On absolutely nonmeasurable sets and functions

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Abstract. We consider some properties of sets and functions which are measurable (nonmeasurable) with respect to certain classes of measures. In this context, the notion of an absolutely nonmeasurable set (function) is examined. Sierpinski-Zygmund type functions are constructed with additional properties closely connected with the above-mentioned notion. Also, some small subsets of uncountable commutative groups are discussed whose algebraic sum turns out to be an absolutely nonmeasurable set.

2000 Mathematics Subject Classification: 28A05, 28D05

Key words and phrases: Absolutely nonmeasurable set, absolutely nonmeasurable function, universal measure zero set, extension of measure, Sierpinski-Zygmund function.

Let \( E \) be a set and let \( M \) be a class of measures on \( E \) (in general, we do not assume that measures belonging to \( M \) are defined on the same \( \sigma \)-algebra of subsets of \( E \)).

We say that a function \( f : E \to \mathbb{R} \) is relatively measurable with respect to \( M \) if there exists at least one measure \( \mu \in M \) such that \( f \) is \( \mu \)-measurable.

Otherwise, we say that \( f \) is absolutely nonmeasurable with respect to \( M \).

Accordingly, we say that a set \( X \subset E \) is relatively measurable (absolutely nonmeasurable) with respect to \( M \) if its characteristic function \( \chi_X \) is relatively measurable (absolutely nonmeasurable) with respect to \( M \).

Let us give some typical examples.

Example 1. Any Vitali’s subset of the real line \( \mathbb{R} \) is absolutely nonmeasurable with respect to the class of all translation-invariant extensions of the Lebesgue measure on \( \mathbb{R} \). On the other hand, there exist Vitali’s subsets of \( \mathbb{R} \) which are relatively measurable with respect to the class of all translation-quasi-invariant extensions of the Lebesgue measure on \( \mathbb{R} \).

Example 2. Let \( E \) be an infinite-dimensional separable Hilbert space and let \( B \) denote the unit ball of this space. Then \( B \) is absolutely nonmeasurable with respect to the class of all nonzero \( \sigma \)-finite \( E \)-quasi-invariant measures on \( E \). Consequently, there exists no nonzero \( \sigma \)-finite translation-quasiinvariant Borel measure on \( E \) (see, e.g., [1]).
Example 3. Any Bernstein subset of an uncountable Polish space $E$ is absolutely nonmeasurable with respect to the class of the completions of all nonzero $\sigma$-finite continuous Borel measures on $E$.

Example 4. Any Sierpinski-Zygmund function on $\mathbb{R}$ is absolutely nonmeasurable with respect to the class of the completions of all nonzero $\sigma$-finite continuous Borel measures on $\mathbb{R}$.

Let $E$ be an uncountable set and let $M(E)$ denote the class of all nonzero $\sigma$-finite continuous measures on $E$. It is well known that there are no absolutely nonmeasurable sets with respect to $M(E)$. Moreover, for any measure $\mu \in M(E)$ and for each subset $X$ of $E$, there exists a measure $\mu'$ on $E$ extending $\mu$ and satisfying the relation $X \in \text{dom}(\mu')$.

The case of real-valued functions defined on $E$ essentially differs from the case of subsets of $E$. Let $f : E \to \mathbb{R}$ be a function. The following two assertions are equivalent:

(1) $f$ is absolutely nonmeasurable with respect to $M(E)$;

(2) $\text{ran}(f)$ is universal measure zero and $\text{card}(f^{-1}(t)) \leq \omega$ for each $t \in \mathbb{R}$.

The equivalence of (1) and (2) implies, in particular, that if $f : E \to \mathbb{R}$ is absolutely nonmeasurable with respect to the class $M(E)$, then the restriction of $f$ to any uncountable set $E' \subset E$ is absolutely nonmeasurable with respect to the class $M(E')$.

Also, the existence of uncountable universal measure zero subsets of $\mathbb{R}$ (see, e.g., [2] - [6]) implies that, for any set $E$ with $\text{card}(E) = \omega_1$, there exist injective functions $f : E \to \mathbb{R}$ which are absolutely nonmeasurable with respect to $M(E)$.

Example 5. Under Martin’s Axiom, there exists a Sierpinski-Zygmund function on $\mathbb{R}$ absolutely nonmeasurable with respect to the class $M(\mathbb{R})$. Moreover, such a function can be additive, i.e. can be a nontrivial solution of the Cauchy functional equation (see [7]).

Example 6. There exists a Sierpinski-Zygmund function on $\mathbb{R}$ which is relatively measurable with respect to the class of all translation-invariant extensions of the Lebesgue measure on $\mathbb{R}$ (see [8]).

Our goal is to strengthen (under some additional set-theoretical assumption) the result presented in Example 6.

Theorem 1. Under Martin’s Axiom, there exists a Sierpinski-Zygmund function $f : \mathbb{R} \to \mathbb{R}$ such that:

(a) $f$ is relatively measurable with respect to the class of all translation-invariant extensions of the Lebesgue measure on $\mathbb{R}$;

(b) for every set $X \subset \mathbb{R}$ of cardinality continuum, the restriction of $f$ to $X$ is relatively measurable with respect to $M(X)$. 

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The proof of this statement is based on the following auxiliary proposition (the symbol $T$ below stands for the one-dimensional unit torus and $\mathbb{R}$ is considered as a commutative group).

**Lemma 1.** Under Martin’s Axiom, there exists an injective homomorphism $g : \mathbb{R} \rightarrow T$ satisfying the relations:

1. $g$ is a Sierpinski-Zygmund function;
2. the graph of $g$ is thick in the product space $\mathbb{R} \times T$;
3. the range of $g$ is a Sierpinski subset of $T$.

Let $(G, +)$ be an uncountable commutative group and let $X$ be a subset of $G$. We say that $X$ is $G$-absolutely negligible in $G$ if, for every $\sigma$-finite $G$-invariant (respectively, $G$-quasiinvariant) measure $\mu$ on $G$, there exists a $G$-invariant (respectively, $G$-quasiinvariant) extension $\mu'$ of $\mu$ such that $\mu'(X) = 0$.

We say that a set $Y \subset G$ is $G$-absolutely nonmeasurable in $G$ if $Y$ is absolutely nonmeasurable with respect to the class of all nonzero $\sigma$-finite $G$-quasiinvariant measures on $G$.

**Lemma 2.** If $E$ is a vector space over the field $\mathbb{Q}$ of all rational numbers and $\text{card}(E)$ is greater than or equal to the cardinality of the continuum, then there exists an $E$-absolutely negligible set $X \subset E$ such that the vector sum $X + X$ is $E$-absolutely nonmeasurable in $E$.

The proof of this lemma is based on the $H$-absolute nonmeasurability of the unit ball in a separable infinite-dimensional Hilbert space $H$ (see Example 2).

As a consequence of Lemma 2, we have the following statement.

**Theorem 2.** If the Continuum Hypothesis holds and $E$ is an arbitrary uncountable vector space over $\mathbb{Q}$, then there exists an $E$-absolutely negligible set $X \subset E$ such that $X + X$ is absolutely nonmeasurable in $E$.

A direct analog of this statement is true for a wide class of uncountable commutative groups. However, we do not know whether the same result remains valid for an arbitrary uncountable commutative group. In this context, let us mention that, for obtaining the required result for all uncountable commutative groups, it suffices to establish its validity only for commutative groups of cardinality $\omega_1$.

**References**


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