

SEMIGROUPS OF DISTRIBUTIONS WITH LINEAR JACOBI PARAMETERS

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ABSTRACT. We show that a convolution semigroup $\{\mu_t\}$ of measures has Jacobi parameters linear in the convolution parameter t if and only if the measures come from the Meixner class. Moreover, we prove the parallel result, in a more explicit way, for the free convolution and the free Meixner class. We then construct the class of measures satisfying the same property for the two-state free convolution. This class has not been considered explicitly before, but we describe its relation to the two-state free Laha-Lukacs characterization, and to the $q = 0$ case of quadratic harnesses.

1. INTRODUCTION.

Any probability measure μ on the real line, all of whose moments are finite, has associated to it two sequences of Jacobi parameters: for example, μ is the spectral measure of the tridiagonal matrix

$$\begin{pmatrix} \beta_0 & \gamma_0 & 0 & 0 & \ddots \\ 1 & \beta_1 & \gamma_1 & 0 & \ddots \\ 0 & 1 & \beta_2 & \gamma_2 & \ddots \\ 0 & 0 & 1 & \beta_3 & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}.$$

We will denote this fact by

$$J(\mu) = \begin{pmatrix} \beta_0, & \beta_1, & \beta_2, & \beta_3, & \dots \\ \gamma_0, & \gamma_1, & \gamma_2, & \gamma_3, & \dots \end{pmatrix}.$$

These parameters are related to the moments of the measure via the Viennot-Flajolet [Fla80, Vie84] and Accardi-Bożejko [AB98] formulas. On the other hand, in probability theory and other applications, measures frequently come in time-dependent convolution semigroups. In general, the time dependence of the Jacobi parameters is complicated (they are rational functions of t). However, for the Gaussian convolution semigroup

$$\mu_t(x) = \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t},$$

the Jacobi parameters are simply

$$\beta_n(t) = 0, \quad \gamma_n(t) = (n+1)t,$$

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while for the Poisson convolution semigroup

$$\mu_t(x) = e^{-t} \sum_{k=0}^{\infty} \frac{1}{k!} t^k \delta_k(x),$$

they are

$$\beta_n(t) = t + n, \quad \gamma_n(t) = (n + 1)t.$$

So it is natural to ask, what are all the measures whose Jacobi parameters are linear (in the calculus sense, that is, affine) functions of the convolution parameter? In this paper we provide the answer: such measures form precisely the Meixner class [Mei34]. Thus we add a new one to already numerous known characterizations of this class, see for example [Mor82] for a survey.

The usual convolution operation has a number of relatives which arise in non-commutative probability theories. In many ways, these operations are more complicated than the usual convolution; notably, the usual operation is distributive,

$$\mu * (\nu_1 + \nu_2) = \mu * \nu_1 + \mu * \nu_2$$

while the other ones are not. Nevertheless, in other ways they appear to be simpler. The combinatorial theory of such convolutions is typically based on an appropriate sequence of cumulants which linearize it; for example, the classical cumulants $r_n^*(\mu)$ have the property that

$$(1) \quad r_n^*(\mu * \nu) = r_n^*(\mu) + r_n^*(\nu).$$

In particular, $r^*(\mu^{*t}) = t r^*(\mu)$: cumulants are always proportional to the convolution parameter t . While, as pointed out above, there is a nice relation between Jacobi parameters and moments, as well as a relation between cumulants and moments (see Section 2.3), we are not aware of a simple relation between Jacobi parameters and cumulants. However, in [Mł09], the second author found a formula relating Jacobi parameters and *free* cumulants, which linearize the *free convolution* [NS06]. This allows us to provide a constructive proof that Jacobi parameters are linear in the free convolution parameter if and only if the measure belongs to the free Meixner class considered in [SY01, Ans03, BB06] and numerous other sources. We then give a simple but indirect argument which provides the corresponding characterization for the Meixner class. Another consequence of the analysis is that there are no measures whose Jacobi parameters are polynomial functions of the convolution parameter of degree greater than one.

Yet another convolution operation was introduced in [BLS96] in relation to what the authors called “conditionally free probability”, but is better called two-state free probability theory. As the name indicates, this is a convolution operation on pairs of measures $(\tilde{\mu}, \mu)$, and as such does not really have a classical analogue. The techniques from [Mł09] allow us to find all (pairs of) measures whose Jacobi parameters are linear with respect to this convolution. Unlike in the analysis above, this class has not been explicitly described before. It consists of measures whose Jacobi parameters do not depend on n for $n \geq 3$. We show that, without being so identified, this class has in fact appeared in applications: in the two-state free Poisson limit theorem, in the two-state Laha-Lukacs characterization [BB09], and as a subclass of the $q = 0$ case of quadratic harnesses [BW05]. The paper ends with an appendix showing that in the case of a two-state free convolution semigroup $(\tilde{\mu}, \mu)^{\boxplus ct}$, it suffices to only require that the Jacobi parameters of $\tilde{\mu}$ are polynomial in t , and the linearity of the Jacobi parameters of both $\tilde{\mu}$ and μ then follows automatically.

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2. BACKGROUND.

2.1. Partitions. A *partition* of a linearly ordered set X is a family π of nonempty, pairwise disjoint subsets of X , called *blocks* of π , such that $\bigcup \pi = X$. A partition is *noncrossing* if whenever $x_1 < x_2 < x_3 < x_4$, $x_1, x_3 \in V_1 \in \pi$ and $x_2, x_4 \in V_2 \in \pi$ then $V_1 = V_2$. Every noncrossing partition admits a natural partial order: $U \preceq V$ if there are $r, s \in V$ such that $r \leq k \leq s$ holds for every $k \in U$. Now we can define *depth* of a block $U \in \pi$, namely

$$d(U, \pi) := |\{V \in \pi : U \preceq V \neq U\}|.$$

If $d(U, \pi) \geq 1$ then we define the *derivative* of U as the unique block $U' \in \pi$ such that $U \preceq U'$ and $d(U', \pi) = d(U, \pi) - 1$. The derivatives of higher orders are defined by putting $V^{(k)} := (V^{(k-1)})'$.

In particular, a block of a noncrossing partition with $d(U, \pi) = 0$ is called *outer*, and a block with $d(U, \pi) \geq 1$ is called *inner*. An *interval partition* is a non-crossing partition with only outer blocks. For the set $\{1, 2, \dots, m\}$, we will denote the lattice of all partitions by $\mathcal{P}(m)$, the lattice of all noncrossing partitions by $\text{NC}(m)$, and the lattice of all interval partitions by $\text{Int}(m)$. In addition, $\text{NC}_{1,2}(m)$ will stand for the class of all partitions $\pi \in \text{NC}(m)$ such that $|V| \leq 2$ holds for every $V \in \pi$.

2.2. Jacobi parameters. Throughout the paper, μ will be a probability measure on \mathbb{R} all of whose *moments*

$$(2) \quad s_m := \int_{\mathbb{R}} x^m d\mu(x)$$

are finite. Then there is a sequence $\{P_m\}_{m=0}^{\infty}$ of monic polynomials, with $\deg P_m = m$, which are orthogonal with respect to μ . They satisfy a recurrence relation: $P_0(x) = 1$ and for $m \geq 0$

$$(3) \quad xP_m(x) = P_{m+1}(x) + \beta_m P_m(x) + \gamma_{m-1} P_{m-1}(x),$$

under convention that $P_{-1}(x) = 0$, where the *Jacobi parameters* [Chi78] satisfy $\beta_m \in \mathbb{R}$ and $\gamma_m \geq 0$. Then we will write

$$J(\mu) = \begin{pmatrix} \beta_0, & \beta_1, & \beta_2, & \beta_3, & \dots \\ \gamma_0, & \gamma_1, & \gamma_2, & \gamma_3, & \dots \end{pmatrix}.$$

$\{P_m\}$ are unique for $m \leq |\text{supp}(\mu)|$. Moreover $N := |\text{supp}(\mu)| < \infty$ if and only if $\gamma_{N-1} = 0$ and $\gamma_m > 0$ for $m < N - 1$. In this case for $m \geq N$, P_{m+1} are not uniquely determined and in fact β_m, γ_m can be chosen arbitrarily.

The Viennot-Flajolet theory [Fla80, Vie84] gives the relation between moments of a measure and its Jacobi parameters in terms of Motzkin paths. We will use a related formula of Accardi and Bożejko [AB98] expressing the same relation using non-crossing partitions:

$$(4) \quad s_m = \sum_{\sigma \in \text{NC}_{1,2}(m)} \prod_{\substack{V \in \sigma \\ |V|=1}} \beta_{d(V,\sigma)} \cdot \prod_{\substack{V \in \sigma \\ |V|=2}} \gamma_{d(V,\sigma)}.$$

This formula should be compared with the formula (6) below.

2.3. Cumulants and convolutions. The classical cumulants $r_n^*(\mu)$ [Shi96], free cumulants

$$r_n(\mu) = r_n^{\boxplus}(\mu)$$

[Spe90, Spe94, NS06], Boolean cumulants $r_n^{\boxplus}(\mu)$ [SW97], and two-state free cumulants

$$R_n(\tilde{\mu}, \mu) = r_n^{\boxplus_c}(\tilde{\mu}, \mu)$$

[BLS96] are defined via the following moment-cumulant formulas, which express them implicitly in terms of the moments $s_m(\mu)$:

$$(5) \quad s_m = \sum_{\pi \in \mathcal{P}(m)} \prod_{V \in \pi} r_{|V|}^*$$

$$(6) \quad s_m = \sum_{\pi \in \text{NC}(m)} \prod_{V \in \pi} r_{|V|},$$

$$(7) \quad s_m = \sum_{\pi \in \text{Int}(m)} \prod_{V \in \pi} r_{|V|}^{\boxplus},$$

and

$$(8) \quad s_m(\tilde{\mu}) = \sum_{\pi \in \text{NC}(m)} \prod_{V \in \text{Out}(\pi)} R_{|V|}(\tilde{\mu}, \mu) \prod_{U \in \text{Inn}(\pi)} r_{|U|}(\mu).$$

Now using cumulants, we can define in a uniform way the classical convolution $*$, the free convolution \boxplus , the Boolean convolution \boxplus , and the two-state free convolution \boxplus_c via equation (1) and its analogs. Note that the classical convolution defined in this way does coincide with the more familiar formula

$$(\mu * \nu)(A) = \int \mu(A - x) d\nu(x),$$

but there are no such explicit formulas for the other operations. Instead, each of them is related to an appropriate notion of independence, see the references above.

For any of the convolution operations, for example for $*$, a *convolution semigroup* generated by μ is a family of measures $\{\mu_t\}$ such that $\mu_1 = \mu$ and $\mu_t * \mu_s = \mu_{t+s}$. A priori, our semigroups will be indexed by $t \in \mathbb{N}$. For free convolution \boxplus and two-state free convolution, one can always extend the semigroup to $t \in [1, \infty)$. For the Boolean convolution \boxplus , or for any of our convolutions if μ is infinitely divisible in the appropriate sense, one has a full convolution semigroup for $t \in [0, \infty)$.

2.4. Generating functions. It is frequently more convenient to work with generating functions instead of moments and cumulants. For example, the Fourier transform

$$\mathcal{F}(z) = \sum_{m=0}^{\infty} \frac{1}{m!} s_m(\mu) (iz)^m$$

is the exponential moment generating function of μ , and

$$\log \mathcal{F}(z) = \sum_{m=1}^{\infty} \frac{z^m}{m!} r_m^*(\mu)$$

is its (classical) cumulant generating function. The ordinary moment generating function of μ is

$$M^\mu(z) = \sum_{k=1}^{\infty} s_k z^k,$$

and the cumulant generating functions for the other cumulant sequences are denoted by

$$R^\mu(z) = \sum_{k=1}^{\infty} r_k z^k$$

(R -transform, the free cumulant generating function),

$$\eta^\mu(z) = \sum_{k=1}^{\infty} r_k^{\uplus} z^k$$

(eta-transform, the Boolean cumulant generating function), and

$$R^{\tilde{\mu}, \mu}(z) = \sum_{k=1}^{\infty} R_k(\tilde{\mu}, \mu) z^k$$

(the two-state free cumulant generating function). The combinatorial relations (6), (7), (8) translate into the relations between generating functions:

$$(9) \quad M^\mu(z) = R^\mu((1 + M^\mu(z))z),$$

$$(10) \quad 1 + M^\mu(z) = (1 - \eta^\mu(z))^{-1},$$

and

$$(11) \quad \eta^{\tilde{\mu}}(z) = (1 + M^\mu(z))^{-1} R^{\tilde{\mu}, \mu}((1 + M^\mu(z))z).$$

We will also use the following notations. \mathbb{B} is the Boolean-to-free Bercovici-Pata bijection defined in [BN08]; it is determined by

$$\eta^\mu(z) = R^{\mathbb{B}[\mu]}(z).$$

Φ is the Jacobi shift, considered for example in [BN09], determined by

$$\eta^{\Phi[\mu]}(z) = z^2(1 + M^\mu(z)).$$

The name comes from the observation that

$$J(\Phi(\mu)) = \begin{pmatrix} 0, & \beta_0(\mu), & \beta_1(\mu), & \beta_2(\mu), & \dots \\ 1, & \gamma_0(\mu), & \gamma_1(\mu), & \gamma_2(\mu), & \dots \end{pmatrix}.$$

Finally, the two-variable map Φ from [Ans09b] is determined by

$$\eta^{\Phi[\omega, \mu]}(z) = (1 + M^\mu(z))^{-1} R^\omega((1 + M^\mu(z))z)$$

and corresponds to $R^{\tilde{\mu}, \mu} = R^\omega$.

2.5. Free Meixner distributions. The free Meixner distributions $\mu_{b,c}$ with mean zero, variance one, and parameters $b \in \mathbb{R}$, $c \geq -1$ are probability measures with

$$J(\mu_{b,c}) = \begin{pmatrix} 0, & b, & b, & b, & \dots \\ 1, & 1+c, & 1+c, & 1+c, & \dots \end{pmatrix}.$$

The general free Meixner distributions are affine transformations of these. More explicitly, the distribution with parameters b, c is

$$\frac{1}{2\pi} \cdot \frac{\sqrt{\left(4(1+c) - (x-b)^2\right)_+}}{1+bx+cx^2} dx + 0, 1, \text{ or } 2 \text{ atoms,}$$

see [SY01, Ans03, BB06]. There are numerous characterizations of this class in free probability.

3. THE FREE CONVOLUTION.

Formulas (4) and (6) relate moments of a measure to its Jacobi parameters, resp. free cumulants. It is also possible to find a direct relation between free cumulants and Jacobi parameters, see [Mł09]. For this purpose we will need some additional notions.

A *labelling* of a noncrossing partition σ is a function κ on σ such that for any $V \in \sigma$ we have $\kappa(V) \in \{0, 1, \dots, d(V, \sigma)\}$. For a labelling κ of a noncrossing partition σ we denote by $\mathcal{R}(\sigma, \kappa)$ the smallest equivalence relation on σ containing all the pairs $(V^{(i)}, V^{(j)})$ with $V \in \sigma$, $0 \leq i, j \leq \kappa(V)$. By $\text{NCL}_{1,2}^1(m)$ (not to be confused with non-crossing linked partitions introduced by Dykema [Dyk07]) we will denote the family of all pairs (σ, κ) such that $\sigma \in \text{NC}_{1,2}(m)$, κ is a labelling of σ and $\mathcal{R}(\sigma, \kappa) = \sigma \times \sigma$. In particular, σ has only one outer block. Then, in view of [Mł09], we have the following relation between free cumulants and the Jacobi parameters:

$$(12) \quad r_m = \sum_{(\sigma, \kappa) \in \text{NCL}_{1,2}^1(m)} \prod_{V \in \sigma} w(V, \kappa(V)),$$

where

$$(13) \quad w(V, k) := \begin{cases} \beta_k - \beta_{k-1} & \text{if } |V| = 1, \\ \gamma_k - \gamma_{k-1} & \text{if } |V| = 2, \end{cases}$$

under convention that $\beta_{-1} = \gamma_{-1} = 0$. In particular,

$$(14) \quad r_1 = \beta_0,$$

$$(15) \quad r_2 = \gamma_0,$$

$$(16) \quad r_3 = \gamma_0(\beta_1 - \beta_0),$$

$$(17) \quad r_4 = \gamma_0[(\beta_1 - \beta_0)^2 + (\gamma_1 - \gamma_0)],$$

$$(18) \quad r_5 = \gamma_0[(\beta_1 - \beta_0)^3 + 3(\gamma_1 - \gamma_0)(\beta_1 - \beta_0) + \gamma_1(\beta_2 - \beta_1)],$$

$$(19) \quad r_6 = \gamma_0[(\beta_1 - \beta_0)^4 + 6(\gamma_1 - \gamma_0)(\beta_1 - \beta_0)^2 + 4\gamma_1(\beta_2 - \beta_1)(\beta_1 - \beta_0) + \gamma_1(\beta_2 - \beta_1)^2 + 2(\gamma_1 - \gamma_0)^2 + \gamma_1(\gamma_2 - \gamma_1)].$$

Theorem 1. Assume that the measure μ , with

$$J(\mu) = \begin{pmatrix} \beta_0, & \beta_1, & \beta_2, & \beta_3, & \dots \\ \gamma_0, & \gamma_1, & \gamma_2, & \gamma_3, & \dots \end{pmatrix},$$

is such that for the free powers

$$J(\mu^{\boxplus t}) = \begin{pmatrix} \beta_0(t), & \beta_1(t), & \beta_2(t), & \beta_3(t), & \dots \\ \gamma_0(t), & \gamma_1(t), & \gamma_2(t), & \gamma_3(t), & \dots \end{pmatrix},$$

$t \geq 1$, all the parameters $\beta_m(t)$, $\gamma_m(t)$ are polynomials on t . Then $\beta_1 = \beta_2 = \beta_3 = \dots$ and $\gamma_1 = \gamma_2 = \gamma_3 = \dots$, so that μ is a free Meixner distribution. If this holds then, putting $b := \beta_1 - \beta_0$, $c := \gamma_1 - \gamma_0$ we have

$$J(\mu^{\boxplus t}) = \begin{pmatrix} \beta_0 t, & b + \beta_0 t, & b + \beta_0 t, & b + \beta_0 t, & \dots \\ \gamma_0 t, & c + \gamma_0 t, & c + \gamma_0 t, & c + \gamma_0 t, & \dots \end{pmatrix}.$$

In particular, μ is \boxplus -infinitely divisible if and only if $c \geq 0$. Then

$$R^\mu(z) = \beta_0 z + \gamma_0 z^2 \int_{\mathbb{R}} \frac{d\rho(x)}{1 - zx},$$

where

$$(20) \quad J(\rho) = \begin{pmatrix} b, & b, & b, & b, & \dots \\ c, & c, & c, & c, & \dots \end{pmatrix},$$

so that ρ is a semicircular distribution.

Proof. Putting in formulas (12)–(19) $r_m t$, $\beta_m(t)$, $\gamma_m(t)$, w_t instead of r_m , β_m , γ_m and w respectively we see that

$$\beta_0(t) = \beta_0 t, \quad \gamma_0(t) = \gamma_0 t.$$

Then, from (16) we get $r_3 t = \gamma_0 t(\beta_1(t) - \beta_0 t)$ which yields

$$\beta_1(t) = \beta_1 - \beta_0 + \beta_0 t.$$

Similarly, from (17) we get

$$\gamma_1(t) = \gamma_1 - \gamma_0 + \gamma_0 t.$$

Now we claim that $\beta_k(t) = \beta_1 - \beta_0 + \beta_0 t$ and $\gamma_k(t) = \gamma_1 - \gamma_0 + \gamma_0 t$ for all $k \geq 1$. Fix $d \geq 2$ and assume that this holds for all $1 \leq k < d$. Then for a block V , with $1 \leq |V| \leq 2$ and for $1 < k < d$ we have $w_t(V, k) = 0$. Now we consider formula (12) for $r_{2d+1} t$.

Put

$$\begin{aligned} \sigma_1 &:= \{\{1, 2d+1\}, \{2, 2d\}, \{3, 2d-1\}, \dots, \{d, d+2\}, \{d+1\}\}, \\ \mathcal{K}_1 &:= \{\kappa : (\sigma_1, \kappa) \in \text{NCL}_{1,2}^1(2d+1), \kappa(\{d+1\}) = d\}. \end{aligned}$$

Note that if $(\sigma, \kappa) \in \text{NCL}_{1,2}^1(2d+1) \setminus (\{\sigma_1\} \times \mathcal{K}_1)$, and $\prod_{V \in \sigma} w_t(V, \kappa(V)) \neq 0$ then $\kappa(V) = 1$ for all $V \in \sigma \setminus \{\{1, 2d+1\}\}$ and then

$$\prod_{V \in \sigma} w_t(V, \kappa(V)) = \gamma_0 t \prod_{\substack{V \in \sigma \\ |V|=1}} (\beta_1 - \beta_0) \times \prod_{\substack{V \in \sigma, |V|=2, \\ V \neq \{1, 2d+1\}}} (\gamma_1 - \gamma_0).$$

Now we observe that if $\prod_{V \in \sigma} w_t(\sigma_1, \kappa(V)) \neq 0$, $\kappa \in \mathcal{K}_1$, then $\kappa(V) \in \{0, 1\}$ for all inner blocks $V \in \sigma$, $V \neq \{d+1\}$. Therefore

$$\begin{aligned} & \sum_{\kappa \in \mathcal{K}_1} \prod_{V \in \sigma_1} w_t(V, \kappa) \\ &= \gamma_0 t (\beta_d(t) - \beta_{d-1}(t)) \prod_{k=1}^{d-1} \sum_{i=0}^1 (\gamma_i(t) - \gamma_{i-1}(t)) \\ &= \gamma_0 t (\beta_d(t) - \beta_{d-1}(t)) \gamma_1(t)^{d-1}, \end{aligned}$$

which is a polynomial of degree at least $d \geq 2$, unless $\beta_d(t) = \beta_{d-1}(t) = \beta_1 - \beta_0 + \beta_0 t$.

Now we consider the formula for $r_{2d+2}t$. Put

$$\begin{aligned} \sigma_2 &:= \{\{1, 2d+2\}, \{2, 2d+1\}, \dots, \{d+1, d+2\}\}, \\ \mathcal{K}_2 &:= \{\kappa : (\sigma_2, \kappa) \in \text{NCL}_{1,2}^1(2d+2), \kappa(\{d+1, d+2\}) = d\}. \end{aligned}$$

Similarly as before we conclude that for $(\sigma, \kappa) \in \text{NCL}_{1,2}^1(2d+2) \setminus (\{\sigma_2\} \times \mathcal{K}_2)$ we have $\kappa(V) = 1$ for all $V \in \sigma \setminus \{\{1, 2d+2\}\}$ and then, using the previous step,

$$\prod_{V \in \sigma} w_t(V, \kappa(V)) = \gamma_0 t \prod_{\substack{V \in \sigma \\ |V|=1}} (\beta_1 - \beta_0) \times \prod_{\substack{V \in \sigma, |V|=2, \\ V \neq \{1, 2d+2\}}} (\gamma_1 - \gamma_0).$$

Now we observe that if $w_t(\sigma_2, \kappa) \neq 0$, $\kappa \in \mathcal{K}_2$, then $\kappa(V) \in \{0, 1\}$ for all inner blocks $V \in \sigma$, $V \neq \{d+1, d+2\}$. Therefore

$$\begin{aligned} & \sum_{\kappa \in \mathcal{K}_2} \prod_{V \in \sigma_2} w_t(V, \kappa) \\ &= \gamma_0 t (\gamma_d(t) - \gamma_{d-1}(t)) \prod_{k=1}^{d-1} \sum_{i=0}^1 (\gamma_i(t) - \gamma_{i-1}(t)) \\ &= \gamma_0 t (\gamma_d(t) - \gamma_{d-1}(t)) \gamma_1(t)^{d-1}, \end{aligned}$$

which is again a polynomial of degree at least $d \geq 2$, unless $\gamma_d(t) = \gamma_{d-1}(t) = \gamma_1 - \gamma_0 + \gamma_0 t$.

Put $b := \beta_1 - \beta_0$, $c := \gamma_1 - \gamma_0$. As we have already noticed, if $(\sigma, \kappa) \in \text{NCL}_{1,2}^1(m)$, $V \in \sigma$ and $w(V, \kappa(V)) \neq 0$ then either V is the outer block of sigma (so $\kappa(V) = 0$) or $\kappa(V) = 1$. Therefore for $m \geq 0$ we can write

$$r_{m+2} = \gamma_0 \sum_{\sigma \in \text{NC}_{1,2}(m)} \prod_{\substack{V \in \sigma \\ |V|=1}} b \cdot \prod_{\substack{V \in \sigma \\ |V|=2}} c$$

Therefore, if $c \geq 0$, then $\{r_{m+2}/\gamma_0\}_{m=0}^\infty$ is the moment sequence of the measure ρ satisfying (20) (c.f. [SY01]). \square

4. THE CLASSICAL CONVOLUTION.

In [HS09], the authors define cumulants $\{r_n^*\}$ for an abstract convolution operation \star via the following three properties. All convolutions and cumulants considered in this paper satisfy them.

- (a) $r_n^*(\mu^{*t}) = t r_n^*(\mu)$, where $t \in \mathbb{N}$.
- (b) $r_n^*(D_\lambda \mu) = \lambda^n r_n^*(\mu)$, where $(D_\lambda \mu)(A) = \mu(\lambda^{-1}A)$.

(c) For any n , there is a polynomial Q_n in $n - 1$ variables such that

$$s_n(\mu) = r_n^*(\mu) + Q_n(r_1^*(\mu), \dots, r_{n-1}^*(\mu)).$$

Lemma 2. *Suppose measures $\{\mu_t\}$ form a convolution semigroup, with corresponding cumulants $\{r_k^*(t)\}$ satisfying axioms above. Suppose moreover that the Jacobi parameters of $\{\mu_t\}$ are linear in the convolution parameter t . Then*

$$\begin{aligned} \beta_0(t) &= \beta_0 t, & \beta_1(t) &= \alpha_1 \beta_0 t + b_1, \\ \gamma_0(t) &= \gamma_0 t, & \gamma_1(t) &= (\alpha_2 b_1 \beta_0 + \alpha_3 \gamma_0) t + c_2 \end{aligned}$$

for some $\beta_0, b_1 \in \mathbb{R}$, $\gamma_0, c_2 \geq 0$ and $\alpha_1, \alpha_2, \alpha_3$. Also, the cumulants are expressed in terms of the same parameters as

$$\begin{aligned} r_1^* &= \beta_0, & r_3^* &= \gamma_0 b_1, \\ r_2^* &= \gamma_0, & r_4^* &= \gamma_0 (c_2 + b_1^2), \end{aligned}$$

and the moment-cumulant formulas are

$$\begin{aligned} s_1 &= r_1^*, \\ s_2 &= r_2^* + (r_1^*)^2, \\ s_3 &= r_3^* + (2 + \alpha_1) r_2^* r_1^* + (r_1^*)^3, \\ s_4 &= r_4^* + (2 + 2\alpha_1 + \alpha_2) r_3^* r_1^* + (1 + \alpha_3) (r_2^*)^2 + (3 + 2\alpha_1 + \alpha_1^2) r_2^* (r_1^*)^2 + (r_1^*)^4. \end{aligned}$$

The proof is easy and is left to the interested reader.

Proposition 3. *In the setting of the preceding lemma, suppose we have another family $\{\mu_t^M\}$ of the same type, with $\beta_0(t) = \beta_0 t = \beta_0^M t$, $\gamma_0(t) = \gamma_0 t = \gamma_0^M t$, $\beta_1(t) = \beta_1^M(t)$, $\gamma_1(t) = \gamma_1^M(t)$. Suppose also that $\gamma_1(t)$ varies with t . Then μ_t and μ_t^M have the same moments.*

Proof. We will prove by induction on n that the rest of the Jacobi parameters of the measure μ_t are the same as for μ_t^M . Suppose this is true up to $n - 1$. Then using formula (4),

$$s_{2n+1}(\mu_t) = s_{2n+1}(\mu_t^M) + (\beta_n(\mu_t) - \beta_n(\mu_t^M)) \gamma_{n-1}(t) \dots \gamma_1(t) \gamma_0(t),$$

so if

$$s_k = r_k^* + Q_k(r_1^*, r_2^*, \dots, r_{k-1}^*),$$

then

$$r_{2n+1}^*(\mu_t) = r_{2n+1}^*(\mu_t^M) + (\beta_n(\mu_t) - \beta_n(\mu_t^M)) \gamma_{n-1}(t) \dots \gamma_1(t) \gamma_0(t).$$

By assumption, $\gamma_0(t) = \gamma_0 t$ and $\gamma_1(t)$ has degree at least one in t , while the other $\gamma_i(t)$ are polynomial in t , from which it follows that $\beta_n(\mu_t) = \beta_n(\mu_t^M) = \beta_n(t)$.

Similarly, from

$$s_{2n+2}(\mu_t) = s_{2n+2}(\mu_t^M) + (\gamma_n(\mu_t) - \gamma_n(\mu_t^M)) \gamma_{n-1}(t) \dots \gamma_1(t) \gamma_0(t),$$

it follows that $\gamma_n(\mu_t) = \gamma_n(\mu_t^M) = \gamma_n(t)$. □

Corollary 4. *The Meixner distributions are the only convolution semigroups whose Jacobi parameters are linear in the convolution parameter.*

Proof. For the usual cumulants, $\alpha_1 = 1, \alpha_2 = 0, \alpha_3 = 2$, so $\beta_1(t) = \beta_0 t + b$, and $\gamma_1(t) = 2\gamma_0 t + 2c$ is not constant. The Meixner distribution with these initial values of the Jacobi parameters has

$$J(\mu^{*t}) = \begin{pmatrix} \beta_0 t, & \beta_0 t + b, & \beta_0 t + 2b, & \beta_0 t + 3b, & \dots \\ \gamma_0 t, & 2(\gamma_0 t + c), & 3(\gamma_0 t + 2c), & 4(\gamma_0 t + 3c), & \dots \end{pmatrix}$$

for $\beta_0, b, c \in \mathbb{R}, \gamma_0 > 0, t \in (-c/\gamma_0)\mathbb{N}$ if $c < 0$ and $t \geq 0$ otherwise. See [Sch00] for more details. \square

Example 1. For the free cumulants, $\alpha_1 = 1, \alpha_2 = 0, \alpha_3 = 1$, so $\beta_1(t) = \beta_0 t + b$, and $\gamma_1(t) = \gamma_0 t + c$ is not constant. Therefore we re-prove our result that the free Meixner distributions are the only free convolution semigroups whose Jacobi parameters are linear in the convolution parameter.

Example 2. For the Boolean cumulants, $\alpha_1 = \alpha_2 = \alpha_3 = 0$, so that $\gamma_1(t) = c_2$ is independent of t . Therefore Proposition 3 does not apply. In fact, for any Boolean convolution semigroup, the Jacobi parameters are

$$J(\mu^{\uplus t}) = \begin{pmatrix} \beta_0 t, & \beta_1, & \beta_2, & \dots \\ \gamma_0 t, & \gamma_1, & \gamma_2, & \dots \end{pmatrix},$$

see [BW01] and [Ans09a].

Example 3. For the monotone cumulants [Mur97, HS09], $\alpha_1 = \frac{1}{2}, \alpha_2 = 0, \alpha_3 = \frac{1}{2}$. So Proposition 3 applies. However, we do not know what the ‘‘monotone Meixner’’ distributions should be. In fact, in this case it is not clear if we have a 4-parameter family of distributions with linear Jacobi parameters; for example, this condition implies that the mean $\beta_0 = 0$.

5. THE TWO-STATE FREE CONVOLUTION.

Recall from Section 2.1 that a block $U \in \pi \in \text{NC}(X)$ will be called *outer* (resp. *inner*) if $d(U, \pi) = 0$ (resp. $d(U, \pi) > 0$). The family of all outer (resp. inner) blocks of π will be denoted by $\text{Out}(\pi)$ (resp. $\text{Inn}(\pi)$).

Suppose we have an additional measure $\tilde{\mu}$, with moments \tilde{s}_m and Jacobi parameters $\tilde{\gamma}_m, \tilde{\beta}_m$. Recall from Section 2.3 that the *conditionally free cumulants* $R_m = R_m(\tilde{\mu}, \mu)$ of the pair $(\tilde{\mu}, \mu)$ are defined by

$$(21) \quad \tilde{s}_m = \sum_{\pi \in \text{NC}(m)} \prod_{U \in \text{Out}(\pi)} R_{|U|}(\tilde{\mu}, \mu) \prod_{U \in \text{Inn}(\pi)} r_{|U|}(\mu),$$

where $r_m(\mu)$ are the free cumulants of μ . We also defined $R^{\tilde{\mu}, \mu}(z) := \sum_{m=1}^{\infty} R_m(\tilde{\mu}, \mu) z^m$.

For $\sigma \in \text{NC}_{1,2}(X)$ and $V \in \sigma$, with label k , we define

$$(22) \quad \tilde{w}(V, k, \sigma) := \begin{cases} \tilde{\beta}_k - \beta_{k-1} & \text{if } |V| = 1 \text{ and } k = d(V, \sigma), \\ \beta_k - \beta_{k-1} & \text{if } |V| = 1 \text{ and } k < d(V, \sigma), \\ \tilde{\gamma}_k - \gamma_{k-1} & \text{if } |V| = 2 \text{ and } k = d(V, \sigma), \\ \gamma_k - \gamma_{k-1} & \text{if } |V| = 2 \text{ and } k < d(V, \sigma), \end{cases}$$

keeping our convention that $\beta_{-1} = \gamma_{-1} = 0$. For $(\sigma, \kappa) \in \text{NCL}_{1,2}(X)$ we put

$$(23) \quad \tilde{w}(\sigma, \kappa) := \prod_{V \in \sigma} \tilde{w}(V, \kappa(V), \sigma).$$

Then, in view of [Mto09], for every $m \geq 1$ we have

$$(24) \quad R_m(\tilde{\mu}, \mu) = \sum_{(\sigma, \kappa) \in \text{NCL}_{1,2}^1(m)} \tilde{w}(\sigma, \kappa).$$

For example:

$$(25) \quad R_1 = \tilde{\beta}_0,$$

$$(26) \quad R_2 = \tilde{\gamma}_0,$$

$$(27) \quad R_3 = \tilde{\gamma}_0(\tilde{\beta}_1 - \beta_0),$$

$$(28) \quad R_4 = \tilde{\gamma}_0[(\tilde{\beta}_1 - \beta_0)^2 + (\tilde{\gamma}_1 - \gamma_0)],$$

$$R_5 = \tilde{\gamma}_0[(\tilde{\beta}_1 - \beta_0)^3 + 2(\tilde{\gamma}_1 - \gamma_0)(\tilde{\beta}_1 - \beta_0) + (\tilde{\gamma}_1 - \gamma_0)(\beta_1 - \beta_0)$$

$$(29) \quad + \tilde{\gamma}_1(\tilde{\beta}_2 - \beta_1)],$$

$$R_6 = \tilde{\gamma}_0[(\tilde{\beta}_1 - \beta_0)^4 + 3(\tilde{\gamma}_1 - \gamma_0)(\tilde{\beta}_1 - \beta_0)^2 + 2(\tilde{\gamma}_1 - \gamma_0)(\tilde{\beta}_1 - \beta_0)(\beta_1 - \beta_0)$$

$$+ (\tilde{\gamma}_1 - \gamma_0)(\beta_1 - \beta_0)^2 + 2\tilde{\gamma}_1(\tilde{\beta}_2 - \beta_1)(\tilde{\beta}_1 - \beta_0) + 2\tilde{\gamma}_1(\tilde{\beta}_2 - \beta_1)(\beta_1 - \beta_0)$$

$$(30) \quad + \tilde{\gamma}_1(\tilde{\beta}_2 - \beta_1)^2 + (\tilde{\gamma}_1 - \gamma_0)^2 + (\tilde{\gamma}_1 - \gamma_0)(\gamma_1 - \gamma_0) + \tilde{\gamma}_1(\tilde{\gamma}_2 - \gamma_1)].$$

The conditionally free power of a pair of measures: $(\tilde{\mu}, \mu)^{\boxplus ct} = (\tilde{\mu}_t, \mu_t)$ is defined by: $\mu_t = \mu^{\boxplus t}$ and $R_m(\tilde{\mu}_t, \mu_t) = t \cdot R_m(\tilde{\mu}, \mu)$.

Theorem 5. Assume that the pair of measures $(\tilde{\mu}, \mu)$, with Jacobi parameters $\tilde{\beta}_m, \tilde{\gamma}_m$ and β_m, γ_m respectively, is such that for the conditionally free powers $(\tilde{\mu}_t, \mu_t) := (\tilde{\mu}, \mu)^{\boxplus ct}$, $t \geq 1$, all the Jacobi parameters of $\tilde{\mu}_t$ and μ_t are polynomials on t . Then

$$\beta_1 = \tilde{\beta}_2 = \beta_2 = \tilde{\beta}_3 = \beta_3 = \tilde{\beta}_4 = \dots,$$

$$\gamma_1 = \tilde{\gamma}_2 = \gamma_2 = \tilde{\gamma}_3 = \gamma_3 = \tilde{\gamma}_4 = \dots,$$

so that $\tilde{\mu}$ is a general distribution whose Jacobi parameters do not depend on n for $n \geq 3$, and μ is the corresponding free Meixner distribution. If these hold then, putting $\tilde{b} = \tilde{\beta}_1 - \beta_0$, $b = \beta_1 - \beta_0$, $\tilde{c} = \tilde{\gamma}_1 - \gamma_0$, $c = \gamma_1 - \gamma_0$ we have

$$(31) \quad J(\tilde{\mu}_t) = \begin{pmatrix} \tilde{\beta}_0 t, & \tilde{b} + \beta_0 t, & b + \beta_0 t, & b + \beta_0 t, & \dots \\ \tilde{\gamma}_0 t, & \tilde{c} + \gamma_0 t, & c + \gamma_0 t, & c + \gamma_0 t, & \dots \end{pmatrix}$$

and

$$(32) \quad J(\mu_t) = \begin{pmatrix} \beta_0 t, & b + \beta_0 t, & b + \beta_0 t, & b + \beta_0 t, & \dots \\ \gamma_0 t, & c + \gamma_0 t, & c + \gamma_0 t, & c + \gamma_0 t, & \dots \end{pmatrix}.$$

In particular, the pair $(\tilde{\mu}, \mu)$ is \boxplus_c -infinitely divisible if and only if $c \geq 0$ and $\tilde{c} \geq 0$. Then

$$R^{\tilde{\mu}, \mu}(z) = \tilde{\beta}_0 z + \tilde{\gamma}_0 z^2 \int_{\mathbb{R}} \frac{d\tilde{\rho}(x)}{1 - xz},$$

where $\tilde{\rho}$ is the free Meixner probability measure which satisfies

$$(33) \quad J(\tilde{\rho}) = \begin{pmatrix} \tilde{b}, & b, & b, & b, & \dots \\ \tilde{c}, & c, & c, & c, & \dots \end{pmatrix}.$$

The proof is similar to that of Theorem 1. Theorem 5 actually holds under weaker assumptions, i.e. without assuming that the Jacobi parameters of μ_t are polynomials on t . See the Appendix for the proof.

Theorem 6. *Assume that the pair of measures $(\tilde{\mu}, \mu)$, with Jacobi parameters $\tilde{\beta}_m, \tilde{\gamma}_m$ and β_m, γ_m respectively, is such that for the conditionally free powers $(\tilde{\mu}_t, \mu_t) := (\tilde{\mu}, \mu)^{\boxplus ct}$, $t \geq 1$, all the Jacobi parameters of $\tilde{\mu}_t$ are polynomials on t . Then conclusions (31) and (32) hold.*

Remark 1. An explicit formula for $\tilde{\mu}_t$ can be obtained from the continued fraction expansion of its Cauchy transform:

$$G_{\tilde{\mu}_t}(z) = \frac{1}{z - \tilde{\beta}_0 t - \frac{\tilde{\gamma}_0 t}{z - \beta_0 t - \tilde{b} - (\gamma_0 t + \tilde{c})G_{\sigma_t}(z)}},$$

where σ_t is the semicircular distribution with mean $\beta_0 t + b$ and variance $\gamma_0 t + c$. The corresponding measure belongs to the Bernstein-Szegő class, and has the form

$$\tilde{\mu}_t = \frac{\sqrt{4(\gamma_0 t + c) - (x - \beta_0 t - b)^2}}{\text{cubic polynomial}} dx + \text{at most 3 atoms.}$$

Proposition 7. *Let $\tilde{\rho}$ be a probability measure, $a, c \in \mathbb{R}$, $b \geq 0$, $d \geq 0$. Define μ and ω via*

$$\mathbb{B}[\tilde{\rho}] = \mu^{\boxplus b} \boxplus \delta_a$$

and

$$\omega = \mathbb{B}[\Phi[\tilde{\rho}]]^{\boxplus d} \boxplus \delta_c.$$

Denote $\omega_t = \omega^{\boxplus t}$, $\mu_t = \mu^{\boxplus t}$, and define

$$\tilde{\mu}_t = \Phi[\omega_t, \mu_t],$$

so that $R^{\tilde{\mu}_t, \mu_t} = R^{\omega_t}$ and the pairs $\{(\tilde{\mu}_t, \mu_t)\}$ form a two-state free convolution semigroup.

Suppose that

$$J(\mu_t) = \begin{pmatrix} \beta_0(t), & \beta_1(t), & \beta_2(t), & \dots \\ \gamma_0(t), & \gamma_1(t), & \gamma_2(t), & \dots \end{pmatrix}.$$

Then

$$R^{\tilde{\mu}, \mu}(z) = R^\omega(z) = cz + \frac{dz^2}{1 - az - bR^\mu(z)}$$

and

$$J(\tilde{\mu}_t) = \begin{pmatrix} ct, & a + (1 + \frac{b}{t})\beta_0(t), & \beta_1(t), & \beta_2(t), & \dots \\ dt, & (1 + \frac{b}{t})\gamma_0(t), & \gamma_1(t), & \gamma_2(t), & \dots \end{pmatrix}.$$

Proof. Using definitions of $\omega, \mathbb{B}, \Phi, \eta, \mu$ in succession, we transform

$$\begin{aligned} R^\omega(z) &= cz + dR^{\mathbb{B}[\Phi[\tilde{\rho}]]}(z) = cz + d\eta^{\Phi[\tilde{\rho}]}(z) = cz + dz^2(1 + M^{\tilde{\rho}}(z)) \\ &= cz + \frac{dz^2}{1 - \eta^{\tilde{\rho}}(z)} = cz + \frac{dz^2}{1 - R^{\mathbb{B}[\tilde{\rho}]}(z)} = cz + \frac{dz^2}{1 - az - bR^\mu(z)}. \end{aligned}$$

Therefore by definition of $\tilde{\mu}_t$ and $\Phi[\cdot, \cdot]$,

$$\begin{aligned}\eta^{\tilde{\mu}_t}(z) &= t(1 + M^{\mu_t}(z))^{-1} R^\omega\left((1 + M^{\mu_t}(z))z\right) \\ &= ctz + \frac{dtz^2(1 + M^{\mu_t}(z))}{1 - a((1 + M^{\mu_t}(z))z) - \frac{b}{t}R^{\mu_t}\left((1 + M^{\mu_t}(z))z\right)} \\ &= ctz + \frac{dtz^2(1 + M^{\mu_t}(z))}{1 - a((1 + M^{\mu_t}(z))z) - \frac{b}{t}M^{\mu_t}(z)} \\ &= ctz + \frac{dtz^2}{1 - az - \left(1 + \frac{b}{t}\right)\eta^\mu(z)}.\end{aligned}$$

So writing

$$\eta^\mu(z) = \beta_0(t)z + \gamma_0(t)z^2F,$$

we get

$$\begin{aligned}\eta^{\tilde{\mu}_t}(z) &= ctz + \frac{dtz^2}{1 - az - \left(1 + \frac{b}{t}\right)(\beta_0(t)z + \gamma_0(t)z^2F)} \\ &= ctz + \frac{dtz^2}{1 - \left(a + \left(1 + \frac{b}{t}\right)\beta_0(t)\right)z - \left(1 + \frac{b}{t}\right)\gamma_0(t)z^2F}.\end{aligned}$$

The conclusion follows. □

Corollary 8. *Let μ be a free Meixner distribution,*

$$\mu = \mu_{b,c}^{\boxplus\gamma_0} \boxplus \delta_{\beta_0}$$

and

$$J(\mu) = \begin{pmatrix} \beta_0, & \beta_0 + b, & \beta_0 + b, & \dots \\ \gamma_0, & \gamma_0 + c, & \gamma_0 + c, & \dots \end{pmatrix}.$$

(a) μ_t has the Jacobi parameters in equation (32).

(b) If $c \geq 0$ and $\tilde{c} \geq 0$, then

$$\tilde{\rho} = \mu_{b,c-1}^{\boxplus\tilde{c}} \boxplus \delta_{\tilde{b}}$$

is a free Meixner distribution with

$$J(\tilde{\rho}) = \begin{pmatrix} \tilde{b}, & b, & b, & \dots \\ \tilde{c}, & c, & c, & \dots \end{pmatrix}.$$

For

$$\mathbb{B}[\tilde{\rho}] = \mu^{\boxplus(\tilde{c}/\gamma_0)} \boxplus \delta_{(\tilde{b}\gamma_0 - \beta_0\tilde{c})/\gamma_0}, \quad \omega = \mathbb{B}[\Phi[\tilde{\rho}]]^{\boxplus\tilde{\gamma}_0} \boxplus \delta_{\tilde{\beta}_0}, \quad \tilde{\mu}_t = \Phi[\omega_t, \mu_t],$$

ω is freely infinitely divisible, and $\tilde{\mu}_t$ has the Jacobi parameters in equation (31).

In the case that c or \tilde{c} are negative, note that the operations $\boxplus, \mathbb{B}, \Phi$ can be defined purely combinatorially (in terms of moments), so they are well-defined operations on (not necessarily positive) functionals, in which case \mathbb{B} is even a bijection. So the construction remains valid, as long as $\mu_t, \tilde{\mu}_t$ themselves are positive.

(c) If $\tilde{b} = b$ and $\tilde{c} = c$, so that $\tilde{\rho}$ is a semicircular distribution, then

$$\omega = \mu_{b,c}^{\boxplus \tilde{\gamma}_0} \boxplus \delta_{\tilde{\beta}_0}$$

and

$$J(\tilde{\mu}_t) = \begin{pmatrix} \tilde{\beta}_0 t, & \beta_0 t + b, & \beta_0 t + b, & \dots \\ \tilde{\gamma}_0 t, & \gamma_0 t + c, & \gamma_0 t + c, & \dots \end{pmatrix}$$

so that $\omega, \tilde{\mu}_t$ are also free Meixner distributions.

Free Meixner distributions arise in many situations. A number of appearances of distributions with Jacobi parameters constant after step 3 are described below. Other places where they were encountered include Theorems 11 and 12 of [KW05], examples in [Len07], as well as [HM07, HM08] and [HKM09].

5.1. Limit theorems.

(a) In the two-state free central limit theorem, one gets pairs of distributions $(\tilde{\mu}, \mu)$ such that

$$R^\mu(z) = uz^2, \quad R^{\tilde{\mu}, \mu}(z) = vz^2.$$

Thus $\mu = \mu_{0,0}^{\boxplus u}$ and $\omega = \mu_{0,0}^{\boxplus v}$ are semicircular distributions. In this case

$$\tilde{b} = b = 0, \quad \tilde{c} = c = 0, \quad \tilde{\beta}_0 = \beta_0 = 0, \quad \tilde{\gamma}_0 = v, \quad \gamma_0 = u.$$

Therefore $\tilde{\rho} = \delta_0$ and

$$\tilde{\mu} = \mu_{0,u-v}^{\boxplus v}$$

with

$$J(\tilde{\mu}) = \begin{pmatrix} 0, & 0, & 0 & \dots \\ v, & u, & u, & \dots \end{pmatrix}$$

is a symmetric free Meixner distribution.

(b) In the (centered) two-state free Poisson limit theorem, one gets pairs of distributions $(\tilde{\mu}, \mu)$ such that

$$R^\mu(z) = \frac{uz^2}{1-pz}, \quad R^{\tilde{\mu}, \mu}(z) = \frac{vz^2}{1-rz},$$

Thus $\mu = \mu_{p,0}^{\boxplus u}$ and $\omega = \mu_{r,0}^{\boxplus v}$ are free Poisson distributions. Assuming that $p = r$, in this case

$$\tilde{b} = b = p, \quad \tilde{c} = c = 0, \quad \tilde{\beta}_0 = \beta_0 = 0, \quad \tilde{\gamma}_0 = v, \quad \gamma_0 = u.$$

Therefore

$$\tilde{\rho} = \delta_p$$

with

$$J(\tilde{\rho}) = \begin{pmatrix} p, & p, & p, & \dots \\ 0, & 0, & 0, & \dots \end{pmatrix}$$

and

$$\tilde{\mu} = \mu_{p,u-v}^{\boxplus v}$$

with

$$J(\tilde{\mu}) = \begin{pmatrix} 0, & p, & p, & \dots \\ v, & u, & u, & \dots \end{pmatrix}$$

is a free Meixner distribution.

(c) If $p \neq r$, then

$$\tilde{b} = r, \quad b = p, \quad \tilde{c} = c = 0, \quad \tilde{\beta}_0 = \beta_0 = 0, \quad \tilde{\gamma}_0 = v, \quad \gamma_0 = u.$$

In this case $\tilde{\rho} = \delta_r$ still, but now we identify

$$J(\tilde{\rho}) = \begin{pmatrix} r, & p, & p, & \cdots \\ 0, & 0, & 0, & \cdots \end{pmatrix}$$

and not with a degenerate semicircular distribution. Finally,

$$J(\tilde{\mu}) = \begin{pmatrix} 0, & r, & p, & p, & \cdots \\ v, & u, & u, & u, & \cdots \end{pmatrix}$$

has Jacobi parameters constant after step 3.

5.2. Laha-Lukacs characterization. A classical paper [LL60] characterizes Meixner distributions in terms of certain conditional expectations. In [BB06], the authors obtained a similar characterization of free Meixner distributions. The following is their result for the two-state free independence. Recall that the distribution of an operator X with respect to a state φ is the measure μ such that for all n ,

$$\varphi[X^n] = s_n(\mu) = \int_{\mathbb{R}} x^n d\mu(x).$$

Also, one says that \mathbb{X}, \mathbb{Y} are $(\varphi|\psi)$ -free if all their mixed two-state free cumulants are zero. See the paper quoted below for the rest of the terminology.

Theorem (Theorem 2.1 in [BB09]). *Suppose \mathbb{X}, \mathbb{Y} are self-adjoint $(\varphi|\psi)$ -free and*

$$\varphi[\mathbb{X}^n] = \varphi[\mathbb{Y}^n], \quad \psi[\mathbb{X}^n] = \psi[\mathbb{Y}^n]$$

for all n . Furthermore, assume that $\varphi[\mathbb{X}] = 0, \varphi[\mathbb{X}^2] = 1$. (This can always be achieved by a shift and dilation, as long as $\varphi[\mathbb{X}^2] \neq 0$.) Let $\mathbb{S} = \mathbb{X} + \mathbb{Y}$ and suppose that there are $a, C \in \mathbb{R}$ and $b > -2$ such that

$$(34) \quad \varphi[(\mathbb{X} - \mathbb{Y})^2 \mathbb{S}^n] = C\varphi[(4\mathbb{I} + 2a\mathbb{S} + b\mathbb{S}^2)\mathbb{S}^n], \quad n = 0, 1, 2, \dots$$

Then the φ and ψ -moment generating functions

$$M_{\mathbb{S}}(z) := \sum_{k=1}^{\infty} z^k \varphi[\mathbb{S}^k] \text{ and } m_{\mathbb{S}}(z) := \sum_{k=1}^{\infty} z^k \psi[\mathbb{S}^k],$$

which are defined as formal power series, are related as follows:

$$(35) \quad 1 + M_{\mathbb{S}}(z) = \frac{2 + b - (2az + b)(1 + m_{\mathbb{S}}(z))}{2 + b - (4z^2 + 2az + b)(1 + m_{\mathbb{S}}(z))}.$$

Bożejko and Bryc described the corresponding distributions more explicitly in particular cases corresponding to the Gaussian and Poisson regressions. We now provide a complete description. Recall that if $\varphi = \psi$, then the ψ -distributions of $\mathbb{X}, \mathbb{Y}, \mathbb{S}$ are free Meixner distributions.

Proposition 9. *Denote by $\tilde{\mu}_{\mathbb{S}}, \mu_{\mathbb{S}}$ the distributions of \mathbb{S} with respect to φ, ψ , respectively, and by $\tilde{\mu}, \mu$ the corresponding distributions of \mathbb{X} . Then*

$$\tilde{\mu} = \Phi[\mu_{\mathbb{S}}^{\uplus(1+b/2)} \uplus \delta_a]$$

and

$$J(\tilde{\mu}) = \begin{pmatrix} 0, & a + (1 + b/2)\beta_0(\mu_{\mathbb{S}}), & \beta_1(\mu_{\mathbb{S}}), & \dots \\ 1, & (1 + b/2)\gamma_0(\mu_{\mathbb{S}}), & \gamma_1(\mu_{\mathbb{S}}), & \dots \end{pmatrix}.$$

If moreover $\mu = \mu_{u,v}^{\boxplus d} \boxplus \delta_c$ is a free Meixner distribution, then

$$J(\tilde{\mu}) = \begin{pmatrix} 0, & a + (2 + b)c, & 2c + u, & 2c + u, & \dots \\ 1, & (2 + b)d, & 2d + v, & 2d + v, & \dots \end{pmatrix}.$$

In particular, if (34) also holds with ψ in place of φ , for the same a, b , then $\mu = \mu_{a,b}^{\boxplus d} \boxplus \delta_c$ and

$$J(\tilde{\mu}) = \begin{pmatrix} 0, & (a + 2c) + bc, & (a + 2c), & (a + 2c), & \dots \\ 1, & 2d + bd, & 2d + b, & 2d + b, & \dots \end{pmatrix},$$

so that $\tilde{\mu}$ is a general (up to normalization) distribution with Jacobi parameters constant after step 3.

Proof. From equation (35),

$$\eta^{\tilde{\mu}_{\mathbb{S}}} = -\frac{4z^2}{b + 2az - (2 + b)(M^{\mu_{\mathbb{S}}})^{-1}},$$

so for \mathbb{X} ,

$$\begin{aligned} \eta^{\tilde{\mu}} &= -\frac{2z^2}{b + 2az - (2 + b)(M^{\mu_{\mathbb{S}}})^{-1}} \\ &= -\frac{2z^2}{b + 2az - (2 + b)(1 - \eta^{\mu_{\mathbb{S}}})} \\ &= -\frac{2z^2}{-2 + 2az + (2 + b)\eta^{\mu_{\mathbb{S}}}} \\ &= \frac{z^2}{1 - az - (1 + b/2)\eta^{\mu_{\mathbb{S}}}} \end{aligned}$$

Comparing with the general formula in Proposition 7, we conclude that $\omega = \Phi[\tilde{\rho}], \mathbb{B}[\tilde{\rho}] = \mu^{\boxplus(b/2)} \boxplus \delta_a$, and in terms of the Jacobi parameters of $\mu_{\mathbb{S}}$,

$$J(\tilde{\mu}) = \begin{pmatrix} 0, & a + (1 + b/2)\beta_0, & \beta_1, & \beta_2, & \dots \\ 1, & (1 + b/2)\gamma_0, & \gamma_1, & \gamma_2, & \dots \end{pmatrix},$$

so that

$$\tilde{\mu} = \Phi[\mu_{\mathbb{S}}^{\uplus(1+b/2)} \uplus \delta_a].$$

If \mathbb{X} has, with respect to ψ , a free Meixner distribution $\mu = \mu_{u,v}^{\boxplus d} \boxplus \delta_c$, then

$$\mu_{\mathbb{S}} = \mu_{u,v}^{\boxplus 2d} \boxplus \delta_{2c} = \mu_{2c+u, 2d+v-1}^{\uplus 2d} \uplus \delta_{2c}$$

and

$$\eta^{\mu_{\mathbb{S}}} = 2cz + 2d\eta^{\mu_{2c+u, 2d+v-1}} = 2xz + 2d\eta^{\mathbb{B}_{2c, 2d-1}[\mu_{u,v}]}$$

So

$$\eta^{\tilde{\mu}} = \frac{z^2}{1 - az - (1 + b/2)(2cz + 2d\eta^{\mu_{2c+u, 2d+v-1}})} = \frac{z^2}{1 - (a + (2 + b)c)z - (2 + b)d\eta^{\mu_{2c+u, 2d+v-1}}},$$

and

$$J(\tilde{\mu}) = \begin{pmatrix} 0, & a + (2+b)c, & 2c + u, & 2c + u, & \dots \\ 1, & (2+b)d, & 2d + v, & 2d + v, & \dots \end{pmatrix}.$$

The last conclusion follows. \square

5.3. Free quadratic harnesses. In a series of papers starting with [BW05], Bryc and Wesolowski (along with Matysiak and Szabłowski) have investigated quadratic harnesses. These are square-integrable processes $(X_t)_{t \geq 0}$, with normalization $\mathbb{E}[X_t] = 0$, $\mathbb{E}[X_t X_s] = \min(t, s)$, such that $\mathbb{E}[X_t | \mathcal{F}_{s,u}]$ is a linear function of X_s, X_u and $\text{Var}[X_t | \mathcal{F}_{s,u}]$ is a quadratic function of X_s, X_u . Here $\mathcal{F}_{s,u}$ is the two-sided σ -field generated by $\{X_r : r \in [0, s] \cup [u, \infty)\}$. Then

$$\mathbb{E}[X_t | \mathcal{F}_{s,u}] = \frac{u-t}{u-s} X_s + \frac{t-s}{u-s} X_u$$

and under certain technical assumptions (see [BMW07]),

$$\begin{aligned} \text{Var}[X_t | \mathcal{F}_{s,u}] &= \frac{(u-t)(t-s)}{u(1+\sigma s) + \tau - \gamma s} \left(1 + \sigma \frac{(uX_s - sX_u)^2}{(u-s)^2} + \tau \frac{(X_u - X_s)^2}{(u-s)^2} \right. \\ &\quad \left. + \eta \frac{uX_s - sX_u}{u-s} + \theta \frac{X_u - X_s}{u-s} - (1-\gamma) \frac{(X_u - X_s)(uX_s - sX_u)}{(u-s)^2} \right). \end{aligned}$$

The authors proved the existence of such processes for a large range of parameters $\sigma, \tau, \eta, \theta, \gamma$, in particular connecting the analysis to the Askey-Wilson distributions in [BW09] (the standard Askey-Wilson parameter $q = \gamma + \sigma\tau$). One reason for the interest in this analysis comes from numerous particular cases.

- (a) If $\gamma = 1$ and $\sigma = \eta = 0$, the processes automatically have classically independent increments, and each X_t has a Meixner distribution, see [Wes93].
- (b) For $\gamma = \sigma = \eta = 0$, the processes are classical versions of processes with free independent increments, and have free Meixner distributions.
- (c) For $\sigma = \eta = 0$ and $-1 \leq \gamma = q < 1$, the corresponding orthogonal martingale polynomials have Jacobi parameters

$$\begin{pmatrix} \beta_n(t) & = \theta[n]_q \\ \gamma_n(t) & = [n+1]_q(t + \tau[n]_q) \end{pmatrix},$$

where $[n]_q := 1 + q + \dots + q^{n-1}$ is the q -integer. If $\tau = 0$, the process is a (classical version of a) q -Poisson process from [Ans01]. The case where in addition, $\theta = 0$ was considered even earlier in [Bry01] and corresponds to the q -Brownian motion [BKS97]. The challenge of interpreting the general processes with $\sigma = \eta = 0$ as “processes with q -independent increments” remains open.

- (d) Finally, for $\gamma = \sigma = \tau = 0$, the free bi-Poisson processes from [BW07] are shown, in Section 4 of that paper, to have increments freely independent with respect to a pair of states.

We will now extend the last result above. Proposition 4.3 of [BMW07] states that for

$$q = \gamma + \sigma\tau = 0,$$

there exist orthogonal martingale polynomials for the process. They satisfy recursion relations

$$\begin{aligned} xP_0(x, t) &= P_1(x, t), \\ xP_1(x, t) &= (1 + \sigma t)P_2(x, t) + (ut + v)P_1(x, t) + tP_0(x, t), \\ xP_2(x, t) &= (1 + \sigma t)P_3(x, t) + \left(\frac{u + \sigma v}{1 - \sigma\tau}t + \frac{v + \tau u}{1 - \sigma\tau} \right) P_2(x, t) + \frac{1 + uv}{1 - \sigma\tau}(t + \tau)P_1(x, t), \\ xP_n(x, t) &= (1 + \sigma t)P_{n+1}(x, t) + \left(\frac{u + \sigma v}{1 - \sigma\tau}t + \frac{v + \tau u}{1 - \sigma\tau} \right) P_n(x, t) + \frac{1 + uv}{(1 - \sigma\tau)^2}(t + \tau)P_{n-1}(x, t) \end{aligned}$$

for $n \geq 3$, where

$$u = \frac{\eta + \sigma\theta}{1 - \sigma\tau}, \quad v = \frac{\tau\eta + \theta}{1 - \sigma\tau}$$

and as long as

$$1 + uv > 0, \quad 0 \leq \sigma\tau < 1.$$

The coefficients in this recursion are linear in t and constant for $t \geq 3$, but the corresponding polynomials are not monic. It follows that the Jacobi parameters for the monic orthogonal polynomials for this process (which are not martingale polynomials) are quadratic in t . Therefore they do not form a semigroup with respect to any of the convolutions considered in this paper, unless $\sigma = 0$.

Proposition 10. *Let $q = 0$ and $\sigma = 0$. Denoting by $\tilde{\mu}_t$ the distribution of X_t , for some $\{\mu_t\}$, the pairs $\{(\tilde{\mu}_t, \mu_t)\}$ form a two-state free convolution semigroup. Also in this case, $\tilde{\rho}$ is a free Poisson distribution.*

Proof. Since $\sigma = 0$, we have $u = \eta$, $v = \tau\eta + \theta$, and the identification with parameters in our Theorem 5 gives

$$\tilde{\beta}_0 = 0, \quad \beta_0 = \eta, \quad \tilde{b} = (\tau\eta + \theta), \quad b = 2\tau\eta + \theta$$

and

$$\tilde{\gamma}_0 = 1, \quad \gamma_0 = 1 + \eta(\tau\eta + \theta), \quad \tilde{c} = c = \tau(1 + \eta(\tau\eta + \theta)).$$

Note that this is a distribution with Jacobi parameters constant after step 3, but not the most general one.

Thus if

$$\mu_t = \left(\mu_{2\tau\eta + \theta, \tau(1 + \eta(\tau\eta + \theta))}^{\boxplus(1 + \eta(\tau\eta + \theta))} \boxplus \delta_\eta \right)^{\boxplus t}$$

are free Meixner distributions, then the pairs $\{(\tilde{\mu}_t, \mu_t)\}$ form a two-state free convolution semigroup. Also,

$$J(\tilde{\rho}) = \begin{pmatrix} (\tau\eta + \theta), & 2\tau\eta + \theta, & 2\tau\eta + \theta, & \dots \\ \tau(1 + \eta(\tau\eta + \theta)), & \tau(1 + \eta(\tau\eta + \theta)), & \tau(1 + \eta(\tau\eta + \theta)), & \dots \end{pmatrix}$$

and $\tilde{\rho}$ is a free Poisson distribution. □

Remark 2. Restricting to the $\tau = 0$ case of the free bi-Poisson process gives

$$\mu_t = \left(\mu_{\theta, 0}^{\boxplus(1 + \eta\theta)} \boxplus \delta_\eta \right)^{\boxplus t}$$

a free Poisson distribution, $\tilde{\rho} = \delta_\theta$, $\omega = \mu_{\theta, 0}$ is a free Poisson distribution, and $\tilde{\mu}_t = \mu_{\eta t + \theta, (1 + \eta\theta)t}$ is a free Meixner distribution. Further restriction to $\theta = 0$ gives μ a (non-centered) semicircular distribution, ω the standard semicircular distribution, and $\tilde{\mu}_t = \mu_{\eta t, t}$ a free Meixner distribution.

On the other hand, restriction to $\eta = 0$ gives

$$\mu = \tilde{\mu} = \omega = \mu_{\theta, \tau}$$

and $\tilde{\rho}$ is a semicircular distribution.

APPENDIX A. PROOF OF THEOREM 6.

Proof. Denote by $\tilde{\beta}_m(t)$, $\tilde{\gamma}_m(t)$ and $\beta_m(t)$, $\gamma_m(t)$ the Jacobi parameters of $\tilde{\mu}_t$ and μ_t respectively. Putting in formulas (12)–(19) and (23)–(30)

$$r_mt, \quad R_mt, \quad \beta_m(t), \quad \gamma_m(t), \quad \tilde{\beta}_m(t), \quad \tilde{\gamma}_m(t), \quad w_t, \quad \tilde{w}_t$$

instead of

$$r_m, \quad R_m, \quad \beta_m, \quad \gamma_m, \quad \tilde{\beta}_m, \quad \tilde{\gamma}_m, \quad w, \quad \tilde{w}$$

respectively we see that

$$\beta_0(t) = \beta_0 t, \quad \gamma_0(t) = \gamma_0 t, \quad \beta_1(t) = \beta_1 - \beta_0 + \beta_0 t, \quad \gamma_1(t) = \gamma_1 - \gamma_0 + \gamma_0 t,$$

by (14)–(17), and

$$\tilde{\beta}_0(t) = \tilde{\beta}_0 t, \quad \tilde{\gamma}_0(t) = \tilde{\gamma}_0 t, \quad \tilde{\beta}_1(t) = \tilde{\beta}_1 - \beta_0 + \beta_0 t, \quad \tilde{\gamma}_1(t) = \tilde{\gamma}_1 - \gamma_0 + \gamma_0 t,$$

from (25)–(28). If we apply these formulas to (29) then we get

$$R_5 t = \text{constant} \cdot t + \tilde{\gamma}_0(t) \tilde{\gamma}_1(t) (\tilde{\beta}_2(t) - \beta_1(t)).$$

Since, by assumption, $\tilde{\beta}_2(t)$ is a polynomial, this implies that

$$\tilde{\beta}_2(t) = \beta_1(t) = \beta_1 - \beta_0 + \beta_0 t$$

(for otherwise the right hand side would be a polynomial of degree at least 2). Then in (30) we obtain

$$R_6 t = \text{constant} \cdot t + \tilde{\gamma}_0(t) \tilde{\gamma}_1(t) (\tilde{\gamma}_2(t) - \gamma_1(t)),$$

which, in turn, yields

$$\tilde{\gamma}_2(t) = \gamma_1(t) = \gamma_1 - \gamma_0 + \gamma_0 t.$$

Now we are going to prove by induction that for every $n \geq 1$:

$$(36) \quad \tilde{\beta}_{n+1}(t) = \beta_n(t) = \beta_1 - \beta_0 + \beta_0 t,$$

$$(37) \quad \tilde{\gamma}_{n+1}(t) = \gamma_n(t) = \gamma_1 - \gamma_0 + \gamma_0 t.$$

Fix $d \geq 2$ and suppose that (36)–(37) hold for all n such that $1 \leq n < d$. Now we consider (12) for $r_{2d+1}t$. Put

$$\begin{aligned} \sigma_1^d &:= \{\{1, 2d+1\}, \{2, 2d\}, \{3, 2d-1\}, \dots, \{d, d+2\}, \{d+1\}\}, \\ \mathcal{K}_1^d &:= \{\kappa : (\sigma_1^d, \kappa) \in \text{NCL}_{1,2}^1(2d+1), \kappa(\{d+1\}) = d\}. \end{aligned}$$

By our assumptions, if $1 < k < d$ then $w_t(V, k) = 0$ for any block V , with $1 \leq |V| \leq 2$. Therefore the right hand side of (12) for $r_{2d+1}t$ involves only such $(\sigma, \kappa) \in \text{NCL}_{1,2}^1(2d+1)$ that either $\kappa(V) = 1$ for every inner block $V \in \sigma$ (and then $w_t(V, \kappa(V)) = w_t(V, 1) = \beta_1 - \beta_0$ if

$|V| = 1$ or $\gamma_1 - \gamma_2$ if $|V| = 2$ or $\sigma = \sigma_1^d$, $\kappa \in \mathcal{K}_1^d$ and $\kappa(V) \in \{0, 1\}$ for all inner blocks $V \in \sigma$, $V \neq \{d+1\}$. Accordingly we get

$$r_{2d+1}t = \gamma_0t \sum_{\sigma \in \text{NC}_{1,2}(2d-1)} \prod_{\substack{V \in \sigma \\ |V|=1}} (\beta_1 - \beta_0) \times \prod_{\substack{V \in \sigma \\ |V|=2 \\ V \neq \{1, 2d+1\}}} (\gamma_1 - \gamma_0) + \sum_{\kappa \in \mathcal{K}_1^d} w_t(\sigma_1^d, \kappa)$$

and

$$\begin{aligned} \sum_{\kappa \in \mathcal{K}_1^d} w_t(\sigma_1^d, \kappa) &= \gamma_0t(\beta_d(t) - \beta_{d-1}(t)) \prod_{k=1}^{d-1} \sum_{i=0}^1 (\gamma_i(t) - \gamma_{i-1}(t)) \\ &= \gamma_0t((\beta_d(t) - \beta_1(t))\gamma_1(t))^{d-1}. \end{aligned}$$

This implies that

$$(38) \quad (\beta_d(t) - \beta_1(t))\gamma_1(t)^{d-1} = c_1$$

for some constant c_1 .

Now we consider (24) for $R_{2d+3}t$. Put

$$\begin{aligned} \sigma_1^{d+1} &:= \{\{1, 2d+3\}, \{2, 2d+2\}, \{3, 2d+1\}, \dots, \{d+1, d+3\}, \{d+2\}\}, \\ \mathcal{K}_1^{d+1} &:= \{\kappa : (\sigma_1^{d+1}, \kappa) \in \text{NCL}_{1,2}^1(2d+3), \kappa(\{d+2\}) = d+1\}, \\ \mathcal{L}_1^{d+1} &:= \{\kappa : (\sigma_1^{d+1}, \kappa) \in \text{NCL}_{1,2}^1(2d+3), \kappa(\{d+2\}) = d\}. \end{aligned}$$

By our assumption, if $\tilde{w}_t(\sigma, \kappa) \neq 0$ then either $\kappa(V) = 1$ for every inner block $V \in \sigma$ or $\sigma = \sigma_1^{d+1}$, $\kappa \in \mathcal{K}_1^{d+1}$ and $\kappa(V) \in \{0, 1\}$ for all inner blocks $V \in \sigma$, $V \neq \{d+2\}$ or $\sigma = \sigma_1^{d+1}$, $\kappa \in \mathcal{L}_1^{d+1}$, $\kappa(\{2, 2d+2\}) = 1$ and $\kappa(V) \in \{0, 1\}$ for all inner blocks $V \in \sigma$, $V \neq \{d+2\}, \{2, 2d+2\}$.

Therefore we have

$$\begin{aligned} R_{2d+3}t &= \tilde{\gamma}_0t \cdot \sum_{\sigma \in \text{NC}_{1,2}(2d+1)} \prod_{\substack{V \in \sigma \\ |V|=1 \\ V \in \text{Out}(\sigma)}} (\tilde{\beta}_1 - \beta_0) \cdot \prod_{\substack{V \in \sigma \\ |V|=1 \\ V \in \text{Inn}(\sigma)}} (\beta_1 - \beta_0) \cdot \prod_{\substack{V \in \sigma \\ |V|=2 \\ V \in \text{Out}(\sigma)}} (\tilde{\gamma}_1 - \gamma_0) \cdot \prod_{\substack{V \in \sigma \\ |V|=2 \\ V \in \text{Inn}(\sigma)}} (\gamma_1 - \gamma_0) \\ &+ \sum_{\kappa \in \mathcal{L}_1^{d+1}} \tilde{w}_t(\sigma_1^{d+1}, \kappa) + \sum_{\kappa \in \mathcal{K}_1^{d+1}} \tilde{w}_t(\sigma_1^{d+1}, \kappa). \end{aligned}$$

In view of (38) we note that

$$\begin{aligned} \sum_{\kappa \in \mathcal{L}_1^{d+1}} \tilde{w}_t(\sigma_1^{d+1}, \kappa) &= \tilde{\gamma}_0t(\tilde{\gamma}_1(t) - \gamma_0(t))(\beta_d(t) - \beta_{d-1}(t))\tilde{\gamma}_2(t) \dots \tilde{\gamma}_d(t) \\ &= \tilde{\gamma}_0t(\tilde{\gamma}_1 - \gamma_0)(\beta_d(t) - \beta_1(t))\gamma_1(t)^{d-1} \\ &= \tilde{\gamma}_0t(\tilde{\gamma}_1 - \gamma_0)c_1 \end{aligned}$$

and that

$$\begin{aligned} \sum_{\kappa \in \mathcal{K}_1^{d+1}} \tilde{w}_t(\sigma_1^{d+1}, \kappa) &= \tilde{\gamma}_0t(\tilde{\beta}_{d+1}(t) - \beta_d(t))\tilde{\gamma}_1(t)\tilde{\gamma}_2(t) \dots \tilde{\gamma}_d(t) \\ &= \tilde{\gamma}_0t(\tilde{\beta}_{d+1}(t) - \beta_1(t))\tilde{\gamma}_1(t)\gamma_1(t)^{d-1} + \tilde{\gamma}_0t(\beta_1(t) - \beta_d(t))\tilde{\gamma}_1(t)\gamma_1(t)^{d-1} \\ &= \tilde{\gamma}_0t(\tilde{\beta}_{d+1}(t) - \beta_1(t))\tilde{\gamma}_1(t)\gamma_1(t)^{d-1} - \tilde{\gamma}_0t\tilde{\gamma}_1(t)c_1. \end{aligned}$$

If $\tilde{\beta}_{d+1}(t) \neq \beta_{d-1}(t)$ then the first summand is a polynomial of degree at least $d + 1 \geq 3$ and if $c_1 \neq 0$ then the second one is a polynomial of degree 2. Therefore $c_1 = 0$ and

$$(39) \quad \tilde{\beta}_{d+1}(t) = \beta_d(t) = \beta_1(t).$$

Now we will study $\tilde{\gamma}_{d+1}(t)$ and $\gamma_d(t)$ in a similar way. Consider (12) for $r_{2d+2}t$ and put

$$\begin{aligned} \sigma_2^d &:= \{\{1, 2d+2\}, \{2, 2d+1\}, \{3, 2d\}, \dots, \{d+1, d+2\}\}, \\ \mathcal{K}_2^d &:= \{\kappa : (\sigma_2^d, \kappa) \in \text{NCL}_{1,2}^1(2d+2), \kappa(\{d+1, d+2\}) = d\}. \end{aligned}$$

Then by inductive assumption and by (39) we have

$$r_{2d+2}t = \gamma_0 t \sum_{\sigma \in \text{NC}_{1,2}(2d)} \prod_{\substack{V \in \sigma \\ |V|=1}} (\beta_1 - \beta_0) \cdot \prod_{\substack{V \in \sigma \\ |V|=2}} (\gamma_1 - \gamma_0) + \sum_{\kappa \in \mathcal{K}_2^d} w(\sigma_2^d, \kappa)$$

and similarly as before we see that

$$\sum_{\kappa \in \mathcal{K}_2^d} w_t(\sigma_2^d, \kappa) = \gamma_0 t ((\gamma_d(t) - \gamma_1(t)) \gamma_1(t)^{d-1},$$

which implies that

$$(40) \quad ((\gamma_d(t) - \gamma_1(t)) \gamma_1(t)^{d-1} = c_2$$

for some constant c_2 .

Now we consider (24) for $R_{2d+4}t$. Put

$$\begin{aligned} \sigma_2^{d+1} &:= \{\{1, 2d+4\}, \{2, 2d+3\}, \{3, 2d+2\}, \dots, \{d+2, d+3\}\}, \\ \mathcal{K}_2^{d+1} &:= \{\kappa : (\sigma_2^{d+1}, \kappa) \in \text{NCL}_{1,2}^1(2d+4), \kappa(\{d+2, d+3\}) = d+1\}, \\ \mathcal{L}_2^{d+1} &:= \{\kappa : (\sigma_2^{d+1}, \kappa) \in \text{NCL}_{1,2}^1(2d+4), \kappa(\{d+2, d+3\}) = d\}. \end{aligned}$$

By our assumption and by (39) we have

$$\begin{aligned} R_{2d+4}t &= \tilde{\gamma}_0 t \cdot \sum_{\sigma \in \text{NC}_{1,2}(2d+2)} \prod_{\substack{V \in \sigma \\ |V|=1 \\ V \in \text{Out}(\sigma)}} (\tilde{\beta}_1 - \beta_0) \cdot \prod_{\substack{V \in \sigma \\ |V|=1 \\ V \in \text{Inn}(\sigma)}} (\beta_1 - \beta_0) \cdot \prod_{\substack{V \in \sigma \\ |V|=2 \\ V \in \text{Out}(\sigma)}} (\tilde{\gamma}_1 - \gamma_0) \cdot \prod_{\substack{V \in \sigma \\ |V|=2 \\ V \in \text{Inn}(\sigma)}} (\gamma_1 - \gamma_0) \\ &+ \sum_{\kappa \in \mathcal{L}_2^{d+1}} \tilde{w}_t(\sigma_2^{d+1}, \kappa) + \sum_{\kappa \in \mathcal{K}_2^{d+1}} \tilde{w}_t(\sigma_2^{d+1}, \kappa). \end{aligned}$$

Now similarly as before we note from (40) that

$$\sum_{\kappa \in \mathcal{L}_1^{d+1}} \tilde{w}_t(\sigma_1^{d+1}, \kappa) = t \tilde{\gamma}_0 (\tilde{\gamma}_1 - \gamma_0) c_2,$$

and that

$$\sum_{\kappa \in \mathcal{K}_2^{d+1}} \tilde{w}_t(\sigma_2^{d+1}, \kappa) = \tilde{\gamma}_0 t (\tilde{\gamma}_{d+1}(t) - \gamma_1(t)) \tilde{\gamma}_1(t) \gamma_1(t)^{d-1} - \tilde{\gamma}_0 t \tilde{\gamma}_1(t) c_2.$$

If $\tilde{\gamma}_{d+1}(t) \neq \gamma_{d-1}(t)$ then the first summand is a polynomial of degree at least $d + 1 \geq 3$ and if $c_2 \neq 0$ then the second is a polynomial of degree 2. Therefore $c_2 = 0$ and

$$(41) \quad \tilde{\gamma}_{d+1}(t) = \gamma_d(t) = \gamma_1(t),$$

which completes the whole proof. □

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