Lipschitz with respect to the topological entropy such that $\Psi_i(f_i) \neq 0$. This implies that each f_i has a positive topological entropy.

(3) Since every autonomous diffeomorphism of a surface has zero entropy, entropy norm is bounded from above by the autonomous norm. It would be interesting to know whether these norms are equivalent. Note that the existence of a homogeneous quasimorphism on $\mathrm{Ham}(S)$ which does not vanish on the set of entropy-zero diffeomorphisms, but vanishes on every autonomous diffeomorphism, would imply that these norms are not equivalent.

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