Health Insurance Mathematics: Assignments (1)

1. (10p) Describe examples 2-5 (see attached scan).
2. (10p) Describe examples 6-8 (see attached scan).
3. (10p) Describe examples 9-11 (see attached scan).
4. (5p) Let $\{S(t)\}$ be a Markov chain. Check that

$$\mathbb{P}(S(u) = j, S(w) = k | S(t_0) = i_0, ..., S(t_n) = i_n) = \mathbb{P}(S(u) = j | S(t_n) = i_n)\mathbb{P}(S(w) = k | S(u) = j)$$

   for $t_0 < t_1 < ... < t_n < u < w$.

5. (5p) Check that (Chapman – Kolmogorov equation)

$$\mathbb{P}_{i,j}(t, u) = \sum_{k \in \mathcal{S}} \mathbb{P}_{i,k}(t, w)\mathbb{P}_{k,j}(w, u) \quad \text{for} \quad t \leq w \leq u.$$

6. (10p) Prove that Kolmogorov differential equations hold.
7. (15p) Derive all transition probabilities for the model described in example 6. Then, assume that the transition intensities are constant and derive formulas for the corresponding transition probabilities (hint: see the attached scan).
8. (15p) Derive all transition probabilities for the model described in example 7. Then, assume that the transition intensities are constant and derive formulas for the corresponding transition probabilities (hint: see the attached scan).
9. (15p) Derive all transition probabilities for the model described in example 8. Then, assume that the transition intensities are constant and derive formulas for the corresponding transition probabilities (hint: see the attached scan).
10. (20p) Derive all transition probabilities for the model described in example 9. Then, assume that the transition intensities are constant and derive formulas for the corresponding transition probabilities (hint: see the attached scan).
11. (15p) Derive all transition probabilities for the model described in example 10. Then, assume that the transition intensities are constant and derive formulas for the corresponding transition probabilities (hint: see the attached scan).
12. (15p) Derive all transition probabilities for the model described in example 11. Then, assume that the transition intensities are constant and derive formulas for the corresponding transition probabilities (hint: see the attached scan).

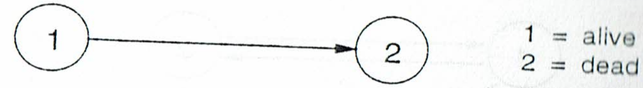

Fig. 1.3 A two-state model.

It is understood that the functions which do not appear in the definition of benefits and premiums (for example, $p_2(t)$) must be considered identically equal to zero. In the case of a single premium, on the other hand, we have $p_1(t) = 0$ and $\pi_1(0) = \pi$.

Example 2

Consider an **endowment assurance**, with c as sum assured in the case of death and in the case of survival to maturity as well. The graph is still given by Fig. 1.3 and we have (for the case of a continuous premium):

$$c_{12}(t) = c \quad (0 < t \leq n)$$

$$d_1(n) = c$$

$$p_1(t) = \begin{cases} p & \text{if } 0 \leq t < n \\ 0 & \text{if } t \geq n \end{cases}.$$

Example 3

Consider a **deferred annuity**. Premiums are assumed to be paid continuously at a rate p over $[0, m)$ when the contract stays in state 1 (i.e. the insured is alive). The benefit is a continuous annuity at a rate b after m when the contract stays in state 1, i.e. until the death of the insured. Thus, benefit and premium functions are as follows:

$$b_1(t) = \begin{cases} 0 & \text{if } 0 \leq t < m \\ b & \text{if } t \geq m \end{cases}$$

$$p_1(t) = \begin{cases} p & \text{if } 0 \leq t < m \\ 0 & \text{if } t \geq m \end{cases}.$$

Example 4

As the next step in building up more complex models, consider a **temporary assurance with a rider benefit in the case of accidental death**. In this case, we have to distinguish between death due to an accident and death due to other causes. The graphical representation is given by Fig. 1.4.

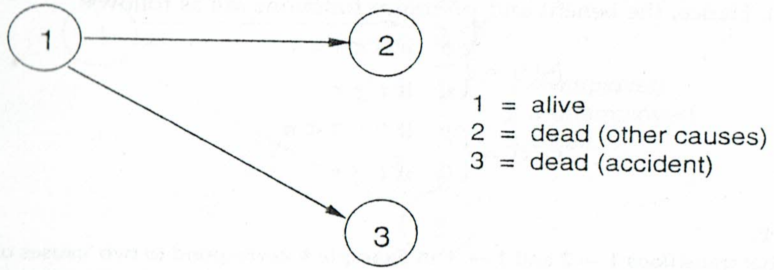

Fig. 1.4 A three-state model with two causes of 'decrement'.

The benefit and premium functions are specified as follows:

$$c_{12}(t) = c \quad (0 < t \leq n)$$
$$c_{13}(t) = c' \quad (0 < t \leq n)$$

$$p_1(t) = \begin{cases} p & \text{if } 0 \leq t < n \\ 0 & \text{if } t \geq n \end{cases},$$

where $c' > c$, and $c' - c$ is the amount of the supplementary benefit in the case of accidental death.

Example 5

Consider now an n-year insurance contract just providing a **lump sum benefit in the case of permanent and total disability**. Also in this case a three-state model is required: states 'active', 'disabled' and 'dead' must be considered. It is important to stress that, since permanent and total disability only is involved, the label 'active' concerns any insured who is alive and non-permanently (or non-totally) disabled. The model is presented in Fig. 1.5. Let c denote the sum assured. Premiums are assumed to be paid continuously at a rate p over $[0, n)$ when the contract stays in state 1 (i.e. when the insured is active).

The benefit and premium functions are as follows:

$$c_{12}(t) = c \quad (0 < t \leq n)$$

$$p_1(t) = \begin{cases} p & \text{if } 0 \leq t < n \\ 0 & \text{if } t \geq n \end{cases}.$$

The above-described model is very simple but rather unrealistic. It is more realistic to assume that the lump sum benefit will be paid out after a qualification period, which is required by the insurer in order to ascertain the permanent character of the disability; the length of the

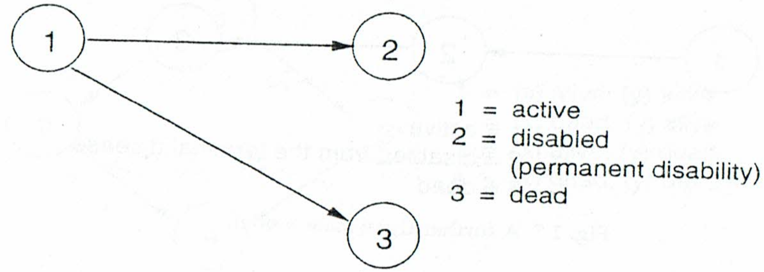

Fig. 1.5 A further three-state model with two causes of 'decrement'.

qualification period would be chosen in such a way that recovery would be practically impossible after that period. A more realistic model will be presented in Chapter 3.

Example 6

Examples 4 and 5 can be generalized to include more than two causes of 'decrement'. This leads to the widely discussed **multiple decrement model**. For example, we may be interested in the benefits provided to members within an occupational pension scheme should they retire, die or leave the scheme. A possible model is presented in Fig. 1.6. Note that in this simplified model it is assumed that there is no impact if death occurs after retirement or withdrawal.

Example 7

A more complicated structure than Example 1 can be used to represent mortality due to a certain disease. In this case, we have the following

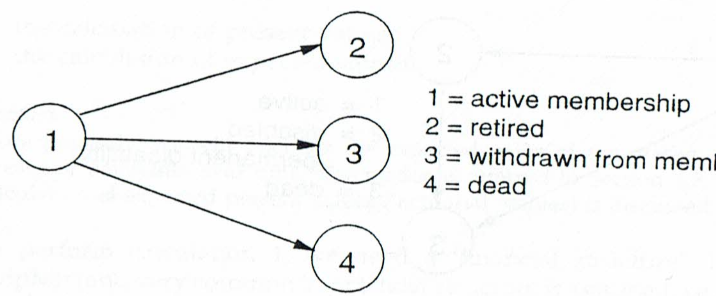

Fig. 1.6 A four-state model with three causes of 'decrement'.

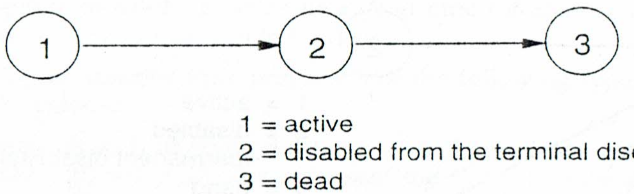

Fig. 1.7 A further three-state model.

three states: 'active', 'disabled from the terminal disease', 'dead'. If the probability of death among those not suffering from the disease is deemed to be sufficiently small for it to be ignored in the model, we can consider only the transitions depicted in Fig. 1.7.

Example 8

A more complicated structure than Example 5 is needed in order to represent an **annuity benefit in the case of permanent and total disability**. In this case, the death of the disabled insured must be considered, and then transition 2 → 3 must be added to the three-state model. The resulting graph is depicted in Fig. 1.8.

Let n denote the policy term; assume that the annuity is payable if the disability inception time belongs to the interval $[0, n)$. Let r denote the stopping time (from policy issue) of the annuity payment, $r \geq n$; for example, if x is the entry age and ξ is the retirement age, then we can assume $r = \xi - x$. (More general and more realistic policy conditions will be considered in Chapter 3.) The annuity benefit is assumed to be paid continuously at a rate b. Premiums are assumed to be paid continuously at a rate p over $[0, n)$ when the contract stays in state 1 (i.e. the insured is

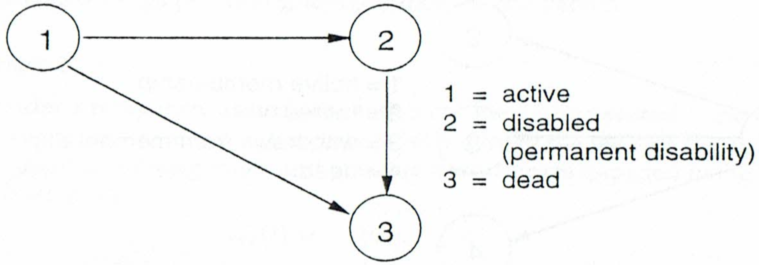

Fig. 1.8 A three-state model with two causes of 'decrement' and with a second-order decrement.

active). Hence, the benefit and premium functions are as follows:

$$b_2(t) = \begin{cases} b & \text{if } 0 < t < r \\ 0 & \text{if } t \geq r \end{cases}$$

$$p_1(t) = \begin{cases} p & \text{if } 0 \leq t < n \\ 0 & \text{if } t \geq n \end{cases}.$$

Remark

Note that transitions 1 → 2 and 1 → 3, in Example 4, correspond to two 'causes of decrements', according to the traditional actuarial terminology (or 'competing risks', according to the statistical terminology); actually, from a collective point of view, transitions 1 → 2 and 1 → 3 correspond to decrements in the number of lives belonging to the 'group of alives'. An analogous interpretation holds with reference to Examples 5 and 6. As regards Example 8, transition 2 → 3 corresponds to a 'second-order cause of decrement', in the sense that it denotes a decremental factor pertaining to a group (the disabled lives) which, in its turn, has originated by decrement from the 'initial' group (the active lives).

Example 9

Let us generalize Example 8, considering an **annuity benefit in the case of total disability**; thus, the permanent character of the disability is not required. Hence, we have to consider the possibility of recovery, and then the transition 2 → 1 must be added to the three-state model depicted in Fig. 1.8. The resulting model is illustrated by Fig. 1.9. If we assume that the policy conditions are as described in Example 6 (as far as policy term, stopping time and premium payment are concerned), then the benefit and premium functions are the same as in Example 6.

Remark

The models presented in Examples 1 to 8 contain only strictly transient states and absorbing states, whilst the model discussed in Example 9 contains (non-strictly)

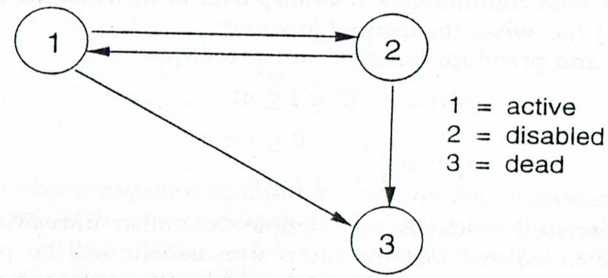

Fig. 1.9 A more general three-state model relating to disability benefits.

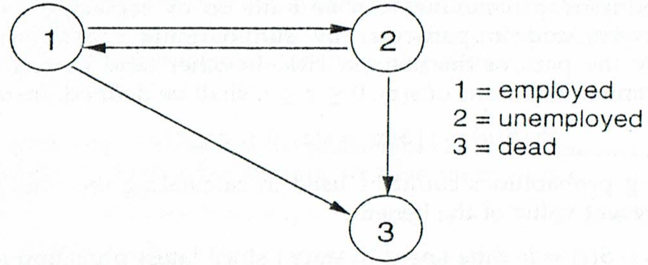

Fig. 1.10 A three-state model relating to an annuity benefit in the case of unemployment.

transient states (states 1 and 2) and an absorbing state (state 3). In section 1.4 we shall consider the difficulties which arise in calculation procedures in the presence of (non-strictly) transient states.

Example 10

Example 9 and the model depicted in Fig. 1.9 can also be adapted to refer to an annuity benefit in the case of unemployment (see Fig. 1.10).

Example 10a

A simplified version of the model depicted in Fig. 1.10 is widely used for practical calculations in respect of annuity benefits paid in the case of unemployment. The model is illustrated in Fig. 1.11, corresponding to Fig. 1.10 but with state 3 omitted completely. This adaptation may be made because the age range covered by such insurance contracts is characterized by low probabilities of death relative to the probabilities of moving from state 1 to state 2 or from state 2 to state 1, or because the financial effects of death may be small relative to that of unemployment.

Example 11

States and transitions can be used as a starting point also to represent insurance contracts with last-survivor benefits. Let us consider a **reversionary**

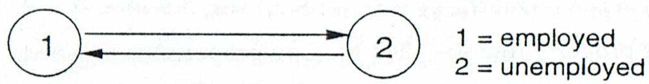

Fig. 1.11 A simplified two-state model relating to an annuity benefit in the case of unemployment.

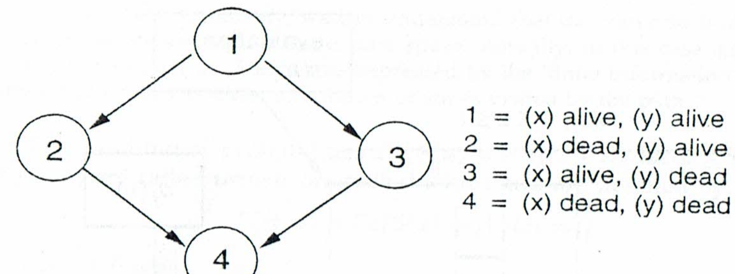

Fig. 1.12 A four-state model.

annuity (i.e. a **widow's pension**). Two lives are involved, say (x) and (y), and then four states must be considered. It is assumed that (x) and (y) cannot die simultaneously. States and transitions are illustrated in Fig. 1.12.

The policy provides a continuous life annuity to (y) at a rate b after the death of (x). Premiums are assumed to be paid continuously at a rate p over $[0, n)$, while both (x) and (y) are alive. Hence, the benefit and premium functions are as follows:

$$b_2(t) = b \quad (t > 0)$$

$$p_1(t) = \begin{cases} p & \text{if } 0 \le t < n \\ 0 & \text{if } t \ge n \end{cases}.$$

1.4 THE TIME-CONTINUOUS MARKOV MODEL

1.4.1 Some preliminary ideas

It is well known that actuarial evaluations (which are needed in order to calculate single premiums, periodic premiums, mathematical reserves, etc.) include:

1. the calculation of present values;
2. the calculation of expected values.

Remark
More precisely, we will calculate the expected value of the present value of a stream of payments over time. The reader is referred to Section 1.9, where the calculation of expected present values ('actuarial' values) is discussed.

To perform calculation 1, we need a 'financial structure'. Here, the simplest (and very common in practice) structure is assumed, i.e. the compound interest model with a non-stochastic, constant force of interest δ.

are involved, i.e. $\mu_{12}(t)$ and $\mu_{13}(t)$. In Example 4 their meaning is:

$$\mu_{12}(t) = \text{intensity of non-accidental mortality}$$
$$\mu_{13}(t) = \text{intensity of accidental mortality;}$$

while in Example 5 the meaning is as follows:

$$\mu_{12}(t) = \text{intensity of (total and permanent) disability}$$
$$\mu_{13}(t) = \text{intensity of mortality for active lives.}$$

Since states 2 and 3 are absorbing states, we have

$$P_{\underline{11}}(z,t) = P_{11}(z,t); \quad 0 \le z \le t.$$

In both cases, the set of simultaneous differential equations is as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t} P_{11}(z,t) = -P_{11}(z,t)(\mu_{12}(t) + \mu_{13}(t)) \tag{1.29a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} P_{12}(z,t) = P_{11}(z,t)\mu_{12}(t) \tag{1.29b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} P_{13}(z,t) = P_{11}(z,t)\mu_{13}(t). \tag{1.29c}$$

From equation (1.29a), using the initial condition $P_{11}(z,z) = 1$, we obtain:

$$P_{11}(z,t) = \exp\left[-\int_z^t (\mu_{12}(u) + \mu_{13}(u))\,\mathrm{d}u\right]. \tag{1.30a}$$

Then, using the initial condition $P_{12}(z,z) = 0$, we have:

$$P_{12}(z,t) = \int_z^t P_{11}(z,u)\mu_{12}(u)\,\mathrm{d}u \tag{1.30b}$$

and finally:

$$P_{13}(z,t) = 1 - P_{11}(z,t) - P_{12}(z,t). \tag{1.30c}$$

1.5.3 Example 6

Here, a four-state model is under consideration, with a strictly transient state and three absorbing states. Three transition intensities are involved:

$$\mu_{12}(t) = \text{intensity of retirement;}$$
$$\mu_{13}(t) = \text{intensity of withdrawal;}$$
$$\mu_{14}(t) = \text{intensity of mortality for active lives.}$$

The set of simultaneous differential equations is as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t} P_{11}(z,t) = -P_{11}(z,t)(\mu_{12}(t) + \mu_{13}(t) + \mu_{14}(t)) \tag{1.31a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} P_{12}(z,t) = P_{11}(z,t)\mu_{12}(t) \tag{1.31b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} P_{13}(z,t) = P_{11}(z,t)\mu_{13}(t) \tag{1.31c}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} P_{14}(z,t) = P_{11}(z,t)\mu_{14}(t) \tag{1.31d}$$

which can be integrated in a similar manner to Examples 4 and 5 in section 1.5.2 above.

1.5.4 Example 7

Here, a three-state model is under consideration, with two strictly transient states and one absorbing state. The transition intensities involved are:

$$\mu_{12}(t) = \text{intensity of incidence of terminal disease}$$
$$\mu_{23}(t) = \text{intensity of mortality of terminal disease.}$$

The set of simultaneous differential equations is as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t} P_{11}(z,t) = -P_{11}(z,t)\mu_{12}(t) \tag{1.32a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} P_{12}(z,t) = P_{11}(z,t)\mu_{12}(t) - P_{12}(z,t)\mu_{23}(t) \tag{1.32b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} P_{13}(z,t) = P_{12}(z,t)\mu_{23}(t) \tag{1.32c}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} P_{22}(z,t) = -P_{22}(z,t)\mu_{23}(t) \tag{1.32d}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} P_{23}(z,t) = P_{22}(z,t)\mu_{23}(t). \tag{1.32e}$$

1.5.5 Example 8

The situation seems to be similar to that considered in Examples 4 and 5, but the transition $2 \to 3$ must be added to the three-state model in order to express the mortality of disabled lives. Then, three transition intensities are involved:

$$\mu_{12}(t) = \text{intensity of (total and permanent) disability}$$
$$\mu_{13}(t) = \text{intensity of mortality for active lives}$$
$$\mu_{23}(t) = \text{intensity of mortality for disabled lives.}$$

Note that state 2 is now a strictly transient state. The set of simultaneous differential equations is as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}P_{11}(z,t) = -P_{11}(z,t)(\mu_{12}(t)+\mu_{13}(t)) \tag{1.33a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}P_{12}(z,t) = P_{11}(z,t)\mu_{12}(t) - P_{12}(z,t)\mu_{23}(t) \tag{1.33b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}P_{13}(z,t) = P_{11}(z,t)\mu_{13}(t) + P_{12}(z,t)\mu_{23}(t) \tag{1.33c}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}P_{23}(z,t) = P_{22}(z,t)\mu_{23}(t) \tag{1.33d}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}P_{22}(z,t) = -P_{22}(z,t)\mu_{23}(t). \tag{1.33e}$$

From equations (1.33a) and (1.33e), using the boundary conditions

$$P_{11}(z,z) = P_{22}(z,z) = 1,$$

we respectively obtain:

$$P_{11}(z,t) = \exp\left[-\int_z^t (\mu_{12}(u)+\mu_{13}(u))\,\mathrm{d}u\right] \tag{1.34a}$$

$$P_{22}(z,t) = \exp\left[-\int_z^t \mu_{23}(u)\,\mathrm{d}u\right] \tag{1.34e}$$

whence

$$P_{23}(z,t) = 1 - P_{22}(z,t). \tag{1.34d}$$

Then, using the relevant boundary conditions and after some manipulations, we obtain:

$$P_{12}(z,t) = \int_z^t P_{11}(z,u)\mu_{12}(u)P_{22}(u,t)\,\mathrm{d}u. \tag{1.34b}$$

Finally

$$P_{13}(z,t) = 1 - P_{11}(z,u) - P_{12}(z,u). \tag{1.34c}$$

As states 1 and 2 are strictly transient states, we have:

$$P_{\underline{11}}(z,t) = P_{11}(z,t) \tag{1.35a}$$

$$P_{\underline{22}}(z,t) = P_{22}(z,t). \tag{1.35b}$$

Remark 1
Example 8 discussed above constitutes a first simple but realistic model for disability annuities. The model is used for instance in Ramlau-Hansen (1991), with Makeham-like transition intensities. The intensities are chosen as

follows:

$$\mu_{12}(t) = 0.0004 + 10^{0.06(x+t)-5.46}$$

$$\mu_{13}(t) = \mu_{23}(t) = 0.0005 + 10^{0.038(x+t)-4.12}.$$

We can in particular note that:

- mortality for active lives and for disabled lives are not discriminated;
- the model is of the aggregate type, since all the transition intensities depend on attained age $x+t$ only.

Remark 2
It is very important to remark that in Examples 1 to 8 a 'sequential' procedure can be adopted in solving the set of simultaneous equations, that is in finding the transition probabilities. It is rather easy to realize that such an opportunity is allowed by the 'hierarchical' structure of the graph which describes the model (see, for instance, Fig. 1.6), i.e. by the fact that the set of nodes is an ordered set. In Example 9 we will deal with a model based on a non-ordered set of nodes; as a consequence, we will see that a sequential procedure cannot be followed for finding the transition probabilities.

1.5.6 Example 9

Now the transition $2 \to 1$ must be added to the model discussed in Example 8, in order to express the possibility of recovery. We will see that in this case it is much more difficult to express the transition probabilities in terms of the transition intensities. Four transition intensities are now involved:

$$\mu_{12}(t) = \text{intensity of (total) disability}$$

$$\mu_{21}(t) = \text{intensity of recovery}$$

$$\mu_{13}(t) = \text{intensity of mortality for active lives}$$

$$\mu_{23}(t) = \text{intensity of mortality for disabled lives.}$$

The set of simultaneous forward differential equations is as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}P_{11}(z,t) = P_{12}(z,t)\mu_{21}(t) - P_{11}(z,t)[\mu_{12}(t)+\mu_{13}(t)] \tag{1.36a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}P_{12}(z,t) = P_{11}(z,t)\mu_{12}(t) - P_{12}(z,t)[\mu_{21}(t)+\mu_{23}(t)] \tag{1.36b}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}P_{13}(z,t) = P_{11}(z,t)\mu_{13}(t) + P_{12}(z,t)\mu_{23}(t) \tag{1.36c}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}P_{21}(z,t) = P_{22}(z,t)\mu_{21}(t) - P_{21}(z,t)[\mu_{12}(t)+\mu_{13}(t)] \tag{1.36d}$$

$$\frac{d}{dt}P_{22}(z,t) = P_{21}(z,t)\mu_{12}(t) - P_{22}(z,t)[\mu_{21}(t) + \mu_{23}(t)] \quad (1.36e)$$

$$\frac{d}{dt}P_{23}(z,t) = P_{22}(z,t)\mu_{23}(t) + P_{21}(z,t)\mu_{13}(t). \quad (1.36f)$$

Note that, unlike Example 8, nodes 1 and 2 are non-strictly transient nodes; hence identities (1.35a) and (1.35b) do not hold. The following differential equations pertain to the occupancy probabilities:

$$\frac{d}{dt}P_{\underline{11}}(z,t) = -P_{\underline{11}}(z,t)[\mu_{12}(t) + \mu_{13}(t)] \quad (1.37a)$$

$$\frac{d}{dt}P_{\underline{22}}(z,t) = -P_{\underline{22}}(z,t)[\mu_{21}(t) + \mu_{23}(t)]. \quad (1.37b)$$

The solution of equations (1.37a) and (1.37b) is trivial; we get:

$$P_{\underline{11}}(z,t) = \exp\left[-\int_z^t [\mu_{12}(u) + \mu_{13}(u)]\,du\right] \quad (1.38a)$$

$$P_{\underline{22}}(z,t) = \exp\left[-\int_z^t [\mu_{21}(u) + \mu_{23}(u)]\,du\right]. \quad (1.38a)$$

The set of differential equations (1.36a) to (1.36f) can be solved as follows. Equations (1.36a) and (1.36b) constitute a pair of simultaneous differential equations with two unknown functions $P_{11}(z,t)$ and $P_{12}(z,t)$. By differentiation and substitution a second-order differential equation can be obtained for $P_{11}(z,t)$ (or for $P_{12}(z,t)$); this equation can be solved by numerical methods, allowing for the relevant boundary conditions. From the solution for $P_{11}(z,t)$, we can by substitution obtain the solution for $P_{12}(z,t)$. With these two solutions, we can solve the differential equation for $P_{13}(z,t)$, i.e. equation (1.36c). The same procedure can be adopted for the subset of equations (1.36d), (1.36e) and (1.36f).

In order to illustrate the procedure, we shall take equations (1.36a) and (1.36b). From (1.36a) we have:

$$P_{12}(z,t) = \frac{1}{\mu_{21}(t)}\left[\frac{d}{dt}P_{11}(z,t) + P_{11}(z,t)[\mu_{12}(t) + \mu_{13}(t)]\right]; \quad (1.36a')$$

substituting into equation (1.36b), after some manipulations we obtain:

$$\frac{d^2}{dt^2}P_{11}(z,t) + \left[\frac{d}{dt}\ln\mu_{21}(t) + \mu_{12}(t) + \mu_{13}(t) + \mu_{21}(t) + \mu_{23}(t)\right]\frac{d}{dt}P_{11}(z,t)$$
$$+ \left[\frac{d}{dt}\ln\mu_{21}(t)[\mu_{12}(t) + \mu_{13}(t)] + \frac{d}{dt}\mu_{12}(t) + \frac{d}{dt}\mu_{13}(t) - \mu_{12}(t)\mu_{21}(t)\right.$$
$$\left. + [\mu_{12}(t) + \mu_{13}(t)][\mu_{21}(t) + \mu_{23}(t)]\right]P_{11}(z,t) = 0. \quad (1.39)$$

The boundary conditions are:

$$P_{11}(z,z) = 1$$

$$\frac{d}{dt}P_{11}(z,t)|_{t=z} = -\mu_{12}(z) - \mu_{13}(z).$$

The second condition is obtained from equation (1.36a) with $P_{12}(z,t) = 0$ when $t = z$.

An interesting alternative expression for probability $P_{12}(z,t)$ is as follows:

$$P_{12}(z,t) = \int_z^t P_{11}(z,u)\mu_{12}(u)P_{\underline{22}}(u,t)\,du. \quad (1.40)$$

It is possible to prove that the right-hand side member of equation (1.40) satisfies the differential equation (1.36b). Equation (1.40) can be easily interpreted by direct reasoning. Moreover, it is interesting to compare it with the analogous equation (1.34b) pertaining to the permanent disability annuity; note that when only permanent disability is considered $P_{\underline{22}} = P_{22}$. Equation (1.40) represents a useful tool in deriving formulae for premium calculations, as we will see in section 1.10.

1.5.7 Example 9 (continued)

Particular assumptions about the transition intensities can lead to substantial simplifications in finding transition probabilities. We now consider some particular cases.

1. Consider the assumption $\mu_{13}(t) = \mu_{23}(t) = \mu(t)$, say, which is often referred to as 'no differential mortality'. In this case we obtain:

$$P_{13}(z,t) = 1 - \exp\left[-\int_z^t \mu(u)\,du\right]. \quad (1.41)$$

This result can be proved as follows. Equation (1.36c) now becomes:

$$\frac{d}{dt}P_{13}(z,t) = \mu(t)[P_{11}(z,t) + P_{12}(z,t)]$$

and then

$$\frac{d}{dt}[1 - P_{13}(z,t)] = -\mu(t)[1 - P_{13}(z,t)]. \quad (1.42)$$

Equation (1.42) can be solved with the boundary condition $P_{13}(z,z) = 0$; so expression (1.41) is obtained.

2. Let us assume that the transition intensities are all constants, independent of age, i.e.

$$\mu_{12}(t) = \mu_{12}, \quad \mu_{13}(t) = \mu_{13}, \quad \mu_{21}(t) = \mu_{21}, \quad \mu_{23}(t) = \mu_{23}.$$

The second-order differential equation for $P_{11}(z,t)$, equation (1.39), then becomes:

$$\frac{d^2}{dt^2}P_{11}(z,t) + (\mu_{12}+\mu_{13}+\mu_{21}+\mu_{23})\frac{d}{dt}P_{11}(z,t) + [(\mu_{12}+\mu_{13})(\mu_{21}+\mu_{23}) - \mu_{12}\mu_{21}]P_{11}(z,t) = 0; \tag{1.43}$$

with boundary conditions:

$$P_{11}(z,z) = 1 \tag{1.44}$$

$$\frac{d}{dt}P_{11}(z,t)|_{t=z} = -\mu_{12} - \mu_{13}. \tag{1.45}$$

A trial solution of the form $P_{11}(z,t) = e^{r(t-z)}$ yields:

$$r^2 + (\mu_{12}+\mu_{13}+\mu_{21}+\mu_{23})r + [(\mu_{12}+\mu_{13})(\mu_{12}+\mu_{23}) - \mu_{12}\mu_{21}] = 0$$

which has solutions r_1 and r_2 where

$$r_1 = \tfrac{1}{2}[-(\mu_{12}+\mu_{13}+\mu_{21}+\mu_{23}) + [(\mu_{12}+\mu_{13}-\mu_{21}-\mu_{23})^2 + 4\mu_{12}\mu_{21}]^{1/2}]$$

$$r_2 = \tfrac{1}{2}[-(\mu_{12}+\mu_{13}+\mu_{21}+\mu_{23}) - [(\mu_{12}+\mu_{13}-\mu_{21}-\mu_{23})^2 + 4\mu_{12}\mu_{21}]^{1/2}].$$

So we have

$$P_{11}(z,t) = A\,e^{r_1(t-z)} + B\,e^{r_2(t-z)} \tag{1.46}$$

and we can determine A and B from the boundary conditions. From equations (1.44) and (1.45) we respectively have:

$$1 = A + B$$

$$-\mu_{12} - \mu_{13} = Ar_1 + Br_2.$$

Solving this pair of simultaneous equations leads to

$$A = \frac{r_2+\mu_{12}+\mu_{13}}{r_2 - r_1} \tag{1.47a}$$

$$B = \frac{r_1+\mu_{12}+\mu_{13}}{r_1 - r_2}. \tag{1.47b}$$

Finally, we have:

$$P_{11}(z,t) = \frac{(r_2+\mu_{12}+\mu_{13})\,e^{r_1(t-z)} - (r_1+\mu_{12}+\mu_{13})\,e^{r_2(t-z)}}{r_2 - r_1}. \tag{1.48}$$

Substituting equation (1.48) into (1.36a′), after some manipulations we obtain:

$$P_{12}(z,t) = \frac{\mu_{12}(e^{r_1(t-z)} - e^{r_2(t-z)})}{r_1 - r_2} \tag{1.49}$$

where r_1 and r_2 are given by the earlier expressions.

3. We now add the assumption of $\mu_{13} = \mu_{23} = \mu$ (that is 'no differential mortality') to the model discussed in case 2. The quadratic equation for r now becomes:

$$r^2 + (\mu_{12}+\mu_{21}+2\mu)r + \mu^2 + \mu(\mu_{12}+\mu_{21}) = 0,$$

which factorizes as

$$(r+\mu)(r+\mu+\mu_{12}+\mu_{21}) = 0,$$

whence

$$r_1 = -\mu, \quad r_2 = -(\mu+\mu_{12}+\mu_{21}).$$

From equation (1.46) we obtain

$$P_{11}(z,t) = A\,e^{-\mu(t-z)} + B\,e^{-(\mu+\mu_{12}+\mu_{21})(t-z)}. \tag{1.50}$$

From equations (1.47a) and (1.47b) we now have:

$$A = \frac{\mu_{21}}{\mu_{12}+\mu_{21}} \tag{1.51a}$$

$$B = \frac{\mu_{12}}{\mu_{12}+\mu_{21}} \tag{1.51b}$$

so

$$P_{11}(z,t) = \frac{1}{\mu_{12}+\mu_{21}}[\mu_{21}\,e^{-\mu(t-z)} + \mu_{12}\,e^{-(\mu+\mu_{12}+\mu_{21})(t-z)}]. \tag{1.52}$$

Similarly we obtain

$$P_{12}(z,t) = \frac{\mu_{12}}{\mu_{12}+\mu_{21}}[e^{-\mu(t-z)} - e^{-(\mu+\mu_{12}+\mu_{21})(t-z)}]. \tag{1.53}$$

By subtraction

$$P_{13}(z,t) = 1 - P_{11}(z,t) - P_{12}(z,t)$$

and we have:

$$P_{13}(z,t) = 1 - e^{-\mu(t-z)}. \tag{1.54}$$

1.5.8 Example 10a

The set of simultaneous differential equations is just a simplified version of equations (1.36) in Example 9, noting the absence of state 3:

$$\frac{d}{dt}P_{11}(z,t) = P_{12}(z,t)\mu_{21}(t) - P_{11}(z,t)\mu_{12}(t) \tag{1.55a}$$

$$\frac{d}{dt}P_{12}(z,t) = P_{11}(z,t)\mu_{12}(t) - P_{12}(z,t)\mu_{21}(t) \tag{1.55b}$$

$$\frac{d}{dt}P_{21}(z,t) = P_{22}(z,t)\mu_{21}(t) - P_{21}(z,t)\mu_{12}(t) \tag{1.55c}$$

$$\frac{d}{dt}P_{22}(z,t) = P_{21}(z,t)\mu_{12}(t) - P_{22}(z,t)\mu_{21}(t). \tag{1.55d}$$

The discussion of Example 9 is also relevant here.

1.5.9 Example 11

In this four-state model four transition intensities must be considered. All the transition intensities are in this case intensities of mortality. Assume that (x) denotes the husband and (y) denotes the wife.

$\mu_{12}(t)$ = intensity of mortality of (x) (married)

$\mu_{13}(t)$ = intensity of mortality of (y) (married)

$\mu_{24}(t)$ = intensity of mortality of (y) (widow)

$\mu_{34}(t)$ = intensity of mortality of (x) (widower).

It is important to stress that, using the Markov modelling approach, it is quite natural to discriminate between the mortality of a married person and that of a widow (or widower), if the remaining lifetimes seem to be correlated. Hence, we can assume:

$$\mu_{12}(t) \neq \mu_{34}(t); \quad \mu_{13}(t) \neq \mu_{24}(t);$$

in particular, if a positive correlation between remaining lifetimes seems to be reasonable, we should assume:

$$\mu_{12}(t) < \mu_{34}(t); \quad \mu_{13}(t) < \mu_{24}(t).$$

Note that the traditional actuarial approach does not allow for correlation; actually, the mortality functions are simply taken from a male ($\mu_{12}(t) = \mu_{34}(t)$) and a female ($\mu_{13}(t) = \mu_{24}(t)$) population life table.

The set of forward differential equations is as follows:

$$\frac{d}{dt}P_{11}(z,t) = -P_{11}(z,t)[\mu_{12}(t) + \mu_{13}(t)] \tag{1.56a}$$

$$\frac{d}{dt}P_{12}(z,t) = P_{11}(z,t)\mu_{12}(t) - P_{12}(z,t)\mu_{24}(t) \tag{1.56b}$$

$$\frac{d}{dt}P_{13}(z,t) = P_{11}(z,t)\mu_{13}(t) - P_{13}(z,t)\mu_{34}(t) \tag{1.56c}$$

$$\frac{d}{dt}P_{22}(z,t) = -P_{22}(z,t)\mu_{24}(t) \tag{1.56d}$$

$$\frac{d}{dt}P_{33}(z,t) = -P_{33}(z,t)\mu_{34}(t) \tag{1.56e}$$

$$\frac{d}{dt}P_{24}(z,t) = P_{22}(z,t)\mu_{24}(t) \tag{1.56f}$$

$$\frac{d}{dt}P_{34}(z,t) = P_{33}(z,t)\mu_{34}(t). \tag{1.56g}$$

As states 1, 2 and 3 are strictly transient states, we have:

$$P_{\underline{11}}(z,t) = P_{11}(z,t) \tag{1.57a}$$

$$P_{\underline{22}}(z,t) = P_{22}(z,t) \tag{1.57b}$$

$$P_{\underline{33}}(z,t) = P_{33}(z,t). \tag{1.57c}$$

The solution of (1.56a), (1.56d), (1.56e) is trivial. We leave as an exercise the problem of solving the remaining equations; note that the solving procedure can follow a 'sequential' path, thanks to the order in the set of nodes.

Remark

Example 11 above has been discussed numerically by Wolthuis (1994), for the case where

$$\mu_{12}(t) = (1-\alpha)\mu_{x+t}$$
$$\mu_{13}(t) = (1-\alpha)\mu_{y+t}$$
$$\mu_{24}(t) = (1+\alpha)\mu_{y+t}$$
$$\mu_{34}(t) = (1+\alpha)\mu_{x+t}$$

with μ_{x+t} and μ_{y+t} based respectively on male and female population life tables and α a parameter representing the dependence of mortality intensities on current marital status.

1.6 THE SEMI-MARKOV MODEL

1.6.1 Some preliminary ideas

The preceding Markov model assumes that transition intensities (and probabilities) at time t depend (at least explicitly) on the current state at that time only. More realistic (and possibly more complicated) models can be built considering, for instance:

1. the dependence of some intensities (and probabilities) on the age x at policy issue;
2. the dependence of some intensities (and probabilities) on the time spent in the current state since the latest transition to that state (see equation (1.4a) in section 1.4);