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Positive Definite Functions on Coxeter Groups with Applications to Operator Spaces and Noncommutative Probability

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Abstract: A new class of positive definite functions related to colour-length function on arbitrary Coxeter group is introduced. Extensions of positive definite functions, called the Riesz–Coxeter product, from the Riesz product on the Rademacher (Abelian Coxeter) group to arbitrary Coxeter group is obtained. Applications to harmonic analysis, operator spaces and noncommutative probability are presented. Characterization of radial and colour-radial functions on dihedral groups and infinite permutation group are shown.

Uffe Haagerup (1949–2015) in Memoriam.

Introduction

In 1979, Uffe Haagerup in his seminal paper [HAA79] proved positive definiteness of the function $P_q(x) := q^{|x|}$, $-1 \le q \le 1$, on the free group $\mathbf{F}_N$ on N generators. Here, $|\cdot|$ denotes the natural length function on $\mathbf{F}_N$. From this he deduced Khinchine type inequalities and showed that the regular C^*-algebra of $\mathbf{F}_N$ admits bounded approximation property and the completely bounded approximation property (CBAP), see [DCH85]. These results had a significant impact on harmonic analysis on free group and also influenced free probability as well as the operator spaces theory, see [HP93].

Note that the Cayley graph of $\mathbf{F}_N$ is the homogeneous tree of order $2N$, so these results can be easily translated into the free Coxeter group $W = \mathbf{Z}/2 * \cdots * \mathbf{Z}/2$. In fact, it was shown in the paper [BJS88] that the function $P_q(x) = q^{|x|}$ is positive definite for $q \in [-1, 1]$ and for every Coxeter group, where $|\cdot|$ is now the natural word length function on a Coxeter group with respect to the set of its Coxeter generators. This implies that Coxeter groups have the Haagerup property (see [CCJ+01]). Recall, that infinite groups with Haagerup property do not have Kazhdan's property (T) (cf [BDLHV08]).

Later, Januszkiewicz [JAN02] and Fendler [FEN02B] applied Haagerup's ideas to prove that the map $W \ni w \mapsto z^{|w|}$ is a coefficient of a uniformly bounded Hilbert representation of W for all $z \in \mathbf{C}$ such that $|z| < 1$. Valette [VAL93] observed that this implies CBAP. For further extension of these Haagerup's results for a big class of groups we refer to the book [BO08].

Bożejko and Speicher [BS96] considered the free product (convolution) of classic normal distribution $N(0, 1)$. They introduced a new length function on the permutation group $\mathfrak{S}_n$ which here we will call the *colour-length* and denote $\|\cdot\|$. It is defined as follows: if $w = s_1 \dots s_k$ is a minimal representation of w as a product of generators $s_i \in S$ then we put $\|w\| = \#\{s_1, s_2, \dots, s_k\}$. In the case of $\mathfrak{S}_n$, S is the set of transpositions $(j, j+1)$, $1 \le j < n$. Moreover, they found formula for the free additive convolution power of the classical normal distribution $\mu_1 := N(0, 1)$ and the Bernoulli distribution $\mu_{-1} := (\delta_{-1} + \delta_1)/2$, namely

$$m_{2n}\left(\mu_{\pm 1}^{\boxplus q}\right) = q^n \sum_{\pi \in \mathcal{P}_2(2n)} (\pm 1)^{|\pi|} q^{-\|\pi\|},$$

$m_{2n+1}\left(\mu_{\pm 1}^{\boxplus q}\right) = 0$, for $q \in \mathbf{N}$. These results motivated us to study the colour length function $\|\cdot\|$ in more details.

For further applications, we will study generalizations of the function $x \mapsto q^{\|x\|}$ on a Coxeter group (W, S), namely Riesz–Coxeter products, which are defined by $R_{\mathbf{q}}(s) := q_s$, for $s \in S$, and

$$R_{\mathbf{q}}(xy) := R_{\mathbf{q}}(x) R_{\mathbf{q}}(y) \quad \text{whenever} \quad \|xy\| = \|x\| + \|y\|,$$

where $\mathbf{q} = (q_s)_{s \in S}$ is a system of real parameters. In particular, if $q_s = q$ for every $s \in S$ then $R_{\mathbf{q}}(x) = q^{\|x\|}$. In one of the most important results of this paper, Theorem 5.2, we provide sufficient conditions for positive definiteness of the function $R_{\mathbf{q}}$.

This implies, in particular, that in an arbitrary Coxeter group (W, S) the set of generators S is a weak Sidon set, i.e. that for every $f : S \to [0, 1]$ there exist positive definite functions $\phi_+, \phi_- : W \to \mathbf{C}$ such that $f(s) = \phi_+(s) - \phi_-(s)$ for every $s \in S$, see Theorem 7.1. These ϕ_+, ϕ_- can be chosen as $R_{\mathbf{q}_+}, R_{\mathbf{q}_-}$ for suitable parameters $\mathbf{q}_+, \mathbf{q}_-$. This result answers a question of Pisier, who was particularly interested in the infinite permutation group $\mathfrak{S}_\infty$. As further consequence we obtain an operator version of the Khinchin inequality for arbitrary Coxeter group, Theorem 7.2, which extend results of [LP86, PIS03].

Let us also mention that the colour-length function on the permutation group $\mathfrak{S}_n$ was also studied in [BBLS11]. Its extension to pairpartitions was applied in the proof that classical normal law $N(0, 1)$ is infinitely divisible under the free additive convolution $\boxplus$. We believe that positive definite functions on Coxeter groups of type B and D, especially these which are colour dependent, may have applications in the development of type B and D versions of free probability (see [BEH15, BEH17]).

The plan of the paper is as follows.

First we recall the definitions of Coxeter groups, the length and the colour-length functions (Sect. 1), and discuss some classes of positive definite functions on such groups, namely the radial, colour-radial and colour-dependent (Sect. 2).

In Sect. 3 we confine ourselves to Abelian Coxeter groups (W, S). In this case both the lengths $|\cdot|$ and $\|\cdot\|$ coincide and we prove that every positive definite radial function on W admits an integral representation. In the next section we extend this results and prove integral representation of positive definite radial functions for some class of Coxeter groups containing the infinite permutation group $\mathfrak{S}_\infty$. This result can be regarded as an analog of the classical de Finetti theorem. A noncommutative version was shown by

Köstler ans Speicher [KS09] (see also [LEH04]). We also show in Theorem 4.3, that the function $\exp(-t|w|^p)$ is positive definite for all $t \geq 0$ if and only if $p \in [0, 1]$.

In Sect. 5 we present the main properties of the colour-dependent positive definite functions on Coxeter groups, in particular we show in Corollary 5.4. that on $\mathfrak{S}_\infty$ and some other Coxeter groups, the function $w \mapsto r^{\|w\|}$ is positive definite if and only if $r \in [0, 1]$.

Section 6 provides characterization of all colour-dependent positive definite functions on the dihedral groups $\mathbf{D}_m$, $m = 1, 2, \ldots, \infty$.

In Sect. 7 we prove that the set S of Coxeter generators is a weak Sidon set in an arbitrary Coxeter group (W, S) with constant 2, and that it is also a completely bounded $\Lambda(p)$ set with constants as $C\sqrt{p}$, for $p > 2$.

In Sect. 8 we prove for arbitrary finitely generated Coxeter group an identity involving both lengths $|\cdot|$ and $\|\cdot\|$ (see Proposition 8.3). We apply it to give a proof of Corollary 7 from [BS96], (see Eq. (8.1)) where the proof, involving probabilistic considerations, was not presented in [BS96].

The paper concludes with Appendix, where we present a short proof of the equivalence of two results concerning positive definite functions on finite Coxeter groups.

1. Coxeter Groups

In this part we recall the basic facts regarding Coxeter groups and introduce notation which will be used throughout the rest of the paper. For more details we refer to [BOU68, HUM90].

A group W is called a *Coxeter group* if it admits the following presentation:

$$W = \left\langle S \,\middle|\, \left\{ (s_1 s_2)^{m(s_1,s_2)} = 1 : s_1, s_2 \in S, m(s_1, s_2) \neq \infty \right\} \right\rangle,$$

where $S \subset W$ is a set and m is a function $m : S \times S \to \{1, 2, 3, \ldots, \infty\}$ such that $m(s_1, s_2) = m(s_2, s_1)$ for all $s_1, s_2 \in S$ and $m(s_1, s_2) = 1$ if and only if $s_1 = s_2$. The pair (W, S) is called a *Coxeter system*. In particular, every generator $s \in S$ has order two and every element $w \in W$ can be represented as

$$w = s_1 s_2 \ldots s_m \tag{1.1}$$

for some $s_1, s_2, \ldots, s_m \in S$. If the sequence $(s_1, \ldots, s_m) \in S^m$ is chosen in such a way that m is minimal then we write $|w| = m$ and call it the *length* of w. In such a case the right hand side of (1.1) is called a *reduced representation* or *reduced word* of w. This is not unique in general, but the set of involved generators is unique [BOU68, Ch. IV, §1, Prop. 7], i.e. if $w = s_1 s_2 \ldots s_m = t_1 t_2 \ldots t_m$ are two reduced representations of $w \in W$ then $\{s_1, s_2, \ldots, s_m\} = \{t_1, t_2, \ldots, t_m\}$. This set $\{s_1, s_2, \ldots, s_m\} \subseteq S$ will be denoted S_w and called the *colour* of w.

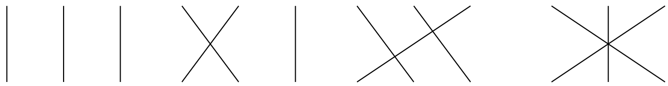
Given a subset $T \subset S$ by W_T we denote the subgroup generated by T and call it the *parabolic subgroup* associated with T. To see that S_w is independent of the reduced representation of w notice that

$$s \in S_w \iff w \notin W_{S \setminus \{s\}}. \tag{1.2}$$

We define the *colour-length* of w putting $\|w\| = \#S_w$ (the cardinality of S_w). Both lengths satisfy the triangle inequality and we have $\|w\| \leq |w|$.

In the case of the permutation group, the colour-length has the following pictorial interpretation. If σ is a permutation in $\mathfrak{S}_{n+1}$ then $\|\sigma\|$ equals n minus the number of connected components of the diagram representing σ. Notice, that $|\sigma|$ equals to the number of crossings in the diagram (the number of pairs of chords that cross).

σ	e	(12)	(12)(23)	(12)(23)(12)



$\lvert\sigma\rvert$	0	1	2	3
$\|\sigma\|$	0	1	2	2

It would be convenient to define

$$\|w\|_s := \begin{cases} 0 & \text{if } s \notin S_w, \\ 1 & \text{if } s \in S_w, \end{cases} \tag{1.3}$$

then, clearly, $\|w\| = \sum_{s \in S} \|w\|_s$.

2. Positive Defined Functions

A complex function φ on a group Γ is called *positive definite* if we have

$$\sum_{x,y \in \Gamma} \varphi(y^{-1}x)\alpha(x)\overline{\alpha(y)} \geq 0$$

for every finitely supported function $\alpha\colon \Gamma \to \mathbf{C}$.

A positive definite φ function is Hermitian and satisfies $|\varphi(x)| \leq \varphi(e)$ for all $x \in \Gamma$. Usually it is assumed, that φ is *normalised*, i.e. that $\varphi(e) = 1$.

In this and the following sections we discuss the *radial functions* on Coxeter groups. These are functions which depend on $|w|$ rather then on w.

We call a function φ on (W, S) *colour-dependent* if $\varphi(w)$ depends only on S_w. We call it *colour-radial* if it depends only on $\|w\|$.

An Abelian Coxeter group generated by S is isomorphic to the direct product $\oplus_{s\in S}\mathbf{Z}/2$. On these groups the lengths $|\cdot|$ and $\|\cdot\|$ coincide and all functions are colour dependent.

The main example of a positive definite function will be the *Riesz–Coxeter function*. Given a sequence $\mathbf{q} = (q_s)_{s\in S}$ we define $R_{\mathbf{q}}(w) = \prod_{s\in S} q_s^{\|w\|_s} = \prod_{s\in S_w} q_s$. We will abuse notation and denote by $R_{\mathbf{q}}$ also the associated operator $\sum_{w\in W} R_{\mathbf{q}}(w)w$. That is

$$R_{\mathbf{q}} = 1 + \sum_{s\in S} q_s s + \sum_{w:S_w=\{s_1,s_2\}} q_{s_1}q_{s_2} w + \sum_{w:S_w=\{s_1,s_2,s_3\}} q_{s_1}q_{s_2}q_{s_3} w + \cdots$$

In the case all $q_s = q$ we get $R_{\mathbf{q}} = \sum q^{\|w\|} w$.

This generalises the classical case of Rademacher–Walsh functions in the Rademacher group Rad_n. If we denote the generator of the i-th factor $\mathbf{Z}/2$ of the latter by the symbol r_i then, by definition, $r_1^2 = 1$, $r_i r_j = r_j r_i$ and

$$R_{\mathbf{q}} = \prod_{i=1}^{n} (1 + q_i r_i).$$

3. Rademacher Groups

In this section we are going to study positive definite radial functions on the Abelian Coxeter groups, $(W, S) = \mathrm{Rad}_S$. Since positive definiteness is tested on functions with finite support, we can assume that S is countable. If $\#S = n$ we will write Rad_n instead of Rad_S. Given $n \in \mathbf{N} \cup \{\infty\}$, we denote by P_n the class of all functions $f : \{0, 1, \ldots, n\} \to \mathbf{R}$ for n finite and $f : \mathbf{N} \to \mathbf{R}$ if $n = \infty$ such that $\varphi(w) = f(|w|)$ is a normalised positive definite function on Rad_n.

The following observation is straightforward.

Proposition 3.1. *Assume that $1 \le m < n \le \infty$ and $f \in P_n$. Then the restriction of f to $\{0, \ldots, m\}$ belongs to P_m. A function f belongs to P_∞ if and only if all its restrictions to $\{0, \ldots, m\}$ for any $m \in \mathbf{N}$ belong to P_m.*

Theorem 3.2. *Assume n is finite. The set P_n form a simplex whose vertices (extreme points) are $f_l^n(k) = \binom{n}{l}^{-1} \sum_{i=0}^{l} (-1)^i \binom{k}{i}\binom{n-k}{l-i}$, where $0 \le l \le n$. Equivalently, every normalised radial positive definite function on the group Rad_n is of the form*

$$\varphi(x) = \sum_{l=0}^{n} \lambda_l f_l^n(|x|),$$

where the sequence of nonnegative numbers $(\lambda_l)_{l=0}^n$ is unique and satisfies $\sum_{l=0}^n \lambda_l = 1$.

Proof. We can identify the dual $\widehat{\mathrm{Rad}_n}$ group of Rad_n with Rad_n via the paring $(x, y) = (-1)^{\sum_{i=1}^n x_i y_i}$. By Bochner's theorem every normalised positive definite function φ on Rad_n is of the form

$$\varphi(x) = \int_{\widehat{\mathrm{Rad}_n}} (x, y)\, \mu(dy),$$

for some probability measure μ. Clearly, such a function is radial if and only if μ is invariant under the action of $\mathfrak{S}_n$.

Among such measures extreme ones are measures μ_l for $0 \le l \le n$, where μ_l is equally distributed among elements of length l. Moreover,

$$\varphi(x) = \int_{\widehat{\mathrm{Rad}_n}} (x, y)\, \mu_l(dy) = f_l^n(|x|)$$

as claimed. □

The following theorem is a version of the classical de Finetti Theorem (see [FEL71, p. 223]) for the infinite Rademacher group.

Theorem 3.3. *Assume that φ is a radial function on the Rademacher group* Rad_∞. *Then φ is a normalised positive definite if and only if there exists a probability measure μ on* $[-1, 1]$ *such that*

$$\varphi(x) = \int_{-1}^{1} q^{|x|}\mu(dq).$$

This measure μ is unique.

Proof. Since the function $q^{|x|}$ is normalised positive definite for $q \in [-1, 1]$, the "if" implication is obvious.

Assume that φ is normalised positive definite. The group Rad_∞ is discrete and Abelian and its dual is the compact group $\widehat{\mathrm{Rad}_\infty} = \prod_{i=1}^{\infty} \mathbf{Z}/2$. By Bochner's theorem, there exists a probability measure η on $\widehat{\mathrm{Rad}_\infty}$ such that

$$\varphi(x) = \int_{\widehat{\mathrm{Rad}_\infty}} (x, y)\, d\eta(y),$$

where for $x = (x_1, x_2, \ldots) \in \mathrm{Rad}_\infty$, $y = (y_1, y_2, \ldots) \in \widehat{\mathrm{Rad}_\infty}$ we put $(x, y) = (-1)^{\sum_{i=1}^{\infty} x_i y_i}$. The function φ is radial if an only if for every permutation $\sigma \in \mathfrak{S}_\infty$ we have $\varphi(x) = \varphi(\sigma(x))$, where $\sigma(x) = (x_{\sigma(1)}, x_{\sigma(2)}, \ldots)$. This, in turn, implies that η is σ-invariant for every $\sigma \in \mathfrak{S}_\infty$, i.e. we have $\eta(A) = \eta(\sigma(A))$ for every Borel subset $A \subset \widehat{\mathrm{Rad}_\infty}$.

For a sequence $\epsilon = (\epsilon_i)_{i=1}^{n} \in \{0, 1\}^n$ we define $C_n(\epsilon) \subseteq \widehat{\mathrm{Rad}_\infty}$ by

$$C_n(\epsilon) = \{y \in \widehat{\mathrm{Rad}_\infty} | y_i = \epsilon_i : 1 \le i \le n\},$$

in particular $C_0(\varnothing) = \widehat{\mathrm{Rad}_\infty}$. Then we have $\eta(C_n(\epsilon)) = \eta(C_n(\epsilon'))$ if $\epsilon'_i = \epsilon_{\sigma(i)}$ for some $\sigma \in \mathfrak{S}_n$ and every $1 \le i \le n$. For $\varepsilon \in \mathbf{R}$ we put

$$\varepsilon^n = \underbrace{\varepsilon, \varepsilon, \ldots, \varepsilon}_{n}$$

and $a_n = \eta(C_n(1^n))$. Moreover, for $n, k \ge 0$ we define the difference operators $\Delta^k a_n$ by induction: $\Delta^0 a_n = a_n$ and $\Delta^{k+1} a_n = \Delta^k a_{n+1} - \Delta^k a_n$. We claim that

$$(-1)^k \Delta^k a_n = \eta\left(C_{n+k}\left(1^n 0^k\right)\right). \tag{3.1}$$

Denoting the right hand side of (3.1) by $c(n, k)$ we note that $c(n, 0) = a_n$ and

$$C_{n+k+1}\left(1^n 0^k 0\right) \cup C_{n+k+1}\left(1^n 0^k 1\right) = C_{n+k}\left(1^n 0^k\right),$$

is a disjoint union. This implies

$$c(n, k+1) = c(n, k) - c(n+1, k).$$

This formula, by induction on k, leads to (3.1).

From (3.1) we see that the sequence (a_n) is *completely monotone*, i.e. that $(-1)^k \Delta^k a_n \geq 0$ for all $n, k \geq 0$. By the celebrated theorem of Hausdorff (see [HAU21, Sätze II and III]), there exists a unique probability measure ρ on $[0, 1]$ such that

$$(-1)^k \Delta^k a_n = \int_0^1 u^n (1-u)^k \, d\rho(u). \tag{3.2}$$

(Note that Eq. (3.2) for arbitrary $k \geq 0$ follows from the case $k = 0$.)

For $x = (1^n 0^\infty) \in \mathrm{Rad}_\infty$ so that $|x| = n$, we have

$$\begin{aligned}
\varphi(x) &= \int_{\widehat{\mathrm{Rad}_\infty}} (x, y)\, d\eta(y) = \int_{\widehat{\mathrm{Rad}_\infty}} (-1)^{\sum_{i=1}^n y_i}\, d\eta(y) \\
&= \sum_{\epsilon \in \{0,1\}^n} (-1)^{\sum_{i=1}^n \epsilon_i} \eta(C_n(\epsilon)) = \sum_{k=0}^n \binom{n}{k} (-1)^k \eta\left(C_n\left(1^k 0^{n-k}\right)\right) \\
&= \sum_{k=0}^n \binom{n}{k} (-1)^k \int_0^1 u^k (1-u)^{n-k}\, d\rho(u) = \int_0^1 (1-2u)^n\, d\rho(u) \\
&= \int_{-1}^1 q^n\, d\mu(q),
\end{aligned}$$

where μ is defined by $\mu(A) = \rho\left(\frac{1}{2} + \frac{1}{2}A\right)$ for a Borel set $A \subseteq [-1, 1]$. $\square$

4. Remarks on Radial Positive Definite Functions on Some Infinitely Generated Coxeter Groups

In this section we extend the last theorem of the previous section to a certain class of Coxeter groups.

Theorem 4.1. *Assume that (W, S) is a Coxeter system and that there is an infinite subset $S_0 \subseteq S$ such that $st = ts$ for $s, t \in S_0$. Assume that φ is a radial function on W with $\varphi(e) = 1$. Then φ is positive definite if and only if there exists a probability measure μ on $[-1, 1]$ such that*

$$\varphi(\sigma) = \int_{-1}^1 q^{|\sigma|} \mu(dq).$$

This measure μ is unique.

Proof. It is sufficient to note that the group generated by S_0 is a parabolic subgroup isomorphic with Rad_∞. $\square$

Example. For $W = \mathfrak{S}_\infty$ we have $S = \{(n, n+1) : n \in \mathbf{N}\}$. Then we can take $S_0 = \{(2n-1, 2n) : n \in \mathbf{N}\}$. Similar S_0 can be found in infinitely generated groups of type B and D.

Problem 4.2. When $-1 \leq q \leq 1, q \neq 0$ is the positive definite function $q^{|x|}$ on $\mathfrak{S}_\infty$ an extreme point in the set of normalised positive definite functions?

Theorem 4.3. *The function $\varphi_p(\sigma) = e^{-t|\sigma|^p}$ is positive definite on $\mathfrak{S}_\infty$ if and only if $0 \leq p \leq 1$.*

Proof. By contraposition, assume that for some $p > 1$ and $t_0 > 0$ the function $\psi_p(\sigma) = e^{-t_0|\sigma|^p}$ is positive definite on $\mathfrak{S}_\infty$.

For $q_0 = e^{-t_0}$, choosing σ such that $|\sigma| = 2n$ we have $q_0^{(2n)^p} = \int_{-1}^1 q^{2n}\, d\mu_0(q)$ for some probability measure μ_0 on $[-1, 1]$. Since $\left(\int_{-1}^1 q^{2n}\, d\mu_0(q)\right)^{1/n}$ tends to $\max\{q^2 | q \in \operatorname{supp} \mu_0\}$ while $\left(q_0^{(2n)^p}\right)^{1/n} \to 0$, we conclude that μ_0 is the Dirac measure at 0, which is a contradiction.

The "if" part is standard. We need to show that $f(x) = e^{-tx^p}$ is the Laplace transform of some probability measure supported on $[0, \infty)$, so f is a convex combination of functions of the form e^{-sx}.

By characterisation of Laplace transforms (see [HAU21, Satz III]) this is equivalent to *complete monotonicity*, that is $(-1)^n f^{(n)} > 0$ for all $n = 0, 1, \dots$. And indeed, by induction, $(-1)^n f^{(n)}$ is a positive linear combination of positive functions of the form $x^{pj-n} f(x)$ for $1 \le j \le n$. □

The measures with Laplace transforms e^{-tx^p} for $t \ge 0$ and $0 \le p \le 1$ are studied in detail in [YOS80, Ch. IX.11] (see Propositions 1 and 2 there).

Let us note that for such groups like $\mathbf{Z}^k$ or $\mathbf{R}^k$ with the Euclidean distance d the functions $\exp(-td^p)$ are positive definite for all $t \ge 0$ and $0 \le p \le 2$ (the case $p = 2$ corresponds to the Gaußian Law).

5. Colour-Dependent Positive Definite Functions on Coxeter Groups

The question which colour-dependent or colour-radial functions are positive functions on Coxeter groups is wide open. In this section we provide some sufficient conditions. In the next section we will examine the dihedral groups in full details.

Lemma 5.1. *Let H be a subgroup of a group Γ of index d. Then the function φ_r defined by $\varphi_r(x) = 1$ if $x \in H$ and $\varphi_r(x) = r$ otherwise is positive definite on Γ if and only if $r \in [-1/(d-1), 1]$, with natural convention that if $d = \infty$ then $-1/(d-1) = 0$.*

Note, that if $H = \{1\}$ then $d = |\Gamma|$.

Proof. First assume that d is finite and let us enumerate the left cosets:

$$\{gH : g \in \Gamma\} = \{H_1, H_2, \dots, H_d\}.$$

Note, that for $x \in H_i$, $y \in H_j$ we have $y^{-1}x \in H$ if and only if $i = j$. Therefore, for $r_0 = -1/(d-1)$ and for a finitely supported complex function f on Γ we have

$$\sum_{x,y\in\Gamma} \varphi_{r_0}(y^{-1}x) f(x)\overline{f(y)} = \frac{1}{d-1} \sum_{1\le i<j\le d} \left| \sum_{x\in H_i} f(x) - \sum_{y\in H_j} f(y) \right|^2,$$

which proves that φ_{r_0} is positive definite. For $r \in [-1/(d-1), 1]$ the function φ_r is positive definite as a convex combination of φ_{r_0} and the constant function φ_1.

On the other hand, if we choose $x_i \in H_i$ for each $i \le d$ and define f as the characteristic function of the set $\{x_1, \dots, x_d\}$ then

$$\sum_{x,y\in\Gamma} \varphi_r(y^{-1}x) f(x)\overline{f(y)} = d + (d^2 - d)r, \tag{5.1}$$

which proves that $r \geq -1/(d-1)$ is a necessary condition for positive definiteness of φ_r.

If $d = \infty$ then $r_0 = 0$ and the function φ_0 is positive definite as the characteristic function of the subgroup H. For "only if" part we chose an arbitrarily long sequence $x_1, \ldots, x_{d'}$ of elements from different left cosets and use (5.1) with d' instead of d. □

Theorem 5.2. *Assume that for every $s \in S$ we are given a number q_s,*

$$\frac{-1}{d_s - 1} \leq q_s \leq 1,$$

where d_s denotes the index if the parabolic subgroup generated by $S \setminus \{s\}$ in W: $d_s = [W : W_{S \setminus \{s\}}]$. Then the Riesz–Coxeter $R_{\mathbf{q}}$ is positive definite on W.

Proof. From Lemma 5.1 the function $w \mapsto q_s^{\|w\|_s}$ is positive definite for $s \in S$ and $-1/(d_s - 1) \leq q_s \leq 1$. Since the pointwise product of positive definite functions is positive definite, the statement holds. □

Example. Take $W = \mathfrak{S}_n$, the permutation group on the set $\{1, 2, \ldots, n\}$. It is generated by the transpositions $S = \{s_i = (i, i+1), 1 \leq i \leq n-1\}$. For $1 \leq i \leq n-1$ the parabolic subgroup generated by $S \setminus \{s_i\}$ is isomorphic with $\mathfrak{S}_{i-1} \times \mathfrak{S}_{n-i-1}$, so its index is $i\binom{n}{i}$.

It would be interesting to determine for which r the function $w \mapsto r^{\|w\|}$ is positive definite. By Proposition 5.2 this holds for $r \in [-1/(d-1), 1]$, where d is the maximal index of the parabolic subgroups of the form $W_{S \setminus \{s\}}$. We note a necessary condition.

Proposition 5.3. *Assume that we have distinct generators $s_0, s_1, \ldots, s_n \in S$ such that $s_0 s_k \neq s_k s_0$ (i.e. $m(s_0, s_k) > 2$) for $1 \leq k \leq n$. If the function $w \mapsto r^{\|w\|}$ is positive definite on W, then ${}^{-1}\!/\!_{(n-1)} \leq r^3 \leq 1$.*

If there is an element $s_0 \in S$ for which there are infinitely many $s \in S$ such that $s_0 s \neq s s_0$ then $r^{\|w\|}$ is positive definite on W if and only if $0 \leq r \leq 1$.

Proof. Consider elements $w_k = s_0 s_k s_0$. Note, that for $k \neq l$ we have $\|w_l^{-1} w_k\| = 3$. If φ_r is positive definite on W then we have

$$0 \leq \sum_{k,l=1}^{n} \varphi_r(x_l^{-1} x_k) = n + (n^2 - n) r^3,$$

which implies $r^3 \geq {}^{-1}\!/\!_{(n-1)}$. □

Corollary 5.4. *The function $w \mapsto q^{\|w\|}$ on $\mathfrak{S}_\infty$ is positive definite if and only if $0 \leq q \leq 1$.*

Problem 5.5. Thus, it is valid to ask the following. Is it true that *every* normalised positive definite colour-length-radial function $\phi \colon \mathfrak{S}_\infty \to \mathbf{R}$ is of the form $\phi(\sigma) = \int_0^1 q^{\|\sigma\|}\, d\mu(q)$ for some probability measure μ on $[0, 1]$?

6. Dihedral Groups

In this part will examine the class of colour-dependent positive definite functions on the case the simplest noncommutative Coxeter groups. Assume that $W = \mathbf{D}_{2n} = \langle s, t | (st)^n \rangle$ (i.e. the group of symmetries of a regular n-gon), and define a colour-dependent function on W:

$$\phi(w) = \begin{cases} 1 & \text{if } w = e, \\ p & \text{if } w = s, \\ q & \text{if } w = t, \\ r & \text{otherwise.} \end{cases} \tag{6.1}$$

If $p = q$ then ϕ is colour radial. We are going to determine for which parameters p, q, r the function ϕ is positive definite on W. It is easy to observe necessary conditions: $p, q, r \in [-1, 1]$. Moreover, since $\langle st \rangle$ is a cyclic subgroup of order n, Lemma 5.1, implies a necessary condition: $-1/(n-1) \leq r \leq 1$.

Finite dihedral groups. Assume that W is a finite dihedral group, $W = \mathbf{D}_{2n}$, so that $(st)^n = 1$. We will use the following version of Bochner's theorem: A function f on a compact group G is positive definite if and only if its *Fourier transform*:

$$\widehat{f}(\pi) = \int_G f(x)\pi(x^{-1})dx$$

is a positive operator for every $\pi \in \widehat{G}$, where $\widehat{G}$ denotes the dual object of G, i.e. the family of all equivalency classes of unitary irreducible representations of G, see [SIM96]. Then we have

$$f(x) = \sum_{\pi \in \widehat{G}} d_\pi \operatorname{tr}\left[\widehat{f}(\pi)\pi(x)\right].$$

Therefore, for every irreducible representation π of $\mathbf{D}_{2n}$ we are going to find

$$\widehat{\phi}(\pi) = \frac{1}{2n} \sum_{g \in G} \phi(g)\pi(g^{-1}).$$

We will identify s with $(0, -1)$ and t with $(1, -1)$. If n is odd then $\mathbf{D}_{2n}$ possesses two characters: $\chi_{+,+}$ such that $\chi_{+,+}(w) = 1$ for every $w \in \mathbf{D}_{2n}$ and $\chi_{-,-}$ such that $\chi_{-,-}(s) = \chi_{-,-}(t) = -1$. If n is even then we have two additional characters $\chi_{+,-}$ and $\chi_{-,+}$ such that $\chi_{+,-}(s) = \chi_{-,+}(t) = 1$ and $\chi_{+,-}(t) = \chi_{-,+}(s) = -1$. It is easy to check that

$$2n\widehat{\phi}(\chi_{+,+}) = 1 + p + q + (2n-3)r,$$
$$2n\widehat{\phi}(\chi_{-,-}) = 1 - p - q + r,$$

which gives

$$-1 - (2n-3)r \leq p + q \leq 1 + r$$

and, for n even,

$$2n\widehat{\phi}(\chi_{+,-}) = 1 + p - q - r,$$
$$2n\widehat{\phi}(\chi_{-,+}) = 1 - p + q - r,$$

which implies

$$|p - q| \leq 1 - r.$$

We have also the family of two dimensional representations U_a:

$$U_a(k, 1) = \begin{pmatrix} e^{2\pi i k a/n} & 0 \\ 0 & e^{-2\pi i k a/n} \end{pmatrix},$$
$$U_a(k, -1) = \begin{pmatrix} 0 & e^{2\pi i k a/n} \\ e^{-2\pi i k a/n} & 0 \end{pmatrix},$$

where $a = 1, 2, \ldots, \lfloor \frac{n-1}{2} \rfloor$. Then for the function given by (6.1) we have

$$\begin{aligned} 2n\widehat{\phi}(U_a) &= (1 - r)\mathrm{Id} + (p - r)U_a(0, -1) + (q - r)U_a(1, -1) \\ &= \begin{pmatrix} 1 - r & p - r + (q - r)e^{2\pi i a/n} \\ p - r + (q - r)e^{-2\pi i a/n} & 1 - r \end{pmatrix}. \end{aligned}$$

This matrix is positive definite if and only if $r \leq 1$ and

$$\left| p - r + (q - r)e^{2\pi i a/n} \right| \leq 1 - r.$$

Therefore we have

Proposition 6.1. *The function ϕ given by (6.1) is positive definite on $\mathbf{D}_{2n}$ if and only if*

$$1 + p + q + (2n - 3)r \geq 0, \qquad 1 - p - q + r \geq 0$$

(plus

$$1 + p - q - r \geq 0, \qquad 1 - p + q - r \geq 0$$

whenever n is even) and

$$\left| p - r + (q - r)e^{2\pi i a/n} \right| \leq 1 - r.$$

for $a = 1, 2, \ldots, \lfloor \frac{n-1}{2} \rfloor$.

Let us confine ourselves to colour-radial functions.

Corollary 6.2. *Assuming that $p = q$, the function ϕ defined by (6.1) is positive definite on $W = \mathbf{D}_{2n}$ if and only if*

$$\max\left\{ \frac{-2p - 1}{2n - 3}, 2p - 1 \right\} \leq r \leq \frac{1 + 2p\cos(\pi/n)}{1 + 2\cos(\pi/n)},$$

i.e. if and only if the point (p, r) belongs to the triangle whose vertices are

$$\left(\frac{1 - n - \cos(\pi/n)}{1 + (2n - 1)\cos(\pi/n)}, \frac{1 - \cos(\pi/n)}{1 + (2n - 1)\cos(\pi/n)} \right), \quad \left(\frac{n - 2}{2n - 2}, \frac{-1}{n - 1} \right), \quad (1, 1).$$

Proof. For $p = q$ the conditions from Proposition 6.1 reduce to

$$2p - 1 \leq r, \quad -1 - 2p \leq (2n-3)r, \quad \text{and} \quad 2\cos(\pi/n)|p - r| \leq 1 - r.$$

It is sufficient to note that $2p - 1 \leq r$ implies $2\cos(\pi/n)(p - r) \leq 1 - r$ for $p \leq 1$. □

Example. For $\mathbf{D}_4$ we have the positive definiteness of ϕ is equivalent to

$$-1 + |p + q| \leq r \leq 1 - |p - q|,$$

which means that the set of all possible (p, q, r) forms a tetrahedron with vertices $(-1, 1, -1)$, $(1, -1, -1)$, $(-1, -1, 1)$, $(1, 1, 1)$. For $p = q$ the condition reduces to $2|p| - 1 \leq r \leq 1$.

In the case of $\mathbf{D}_6$ Proposition 6.1 leads to the following conditions:

$$1 - p - q + r \geq 0, \qquad 1 + p + q + 3r \geq 0,$$
$$1 - r \geq \sqrt{p^2 + q^2 + r^2 - pq - pr - qr},$$

which can be expressed as

$$\max\left\{\frac{-1 - p - q}{3}, p + q - 1\right\} \leq r \leq \frac{1 - p^2 - q^2 + pq}{2 - p - q}.$$

The infinite dihedral group. Here we are going to study $W = \mathbf{D}_\infty$.

Proposition 6.3. *The function ϕ given by (6.1) is positive definite on $W = \mathbf{D}_\infty$ if and only if $0 \leq r$ and $|p - r| + |q - r| \leq 1 - r$, i.e.*

$$\max\{0, p + q - 1\} \leq r \leq \min\left\{1 - |p - q|, \frac{1 + p + q}{3}\right\}. \tag{6.2}$$

Proof. First we note that the set of $(p, q, r) \in \mathbf{R}^3$ satisfying (6.2) constitutes a pyramid which is the convex hull of the points $(\pm 1, 0, 0)$, $(0, \pm 1, 0)$ and $(1, 1, 1)$ (apex). For these particular parameters it is easy to see that ϕ is positive definite: $(1, 1, 1)$ corresponds to the constant function 1, $(1, 0, 0)$ to the characteristic function of the subgroup $\langle s \rangle = \{1, s\}$, and $(-1, 0, 0)$ to the character $\chi_{-,-}$ times the characteristic function of $\langle s \rangle$. Similarly for $(0, \pm 1, 0)$. This, by convexity, proves that (6.2) is a sufficient condition.

On the other hand, we know already that $r \geq 0$ is a necessary condition. Let us fix n and define $W^+(n) = \{x \in W : |sx| < |x| \leq 2n\}$, $W^-(n) = \{x \in W : |tx| < |x| \leq 2n\}$ and

$$f(x) = \begin{cases} \pm 1 & \text{if } x \in W^{\pm}(n), \\ 0 & \text{otherwise.} \end{cases}$$

For $x, y \in W^+(n)$ we have $S_{y^{-1}x} = \varnothing$ in $2n$ cases (namely, if $x = y$) $S_{y^{-1}x} = \{s\}$ in $2n - 2$ cases (namely if $|x| = 2k$, $|y| = 2k + 1$ or vice-versa, $k = 1, \ldots, n - 1$) $S_{y^{-1}x} = \{t\}$ in $2n$ cases (namely if $|x| = 2k$, $|y| = 2k - 1$ or vice-versa, $k = 1, \ldots, n$) and $S_{y^{-1}x} = \{s, t\}$ in all the other $(2n - 1)(2n - 2)$ cases. Similarly, for $x, y \in W^-(n)$ we have $S_{y^{-1}x} = \emptyset$ in $2n$ cases, $S_{y^{-1}x} = \{s\}$ in $2n$ cases, $S_{y^{-1}x} = \{t\}$ in $2n - 2$ cases and $S_{y^{-1}x} = \{s, t\}$ in $(2n-1)(2n-2)$ cases. If $x \in W^+(n)$, $y \in W^-(n)$ or vice-versa then $S_{y^{-1}x} = \{s, t\}$. Summing up, we get

$$\begin{aligned}\sum_{x,y\in W} &\phi(y^{-1}x) f_n(x) f_n(y)\\ &= 4n + (4n-2)p + (4n-2)q + (4n-2)(2n-2)r - 8n^2 r\\ &= 4n + (4n-2)p + (4n-2)q - (12n-4)r.\end{aligned}$$

Therefore for every $n \in \mathbf{N}$ we have a necessary condition

$$1 + \left(1 - \frac{1}{2n}\right) p + \left(1 - \frac{1}{2n}\right) q - \left(3 - \frac{1}{n}\right) r \geq 0.$$

Letting $n \to \infty$ we get $1 + p + q \geq 3r$.

Put $x_k = stst\ldots$, $|x_k| = k$. Fix n and define

$$g(x) = \begin{cases} \chi_{-,+}(x) & \text{if } x = x_k \text{ for } 1 \leq k \leq 4n,\\ 0 & \text{otherwise,}\end{cases}$$

where, as before, $\chi_{-,+}$ is the character on W for which $\chi_{-,+}(s) = -1$, $\chi_{-,+}(t) = 1$. Then

$$\sum_{x,y\in W} \phi(y^{-1}x) g(x) g(y) = \sum_{k,l=1}^{4n} \phi(x_l^{-1} x_k) g(x_k) g(x_l).$$

Denote $c_{k,l} = \phi(x_l^{-1}x_k)g(x_k)g(x_l)$. Then we have $c_{k,k} = 1$, $1 \leq k \leq 4n$, $c_{k,k-1} = q$ if k is even, $c_{k,k-1} = -p$ if k is odd, $2 \leq k \leq 4n$ and $c_{k,l} = c_{l,k}$ for all $1 \leq k, l \leq 4n$. If $1 \leq k, l \leq 4n$ and $|k-l| \geq 2$ then $c_{k,l} = (-1)^j r$, where j is the total number of s appearing in x_k and x_l. Now it is not difficult to check that

$$\sum_{l=1}^{4n} c_{k,l} = \begin{cases} 1 + q - 2r & \text{if } k = 1 \text{ or } k = 4n,\\ 1 - p + q - r & \text{if } 1 < k < 4n,\end{cases}$$

which implies

$$\sum_{x,y\in W} \phi(y^{-1}x) g(x) g(y) = 4n - (4n-2)p + 4nq - (4n+2)r$$

and leads to necessary condition $r \leq 1 - p + q$. In a similar manner we get $r \leq 1 + p - q$.

Finally, define a function h similarly like g, but now we use the character $\chi_{-,-}$:

$$h(x) = \begin{cases} \chi_{-,-}(x) = (-1)^k & \text{if } x = x_k \text{ for } 1 \leq k \leq 4n,\\ 0 & \text{otherwise.}\end{cases}$$

Putting $d_{k,l} = \phi(x_l^{-1}x_k)h(x_k)h(x_l)$ we have $d_{k,k} = 1$, $d_{k,k-1} = -p$ if $2 \leq k \leq 4n$ is even and $d_{k,k-1} = -q$ if k is odd. Moreover, if $|k-l| \geq 2$, $1 \leq k, l \leq 4n$ then $d_{k,l} = (-1)^{k+l} r$. Now one can check that

$$\sum_{l=1}^{4n} d_{k,l} = \begin{cases} 1 - q & \text{if } k = 1 \text{ or } k = 4n,\\ 1 - p - q + r & \text{if } 1 < k < 4n,\end{cases}$$

which yields $1 - p - q + r \geq 0$ and completes the proof that the conditions (6.2) are necessary. □

7. Weak Sidon Sets and Operator Khinchine Inequality

The aim of this section is to show that the set of Coxeter generators S in an arbitrary Coxeter group W is a *weak Sidon set*, ie. an interpolation set for the Fourier–Stieltjes algebra $B(W)$.

Given a group Γ, the Fourier–Stieltjes algebra consists of linear combinations of positive definite functions on Γ, ie. every element of $B(\Gamma)$ is of the form $f = \varphi_1 - \varphi_2 + i(\varphi_3 - \varphi_4)$ for some positive definite functions φ_i $(1 \leq i \leq 4)$ on Γ. The norm on $B(\Gamma)$ is defined as

$$\|f\|_{B(\Gamma)} = \inf\left\{\sum \varphi_i(e) | \text{where } f \text{ decomposes as above}\right\}$$

Theorem 7.1. *The set of Coxeter generators S in an arbitrary Coxeter group W is a weak Sidon set, ie. for every bounded function $f : S \to [-1, 1]$ there exists positive definite functions $\varphi_\pm$, such that $f(s) = \varphi_+(s) - \varphi_-(s)$ for any $s \in S$. One can take $\varphi_\pm = R_{\mathbf{q}^\pm}$ for a suitable choice of $\mathbf{q}^\pm$. Moreover*

$$\|\varphi_+ - \varphi_-\|_{B(W)} \leq 2$$

Proof. Put $S_\pm(f) = \{s \in S | \pm f(s) > 0\}$. Set

$$q_s^\pm = \begin{cases} \pm f(s) & \text{for } s \in S_\pm(f), \\ 0 & \text{otherwise.} \end{cases}$$

Then $f(s) = R_{\mathbf{q}^+}(s) - R_{\mathbf{q}^-}(s)$ as claimed. The rest of the statement hold as the Riesz–Coxeter function at the identity element equals to one. $\square$

Given a matrix $A \in M_n(\mathbf{C})$ and $p \geq 1$ the Schatten p-class norm $\|A\|_{\mathcal{S}_p}$ is defined as $\|A\|_{\mathcal{S}_p}^p = (\mathrm{tr}|A|^p)^{1/p}$, where $|A| = (A^*A)^{1/2}$.

Let λ denote the left regular representation of a group Γ. Given a finite sum $f = \sum c_g \lambda(g) \in \mathbf{C}[\Gamma]$ we define noncommutative L^p-norm

$$\|f\|_{L^p(\Gamma)}^p = \left(\tau\left((f^* * f)^{p/2}\right)\right)^{1/p}$$

where $\tau(f) = c_e$ is the von Neumann trace and $L^p(\Gamma)$ is a completion of $\mathbf{C}[\Gamma]$ with respect to the above norm.

We recall, that a scalar-valued map φ on a group Γ is called a *completely bounded Fourier multiplier* on $L^p(\Gamma)$ if the associated operator

$$M_\varphi(\lambda(g)) = \varphi(g)\lambda(g), \qquad g \in \Gamma$$

extends to a completely bounded operator on $L^p(\Gamma)$.

We let $M_{\mathrm{cb}}(L^p(\Gamma))$ to be an algebra of completely bounded Fourier multipliers equipped with the norm

$$\|\varphi\|_{M_{\mathrm{cb}}(L^p(\Gamma))} = \|M_\varphi \otimes \mathrm{id}_{\mathcal{S}^p}\|.$$

Following Pisier [PIS03], for $a_s \in M_n(\mathbf{C})$, where $s \in S$, we define

$$\|(a_s)_{s\in S}\|_{R\cap C} = \max\left\{\left\|\left(\sum_{s\in S} a_s a_s^*\right)^{1/2}\right\|_{\mathcal{S}^p}, \left\|\left(\sum_{s\in S} a_s^* a_s\right)^{1/2}\right\|_{\mathcal{S}^p}\right\}.$$

For a set $E \subset \Gamma$ we define the *completely bounded* constant $\Lambda_p^{\mathrm{cb}}(E)$ as infimum of C such that

$$\left\| \sum_{s \in S} a_s \otimes \lambda(s) \right\|_{L^p(W)} \leq C \, \|(a_s)_{s \in S}\|_{R \cap C}$$

for all matrices $a_s \in M_n(\mathbf{C})$ and $n \in \mathbf{N}$.

Theorem 7.2. *If $a_s \in M_n(\mathbf{C})$, then for all $p \geq 2$ and any Coxeter system (W, S) we have*

$$\|(a_s)_{s \in S}\|_{R \cap C} \leq \left\| \sum_{s \in S} a_s \otimes \lambda(s) \right\|_{L^p(W)} \leq 2A' \sqrt{p} \, \|(a_s)_{s \in S}\|_{R \cap C} .$$

Proof. It was shown by Harcharras [LP86, Prop. 1.8] that $\Lambda_p^{\mathrm{cb}}(E)$ is finite if and only if E is an interpolation set for $M_{\mathrm{cb}}(L^p(\Gamma))$, i.e. every bounded function on E can be extended to a multiplier, and

$$\Lambda_p^{\mathrm{cb}}(E) \leq \Lambda_p^{\mathrm{cb}}(R) \mu_p^{\mathrm{cb}}(E),$$

where R is the generating set in the Rademacher group Rad_∞ and $\mu_p^{\mathrm{cb}}(E)$ is the interpolation constant.

As shown by Buchholz [BUC05, Thm. 5] for $p = 2n$, and R the standard generating set in Rad_∞, $\Lambda_{2n}^{\mathrm{cb}}(R) = ((2n-1)!!)^{1/2n} \leq A\sqrt{p}$ for some absolute A. This was extended by Pisier [PIS03, Thm. 9.8.2] for any $p \geq 2$, i.e

$$\Lambda_p^{\mathrm{cb}}(R) \leq A' \sqrt{p},$$

for an absolute constant A'.

We have shown in Theorem 7.1 that in an arbitrary Coxeter group W its Coxeter generating set S is a weak Sidon set, i.e. it is interpolation set for the Fourier–Stieltjes algebra $B(W)$. Since for $p \geq 1$, $B(\Gamma)$ is a subalgebra of $M_{\mathrm{cb}}(L^p(\Gamma))$ and

$$\|\varphi\|_{M_{\mathrm{cb}}(L^p(\Gamma))} \leq \|\varphi\|_{B(\Gamma)},$$

we see that $\mu_p^{\mathrm{cb}}(S) \leq 2$. Thus $\Lambda_p^{\mathrm{cb}}(S) \leq 2A'\sqrt{p}$. This finishes the proof of the right inequality.

The left inequality holds for any group Γ and any $S \subset \Gamma$ (see [LP86]). □

Remark 7.3. Fendler [FEN02A] has shown that if for all $s, t \in S$, $s \neq t$, we have $m(s, t) \geq 3$, then

$$\Lambda_p^{\mathrm{cb}}(S) \leq 2\sqrt{2}.$$

See also [BOZ75] and [BUC99] for related results in the case of free Coxeter groups. Also Haagerup and Pisier have shown that $\Lambda_\infty^{\mathrm{cb}}(S) = 2$, where $\Lambda_\infty^{\mathrm{cb}}(E) = \sup_{p \geq 2} \Lambda_p^{\mathrm{cb}}(E)$ [HP93]. See the paper of Haagerup [HAA81] where the best constant was calculated for the set of Coxeter generators of the Rademacher group in case when a_s are scalars.

Remark 7.4. Theorem 7.2 implies that in an arbitrary Coxeter group the set of its Coxeter generators is a weak Sidon set and also it is completely bounded Λ_p^{cb}-set, see Theorems 7.1 and 7.2. Equivalently, the span of the linear operators $\{\lambda(s) | s \in S\}$ in the noncommutative L^p-space $L^p(W)$, for $p > 2$, is completely boundedly isomorphic to row and column operator Hilbert space.

8. Chromatic Length Function for Coxeter Groups and Pairpartitions

Let $[2n] = \{1, \ldots, 2n\}$. Let $2^{[2n]}$ denote the set of subsets of $[2n]$. By a partition of $[2n]$ we mean $\pi \subset 2^{[2n]}$ such that $\bigcup \pi = [2n]$ and if $\pi', \pi'' \in \pi$ then $\pi' = \pi''$ or $\pi' \cap \pi'' = \varnothing$. We say, that partition ϱ is a *coarsening* of a partition π if for any $\pi' \in \pi$ there exists $\varrho' \in \varrho$ such that $\pi' \subset \varrho'$.

A partition is called *crossing* if there exist $1 \le a < b < c < d \le 2n$ and $\pi_1, \pi_2 \in \pi$ with $a, c \in \pi_1 \neq \pi_2 \ni b, d$; otherwise it is called *noncrossing*. For any partition π there exists the smallest noncrossing coarsening $\Phi(\pi)$ of π (ie. if ϱ is a noncrosing coarsening of π then it is a coarsening of $\Phi(\pi)$). We define $\|\pi\| = n - \#\Phi(\pi)$. The notion for the map Φ was introduced in [BY06].

We say that π is a *pairpartition* if every member of π has cardinality two. The set of pairpartitions of $[2n]$ is denoted by $\mathcal{P}_2(2n)$. Given $\pi \in \mathcal{P}_2(2n)$ we write $|\pi|$ to denote the number of ordered quadruples $1 \le a < b < c < d \le 2n$ such that both $\{a, c\}$ and $\{b, d\}$ belong to π. Note, that $|\pi| = 0$ precisely when π is noncrossing. The set of noncrossing pairpartitions is denoted $\mathcal{NC}_2(2n)$.

Given a noncrossing pairpartition ϖ we call $\{b, c\} \in \varpi$ *an inner block* if there exists $\{a, d\} \in \varpi$ with $a < b < c < d$. The number of inner blocks of ϖ we denote as $\mathrm{inn}(\varpi)$.

The aim of this section is to prove the formula by Bożejko and Speicher, which has been presented in [BS96, Cor. 7] without a proof.

$$\sum_{\pi \in \mathcal{P}_2(2n)} (-1)^{|\pi|} q^{\|\pi\|} = \sum_{\varpi \in \mathcal{NC}_2(2n)} (1 - q)^{\mathrm{inn}(\varpi)}. \tag{8.1}$$

Remark 8.1. Let $f_n(q) = \sum_{\varpi \in \mathcal{NC}_2(2n)} (1 - q)^{\mathrm{inn}(\varpi)}$. It is elementary to derive

$$f_n(q) = C_n \; {}_2F_1 \left(\begin{matrix} n, 1-n \\ n+2 \end{matrix} \middle| \, q \right),$$

where $C_n = \binom{2n}{n} - \binom{2n}{n-1} = \#\mathcal{NC}_2(2n)$ denote the n-th Catalan number and ${}_2F_1$ is the classical hypergeometric function. If we write $f(q) = \sum_{j=0}^{n-1} t_j^n q^j$, then the triangle $(t_j^n)_{0 \le j < n}$ appears in [SLO01] as "sequence" A062991. Since we are not going to use this formula, we leave it as an exercise to the reader. For the expansion of $f_n(1 - t)$ and the Delanoy triangle appearing there the reader may consult [BW01, Prop 6.1].

In order to prove Eq. (8.1) we define the *Wick map* $\mathcal{P}_2(2n) \ni \pi \mapsto {:}\pi{:} \in \mathcal{NC}_2(2n)$ (related to the normal order in quantum field theory). Given a pairpartition π we define ${:}\pi{:}$ by repetitive resolving crossings. That is, we replace repetitively every crossing pair $\{a, c\}$ and $\{b, d\}$ with $a < b < c < d$ by $\{a, d\}$ and $\{b, c\}$. In order to see that the result is independent of the order of resolution we describe ${:}\pi{:}$ in an equivalent way.

Let $\Phi(\pi)$ be the smallest noncrossing coarsening of π. For each block β of $\Phi(\pi)$ define $\beta^+ = \{y | (\exists x)\ x \in \beta,\ y > x,\ \{x, y\} \in \pi\}$ and $\beta_1^- = \{x | (\exists y)\ y \in \beta,\ y > x,\ \{x, y\} \in \pi\}$. Order $\beta^+ = \{y_1, \ldots, y_k\}$ in increasing way and $\beta^- = \{x_1, \ldots, x_k\}$ in decreasing way. Then all pairs $\{x_i, y_i\}$ will be parts of ${:}\pi{:}$.

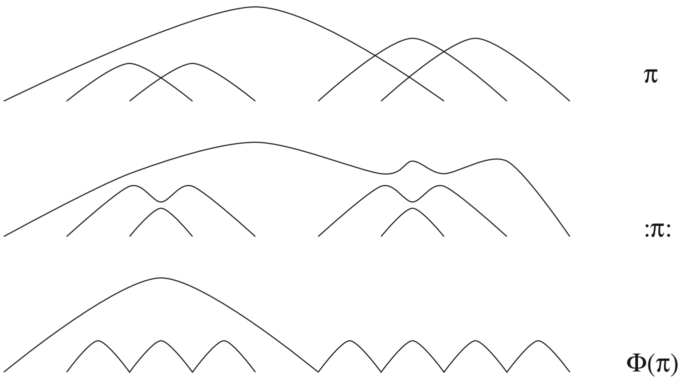


FIGURE. Examples of π, $:\pi:$, and $\Phi(\pi)$

Thus, Eq. (8.1) follows from the following, more refined statement.

Proposition 8.2. *For every* $\varpi \in \mathcal{NC}_2(2n)$

$$\sum_{\substack{\pi \in \mathcal{P}_2(2n) \\ :\pi:=\varpi}} (-1)^{|\pi|} q^{\|\pi\|} = (1-q)^{\mathrm{inn}(\varpi)}. \tag{8.2}$$

Proof. Given a noncrossing partition η we write $\eta_0 = \{\{\min\beta, \max\beta\} | \beta \in \eta\}$. If η is a noncrossing coarsening of ϖ we say that it is *admissible* if $\eta_0 \subset \varpi$. Notice that all outer blocks of ϖ belong to any its admissible coarsening, and for any $\rho \subset \varpi$ containing all outer blocks of ϖ there exists unique admissible coarsening η such that $\rho = \eta_0$.

We need to show that for any admissible coarsening η of ϖ we have

$$\sum_{\substack{\pi \in \mathcal{P}_2(2n) \\ :\pi:=\varpi,\ \Phi(\pi)=\eta}} (-1)^{|\pi|} = (-1)^{n-\#\eta}. \tag{8.3}$$

Indeed, $\Phi(\pi)$ is alway an admissible coarsening of $\varpi = :\pi:$. Thus Eq. (8.2) follows from (8.3) by multiplying by $q^{n-\#\eta}$ and summing over all admissible coarsenings η of ϖ.

Observe, that both sides of Eq. (8.3) factor over blocks of η. Thus it is enough to prove it for η being a single block. In such a case $\varpi = \varpi_n = \{\{i, 2n+1-\sigma(i)\} | 1 \le i \le n\}$.

All admissible coarsenings of ϖ_n are indexed by ordered subsets $\{a_i\}_{i=1}^{k-1}$ of the set $[n-1]$ (we set $a_0 = 0$ and $a_k = n$ for convenience) into parts $\{x | a_{i-1} < x \le a_i$ or $n - a_i < x \le n - a_{i-1}\}$. We call such coarsening η_a. Thus $\eta_\varnothing$ is the coarsening into one part.

By inclusion–exclusion principle we see that

$$\sum_{\substack{\pi \in \mathcal{P}_2(2n) \\ :\pi:=\varpi,\ \Phi(\pi)=\eta_\varnothing}} (-1)^{|\pi|} = \sum_{a \subset [n-1]} (-1)^{\#a} \prod_{i=0}^{\#a} \sum_{\substack{\pi \in \mathcal{P}_2(2(a_{i+1}-a_i)) \\ :\pi:=\varpi_{a_{i+1}-a_i}}} (-1)^{|\pi|} \tag{8.4}$$

Given a permutation $\sigma \in S_k$, we construct a pairpartition $\overline{\sigma} = \{\{i, 2n+1-\sigma(i)\} | 1 \le i \le n\}$. Note, that $|\overline{\sigma}|$ is equal to the length of σ with respect to the Coxeter generators $(1,2), \ldots, (n-1,n)$ of S_n. Therefore, denoting by $|\cdot|$ the Coxeter length will not lead into any confusion.

Clearly, partitions $\pi \in \mathcal{P}_2(2k)$ with $:\pi: = \varpi_k$ are of the form $\overline{\sigma}$ for $\sigma \in S_k$. Thus right hand side of Eq. (8.4) reduces to $(-1)^{n-1}$ as

$$\sum_{\pi \in S_k} (-1)^{|\pi|} = 0$$

for $k > 1$ (see the proof of Corollary 8.4) which is equivalent to the Eq. (8.3). □

Generalisation of Bożejko–Speicher formula to Coxeter groups. The formula (8.2) for $\varpi = \varpi_n$ reads

$$\sum_{\pi \in S_n} (-1)^{|\pi|} q^{\|\pi\|} = (1-q)^{n-1}. \tag{8.5}$$

Below we will generalise it to any finitely geerated Coxeter group.

By $W(t)$ we denote a (formal) growth series of a finitely generated Coxeter group W. That is, a power series $W(t) = \sum_{w \in W} t^{|w|}$. (Note, that the coefficient at t equals to $\#S$. This explains why here and in the rest of this section we consider only finitely generated Coxeter groups. We will not repeat this assumption for short. Moreover, for $X \subset W$ we write $X(t) = \sum_{w \in X} t^{|w|}$.

Let us define a multivariable formal power series (*chromatic length function*). For any $X \subset W$ define

$$X(t, \mathbf{q}) = \sum_{w \in X} t^{|w|} \prod_{s \in S_w} q_s.$$

In particular $X(t) = X(t, \mathbf{1})$, where $\mathbf{1} = (1)_{s \in S}$.

Proposition 8.3. *The polynomial* (*or formal power series, if W is infinite*) $W(t, \mathbf{q})$ *satisfies*

$$W(t, \mathbf{q}) = \sum_{T \subset S} W_T(t) \prod_{r \in T} q_r \prod_{s \in S \setminus T} (1 - q_s).$$

Proof. Let W_R° denote the set of all elements of W_R not contained in any proper parabolic subgroup of W_R, ie. $W_R^\circ := W_R - \bigcup_{T \subsetneq R} W_T$. Then, by inclusion-exclusion principle, $W_R^\circ(t) = \sum_{T \subset R} (-1)^{\#(R-T)} W_T(t)$. Therefore,

$$\begin{aligned} W(t, \mathbf{q}) &= \sum_{w \in W} t^{|w|} \prod_{s \in S_w} q_s = \sum_{R \subset S} W_R^\circ(t) \prod_{r \in R} q_r \\ &= \sum_{R \subset S} \sum_{T \subset R} W_T(t) (-1)^{\#(R-T)} \prod_{r \in R} q_r \\ &= \sum_{T \subset S} W_T(t) \prod_{r \in T} q_r \sum_{T \subset R \subset S} \prod_{s \in R \setminus T} (-q_s) \\ &= \sum_{T \subset S} W_T(t) \prod_{r \in T} q_r \prod_{s \in S \setminus T} (1 - q_s). \end{aligned}$$

□

Corollary 8.4. *If W is a finite Coxeter group then*

$$W(-1,\mathbf{q}) = \prod_{s\in S}(1-q_s). \tag{8.6}$$

In particular, substituting q for every q_s, we obtain

$$\sum_{\pi\in W}(-1)^{|\pi|}q^{\|\pi\|} = (1-q)^{\#S}.$$

Proof. Choose $s \in T$ and put $W_T^{\{s\}} = \{w \in W : |w| < |ws|\}$. Clearly, $W_T = W_T^{\{s\}} W_T$ therefore $W_T(t) = W_T^{\{s\}}(t)W_{\{s\}}(t)$. Since W is a finite group, $W_T^{\{s\}}$ is a polynomial. Thus $W_T(-1) = 0$ if T is nonempty (and $W_\varnothing(-1) = 1$). □

Open problems.

Question 8.5. Both formulae (8.2) and (8.6) generalise (8.5). Is there a common generalisation involving all (finitely generated) Coxeter groups and some analog of pairpartitions?

In the proof of Proposition 8.3 we have not assumed that W was finite. Let us finish this section with a discussion of infinite Coxeter groups. Recall, that -1 does not lie in the radius of convergence on $W(t)$ if W is not finite. Nevertheless, $W(t)$ represents a rational function as follows from the following result.

Proposition 8.6 ([STE68],[SER71, Prop. 26]). *Let (W,S) be an infinite Coxeter system. Then*

$$\frac{1}{W(t)} = \sum_{T\in\mathcal{F}}\frac{(-1)^{\#T}}{W_T(1/t)}. \tag{8.7}$$

Where $\mathcal{F}$ denote the family of subsets $T \subset S$, such that the group W_T generated by T is finite. In particular, $W(t)$ is a series of a rational function (i.e. a quotient of polynomials).

One may ask a question what is the class of (infinite) Coxeter groups such that $W_T(-1) = 0$ for any nonempty subset T of generators. A naïve argument that

$$W(t) = W_{\{s\}}(t)W^{\{s\}}(t) = (1+t)W^{\{s\}}(t)$$

shows, that the question if $W(-1) \neq 0$ is equivalent to whether $W^{\{s\}}(t)$ can have a pole at $t=-1$. On the other hand note, that if W is of type $\tilde{A}_2$, ie. W is given by a presentation $\langle s_i : 1 \le i \le 3 | s_i^2, (s_i sj)^3 : 1 \le i < j \le 3\rangle$ then, by Eq. (8.7), $W(t) = \frac{1+t+t^2}{(1-t)^2}$ and $W(-1) = 1/4$.

More generally, it is known ([BOU68]) that in each coset of W_T there exists the unique shortest element. Let W^T denote the set of those shortest representatives. Moreover if $w = w^T w_T$ with $w^T \in W^T$ and $w_T \in W_T$ then $|w| = |w^T| + |w_T|$. Therefore $W^T(t)W_T(t) = W(t)$. In particular, $W^T(t)$ represents a rational function, and it is legitimate to ask about the value of $W^T(-1)$.

In the case of finite Coxeter group W, Eng [ENG01] observed that

$$W^T(-1) = \#\left\{w \in W^T \middle| w w_0 w \in W_T\right\},$$

where w_0 is the longest element in W. (Eng's proof was case-by-case. Later, a general classification-free proof of Eng's theorem was given in [RSW04]).

Subsequently, Reiner [REI02] has shown that if W is crystallographic (ie. the Weyl group in a compact Lie group G), then both sides of the above equality compute the signature of the corresponding flag variety G/Q_T, where Q_T is a parabolic subgroup associated to T.

Question 8.7. What is the meaning of $W(-1)$ or $W^T(-1)$ for infinite W?

We do not know if it possible for $W(-1)$ to be negative. If one takes $W = \langle s_i : 1 \leq i \leq 4 | s_i^2, (s_i s_j)^3 : 1 \leq i < j \leq 4\rangle$. Then, by Eq. (8.7), $W(t) = \frac{3t^3-2t^2-2t+1}{(1+t)(1+t+t^2)}$ and $W^{s_i}(-1) = {}^{-1}\!/_2$ (in this case $W(-1) = \infty$).

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Appendix: The Longest Element and Radial Positive Definite Functions on Finite Coxeter Groups

If a Coxeter group W is finite, then it contains the unique element $\omega_\circ$ which has the maximal length with respect to $|\cdot|$.
From the definition it is clear, that a function φ on a group Γ with values in the field of complex numbers **C** is *positive definite* if and only $\sum_{g\in\Gamma}\varphi(g)g$ is a nonnegative (bounded if the group is finite) operator on $\ell^2\Gamma$. (We identify $g \in \Gamma$ with $\lambda(g) \in B(\ell^2\Gamma)$, where λ is the left regular representation, for short.)
Let W be a finite Coxeter group. The following two statements are well known.

(A) The function $q^{|w|}$ is positive definite for any $0 \leq q \leq 1$.
(B) The function $\Delta(w) = |\omega_\circ|/2 - |w|$ is positive definite.

The first one was proven in [BJS88] (even for infinite Coxeter groups and also for $-1 \leq q \leq 1$) while the second—in [BS03, Proposition 6]. Here we give a short direct prove of the following.

Proposition. *The above statements* (A) *and* (B) *are equivalent.*

Proof. Let $q = e^{-t}$ (with $t \geq 0$, as we assume $q \leq 1$). The case (A) is equivalent to $\Phi_t = \sum_{w\in W} e^{t\Delta(w)}w = e^{t|\omega_\circ|/2}\sum_{w\in W} q^{|w|}w$ being nonnegative.
Assume (A). Recall first, that $|\omega_\circ w| = |\omega_\circ| - |w| = |w\omega_\circ|$. Therefore $|\omega_\circ|/2 - |\omega_\circ w| = -(|\omega_\circ|/2 - |w|)$, ie. $\Delta(\omega_\circ w) = -\Delta(w)$ and similarly, $\Delta(w\omega_\circ) = -\Delta(w)$.
The equality $\Delta(\omega_\circ w) = \Delta(w\omega_\circ)$ implies that $\omega_\circ$ (and thus $Q = (1-\omega_\circ)/2$) commutes with Δ (and thus Φ_t). Since $Q = Q^2$ is nonnegative we conclude that

$$t^{-1}\Phi_t Q = \sum_{w\in W} \frac{e^{t(|\omega_\circ|/2-|w|)} - e^{t(|\omega_\circ|/2-|w\omega_\circ|)}}{2t} w = \sum_{w\in W} \frac{\sinh(t\Delta(w))}{t} w.$$

is nonnegative. Therefore, taking the limit as $t \to 0$, we obtain that $\sum_{w\in W}\Delta(w)w$ is nonnegative. Thus (B).

Assuming (B) and using the Schur lemma, which says that the (pointwise) product of positive definite functions is positive definite, we get that

$$\Phi_t = \sum_{w \in W} e^{t\Delta(w)} w = \sum_{n \geq 0} \frac{t^n}{n!} \left(\sum_{w \in W} \Delta(w)^n w \right)$$

is nonnegative. Thus (A). □

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