Spectral density of generalized Wishart matrices and free multiplicative convolution

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We investigate the level density for several ensembles of positive random matrices of a Wishart-like structure, $W = XX^\dagger$, where X stands for a non-Hermitian random matrix. In particular, making use of the Cauchy transform, we study the free multiplicative powers of the Marchenko-Pastur (MP) distribution, $\mathrm{MP}^{\boxtimes s}$, which for an integer s yield Fuss-Catalan distributions corresponding to a product of s-independent square random matrices, $X = X_1 \cdots X_s$. New formulas for the level densities are derived for $s = 3$ and $s = 1/3$. Moreover, the level density corresponding to the generalized Bures distribution, given by the free convolution of arcsine and MP distributions, is obtained. We also explain the reason of such a curious convolution. The technique proposed here allows for the derivation of the level densities for several other cases.

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I. INTRODUCTION

Ensembles of non-Hermitian random matrices are of considerable scientific interest [1] in view of their numerous applications in several fields of statistical and quantum physics [2]. On the other hand, any ensemble of non-Hermitian matrices X allows us to write a positive, Hermitian matrix of the *Wishart* form,

$$X \to W = \frac{XX^\dagger}{\mathrm{Tr}XX^\dagger}. \tag{1}$$

The normalization implies that the random matrix satisfies a fixed trace condition, $\mathrm{Tr}W = 1$, so it can be interpreted as a density matrix.

Ensembles of such random density matrices analyzed in Ref. [3] can be obtained by taking a random pure state on a bipartite system and performing a partial trace over a single subsystem. In the case of the isotropic, structureless ensemble of random pure states, generated according to the unique, unitarily invariant measure, the asymptotic level density of the corresponding quantum states is described by the Marchenko-Pastur (MP) distribution $P_{1,c}$ [4]. Here the parameter c is determined by the ratio of the dimensions of the auxiliary and the principal quantum systems.

If the global unitary symmetry of the measure defining the ensemble of pure random states is broken, the partial trace yields *structured* ensembles of random density matrices. They can be constructed combining products of non-Hermitian random Ginibre matrices and sums of random unitary matrices distributed according to the Haar measure. Investigation of these ensembles initiated in Ref. [5] was further developed by Jarosz [6,7].

Random matrices described by the Wishart ensemble corresponding to the product of s Ginibre matrices, $X = G_1 G_2 \cdots G_s$, were found to be useful in the description of the level density of mixed quantum states associated with a graph [8] and states obtained by projection onto the maximally entangled states of a multipartite system [5]. Hence these distributions describe asymptotic statistics of the Schmidt coefficients characterizing entanglement of a random pure state [3].

As the moments of the level density $P_s(x)$ for such ensembles are known to be asymptotically described by the Fuss-Catalan numbers [9,10],

$$C_s(n) = \frac{1}{sn+1}\binom{sn+n}{n}, \quad s > 0, \tag{2}$$

these distributions are called *Fuss-Catalan*. These distributions describe singular values of products of independent Ginibre matrices, see Refs. [11–13], but they are also known [14] to describe the asymptotic distribution of singular values of the s power of a single random Ginibre matrix G^s.

These distributions may be considered as a generalization of the Marchenko-Pastur distribution for square random matrices, $P_1(x)$, which corresponds to the case $s = 1$. The Fuss-Catalan distributions can be interpreted as the free multiplicative convolution product [9] of s copies of the MP distribution $P_1(x)$, written, $P_s(x) = [P_1(x)]^{\boxtimes s}$. The spectral distribution of $P_s(x)$ for a product of an arbitrary number of s random Ginibre matrices was analyzed by Burda *et al.* [15]. The spectral distribution for the general case of random rectangular matrices also has been studied [12,16]. This distribution was expressed as a solution of a polynomial equation and it was conjectured that the finite-size effects can be described by a simple multiplicative correction.

An explicit form of $P_2(x)$ was derived in Ref. [17] in the context of the construction of generalized coherent states from combinatorial sequences. An exact form of the Fuss-Catalan distributions for any integer s was derived in Ref. [18] in terms of hypergeometric functions ${}_sF_{s-1}$. These results were extended in Ref. [19], in which the Mellin transform was used to derive analogous distributions for rational values of the exponent $s = p/q$ in terms of special functions. Free multiplicative powers of the MP distributions

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were investigated by Haagerup and Möller [20], generalized Fuss-Catalan distributions were studied in Ref. [11,21], and power series expansions were recently obtained in Ref. [16].

The main aim of this work is to derive a wide class of new results concerning the level density for generalized Wishart ensembles of random matrices. We, furthermore, want to make a case for the power of free random calculus in handling the problems of quantum information theory. Application of free random variables calculus to the area of quantum information, advocated by the authors in 2010 (Refs. [22] and [5]), is currently becoming a standard calculational tool. This is one of the developments that prompted us to present some old and some new results from the viewpoint of the free random variables formalism. First, we explain how the theorems for so-called isotropic random matrices explain curious combinatorial relations for the class of Bures-like measures. Next, we show that several related results already known in the literature can be put on the same footing by using the resolvent method and the Voiculescu S transform [23]. An analytical expression for the level density can be obtained, provided the corresponding Green's function forms a polynomial equation of a low order. For instance, the higher-order Fuss-Catalan (FC) distribution $P_3(x)$, originally expressed by special functions [18] is shown here to be representable in terms of elementary functions.

The techniques based on the Cauchy transform are applicable for ensembles of random matrices related to the free convolution of the Marchenko-Pastur distribution $P_1(x)$, the Arcsine distribution (AS), and their free powers. In the case of their free product one obtains the Bures distribution [24–26], while higher values of the exponent s lead to its generalization referred to as the s-Bures distribution. It is worth mentioning that these distributions belong to the broader class of *Raney distributions* studied in Refs. [18,19].

This paper is organized as follows. In Sec. II we review basic properties of the Cauchy transform and we inspect how the level density can be derived from the Green's functions. In Sec. III we cover the Haagerup-Larsen theorem for large isotropic random matrices and we demonstrate how one can apply it for the Bures class of measures. Section IV unravels several spectral densities for various powers (including fractional ones) of the Marchenko-Pastur distribution. As an example, we discuss the Marchenko-Pastur distribution $P_{1,c}$ with an arbitrary rectangularity parameter c and the arcsine distributions, for which the Green's function is given as a solution of a quadratic equation. Furthermore, we discuss the generalized Fuss-Catalan distribution $P_{2,c}$ and the generalized Bures distribution, for which the Green's function is given by a Cardano solution of a cubic equation. The third order generalized Fuss-Catalan distribution $P_{3,c}$ and the 2-Bures distribution are studied in another subsection. In these cases the Green's function is given by a Ferrari solution of a quartic equation, which allows us to express the corresponding level density in terms of elementary functions. Some technical details of the derivations are relegated to the Appendix.

II. CAUCHY FUNCTIONS AND LEVEL DENSITIES

To derive the level density corresponding to certain ensembles of random matrices and, more generally, to some free convolutions of the Marchenko-Pastur (MP) distribution, we will use the Voiculescu S transform and the Cauchy functions.

Consider a square random matrix X of size N pertaining to the Ginibre ensemble of non-Hermitian random matrices. The Wishart matrix $W = XX^\dagger$ is positive, and its level density is asymptotically, $N \to \infty$, described by the Marchenko-Pastur distributions [4], with the rectangularity parameter c set to unity,

$$P_1(x) = \frac{1}{2\pi}\sqrt{\frac{4-x}{x}}, \quad x \in [0,4]. \tag{3}$$

The variable x denotes a suitably rescaled eigenvalue λ of W. If a random Wishart matrix is normalized according to the trace condition $\mathrm{Tr}W = 1$, the rescaled variable reads $x = \lambda N$, which implies that the mean value $\langle x\rangle$ is set to unity. Thus the MP distribution describes asymptotically the level density of random quantum states generated with a measure induced by the Hilbert-Schmidt metric [5].

In order to analyze convolutions of the MP distribution, it is convenient to use its Voiculescu S transform [23], defined as a function of a complex variable w,

$$S_{\mathrm{MP}}(w) = \frac{1}{1+w}. \tag{4}$$

The S transform of the multiplicative free convolution is given by the product of the S transforms. For instance, the Fuss-Catalan distribution P_s of an integer order s [9,18], which corresponds to a product of s-independent non-Hermitian random matrices, $X = X_1 \cdots X_s$, can be written as a multiplicative free convolution of the Marchenko-Pastur distribution, $P_s(X) = [P_1(x)]^{\boxtimes s}$. Hence the corresponding S transform reads $S_{C_s}(w) = [S_{\mathrm{MP}}(w)]^s$.

Assume now we are given an S transform $S(w)$, which corresponds to an unknown probability measure at the real axis. To infer this measure and the spectral density $\rho(\lambda)$, we write the S transform as $S(w) = \frac{1+w}{w}\chi(w)$, where

$$\frac{1}{\chi(w)} G\left[\frac{1}{\chi(w)}\right] - 1 = w. \tag{5}$$

To recover the resolvent, we put

$$\frac{1}{\chi(w)} = z, \tag{6}$$

which allows us to write an implicit solution for the Green's function $G(z)$, known also as the *Cauchy* function in the mathematical literature,

$$G(z) \equiv \frac{1}{N}\left\langle \mathrm{tr}\frac{1}{z\mathbf{1}_N - M}\right\rangle = \frac{1+w(z)}{z}. \tag{7}$$

Here M represents a random matrix from the ensemble investigated. In other words, for any given S transform $S(w)$ the corresponding Green function $G(z)$ defined on the complex plane is given as a solution of the following algebraic equation

$$zw(z)\, S[w(z)] = 1 + w(z). \tag{8}$$

Note that the Green's function Eq. (7) acts as a generating function for the spectral moments $m_k = \frac{1}{N}\langle \mathrm{Tr}M^k\rangle = \int d\lambda \lambda^k \rho(\lambda)$, i.e., $G(z) = \sum_{k=0}^{\infty} m_k/z^{k+1}$, as seen by expanding the Green's function at $z = \infty$. Another useful function is the Voiculescu

R transform, defined as a generating function for the free cumulants κ_k, i.e., $R(z) = \sum_{k=1}^{\infty} \kappa_k z^{k-1}$. Both functions G and R are related by functional relations $R[G(z)] + 1/G(z) = z$ [or, equivalently, $G(R(z) + 1/z) = z$]. Finally, R and S transforms can be also related, namely the function $z = yS(y)$ is the composition inverse of $y = zR(z)$ and vice versa, when the expected value is nonzero [27].

The S transform can be used to define the multiplicative free convolution. Let μ_1 and μ_2 be probability measures on $\mathbb{R}_+$ and S_1 and S_2 denote their S transforms. Then the free multiplicative convolution $\mu_1 \boxtimes \mu_2$ is defined as the unique probability measure for which the S transform is given by the product $S_1 \cdot S_2$ - see [23]. Consequently, the free multiplicative power of order s of a probability distribution μ is defined by taking the power S^s of the corresponding transforms of μ.

For several cases, Eq. (8) can be solved analytically with respect to w. This is, for instance, the case for the Fuss-Catalan distribution, as $S(w)$ reads $(1+w)^{-s}$ and Eq. (8) yields a polynomial equation of order $s+1$. It can be solved analytically for $s = 2$ and $s = 3$.

Thus to obtain the spectral density we apply the Stieltjes inversion formula. One needs to analyze all solutions of Eq. (8) to extract the desired information. In the case $s = 2$ the corresponding polynomial has three solutions, one of which is real, and the remaining pair is mutually complex conjugated. On the basis of the Sochocki-Plemelj formula, $\frac{1}{\lambda \pm i\epsilon} = \text{P.V.}\frac{1}{\lambda} \mp i\pi\delta(\lambda)$, where P.V. denotes principal value, the negative imaginary part of the Green's function yields the spectral function

$$\rho(\lambda) = -\frac{1}{\pi} \lim_{\epsilon \to 0} \text{Im} G(z)|_{z=\lambda+i\epsilon}. \tag{9}$$

As analytic solutions of equations of orders three and four contain square roots raised to powers $1/3$ and $-1/3$, care has to be taken by evaluation of the imaginary part of a complex solution along the real axis—for more details see the Appendix.

We mention that the relevant spectral function can be recovered as well from the real part of the resolvent. In this case one uses the maximal entropy argument, yielding

$$\lim_{\epsilon \to 0} [G(\lambda + i\epsilon) + G(\lambda - i\epsilon)] = \frac{\partial V(\lambda)}{\partial \lambda}, \tag{10}$$

where V is the random matrix potential defining the measure, i.e., $d\mu(M) = dM \exp[-N \text{tr} V(M)]$; see Eynard [28].

On the basis of the aforementioned Sochocki-Plemelj formula, the resulting equation is a singular integrodifferential equation. In the case of the spectral support localized on a single, finite interval, one can solve the equation, e.g., by methods developed by Tricomi [29]. Interestingly, one can also view Eq. (10) as an equation for the potential V, provided the spectral density $\rho(\lambda)$ is known. Then the calculation of the Hilbert transform of the spectral density according to Eq. (10) yields the derivative of the potential, which, after integrating the derivative and using the rotational invariance, allows one to infer the form of $V(M)$. The above procedure, although well defined, is complicated at the technical level. In particular, in the case of the spectral functions resulting from the solution of cubic or quartic algebraic equations, integration yields complicated expressions for $V(M)$, which in general are nonpolynomial.

III. ISOTROPIC RANDOM MATRICES AND BURES MEASURE

We define an isotropic random matrix X as an $N \times N$ matrix having a polar decomposition $X = PU$, where P is a positive semidefinite Hermitian random matrix and U is a unitary random matrix distributed with the Haar measure. Such matrices have a spectrum independent of the polar angle on the complex plane. In the large-N limit, the powerful Haagerup-Larsen theorem holds [30], which allows one to infer the radial spectral density of the operator X directly from the spectral properties of the operator P^2, provided P and U are mutually free. In the mathematical literature, the case of infinitely large matrices possessing the above feature is called R diagonal [31]. An important consequence of the Haagerup-Larsen theorem is the so-called single-ring theorem [32], stating that the radially symmetric spectrum of isotropic random operators is always confined to the ring, with known and analytically calculable radii. In particular, the inner radius can be equal to zero, therefore the spectrum of isotropic, large random matrices is always either concentrated on the disk or on the ring. Explicitly, the Haagerup-Larsen theorem says

$$S_{P^2}[F_X(r) - 1] = \frac{1}{r^2}, \tag{11}$$

where $F_X(r)$ is the cumulative radial density for the complex eigenvalues of X, i.e.,

$$F_X(r) = 2\pi \int_0^r ds s \rho_X(s), \tag{12}$$

and $S_{P^2}(z)$ is a Voiculescu S transform for the operator P^2 as defined above.

The aforementioned Bures measures and their generalizations belong to the realm of applicability of the Haagerup-Larsen theorem. Recently, a curious relation has been observed in Ref. [10], stating that the Bures measure is a free multiplication of the arcsine measure and the Marchenko-Pastur measure. The relation has been observed by studying the combinatorial properties of the Bures measure, i.e., it was inferred from some *a priori* unexpected relations between pertinent moments. As far as we know, the general proof of why such factorization holds is missing. In this section, we provide a simple argument, showing that such a feature is just a consequence of the Haagerup-Larsen theorem. First, let us observe that in the large-N limit, we can write down the averages of the products and ratios of the operators as the corresponding products and ratios of the averages of the operators. This means that one can perform all calculations of the spectral properties ignoring the normalization $\text{Tr} XX^\dagger$ in the denominator of Eq. (1) and then, at the end of calculations, perform the rescaling of the argument of the resolvent by the value $a = \frac{1}{N}\langle \text{Tr} XX^\dagger \rangle$, according to the relation

$$G_{\frac{P}{a}}(z) = \frac{1}{N}\left\langle \text{Tr} \frac{1}{z - \frac{P}{a}} \right\rangle = aG_P(za). \tag{13}$$

The Bures measure is constructed from the Wishart measure (modulo above-mentioned normalization) as $X = (U_1 +$

$U_2)G$, where U_i are Haar-distributed unitary matrices measures and G is a Ginibre matrix. Both ingredients of the above product fulfill in the large-N limit the assumptions of the Haagerup-Larsen theorem, i.e., $U_1 + U_2 = P_U U$, where $P_U = |U_1 + U_2|$ and $G = P_G U'$, where P_G is the positive part of the Wigner semicircle (Wigner's semiquarter), and U and U' are Haar measures. The square of the P_G is a Marchenko-Pastur distribution. Since the Marchenko-Pastur distribution, by construction, corresponds to the first moment equal to 1, one does not need to renormalize the spectral density. In the case of the second element of the product, i.e., the operator $|U_1 + U_2|$, its spectral properties come from the special case of the general formula proven in Ref. [30],

$$R_{|U_1+U_2+\ldots+U_k|}(z) = k\frac{\sqrt{1+4z^2}-1}{2z}. \tag{14}$$

Therefore, using the properties of the R transform and choosing $k = 2$, one gets $G_{|U_1+U_2|}(z) = \frac{1}{\sqrt{z^2-4}}$. Since we need the Green's function for the square of the modulus of $|U_1 + U_2|$, we use the symmetry relation $G_{H^2}(z) = G_H(\sqrt{z})/\sqrt{z}$, which yields $G_{|U_1+U_2|^2}(z) = \frac{1}{\sqrt{z(z-4)}}$. Expanding the last relation for $z \to \infty$, we see that the first moment is equal to 2. So final rescaling according to Eq. (13) recovers the Green's function for the arcsine distribution

$$G_{AS}(z) = \frac{1}{\sqrt{z(z-2)}}, \tag{15}$$

which completes the proof. The subscript AS stands for "arcsine"; see Eq. (17) below.

The above construction is easily generalizable for the case of arbitrary long strings of powers of Ginibre ensembles and sums of unitary ensembles. In such a case, one has to use Eq. (14) and the fact that, in the large-N limit, the limiting spectral density of the product of m identically distributed isotropic unitary random matrices is equal to the spectral density of the m-th power of a single matrix from such an ensemble [14,33]. This observation allows us to recover spectral properties of the generalizations of the Bures measures proposed in Refs. [5,7,8]. Further generalizations include the case of strings of Marchenko-Pastur distributions for rectangular matrices X and/or cases of truncated unitary distributions. The general method is always based on the Haagerup-Larsen theorem, but the final formulas are usually more involved compared to the case presented here.

IV. GENERALIZED WISHART MATRICES AND THEIR SPECTRAL DENSITIES

A. Quadratic equation

As a warm-up exercise, we start by recalling simple problems which correspond to a quadratic equation. Consider first the Green's function corresponding to the free binomial distribution, where $\rho(\lambda) = \frac{1}{2}[\delta(\lambda) + \delta(\lambda - 1)]$. The Green's function reads therefore $G(z) = \frac{1}{2}(\frac{1}{z} + \frac{1}{z-1})$. Straightforward manipulations yield the R transform and S transform, given, respectively, by $R(z) = (z - 1 + \sqrt{z^2+1})/(2z)$ and $S(z) = 2(1+y)/(1+2y)$. Anticipating the results needed for the remaining part of this work, we consider now the free sum of two binomial distributions. Since the R transform is additive, we get $R_{AS}(z) = 2R(z) = (z - 1 + \sqrt{z^2+1})/z$. Then the corresponding S transform reads $S_{AS}(z) = (z+2)/(2+2z)$. Substituting it into Eq. (8), we get

$$wz(w+2) = 2(1+w)^2. \tag{16}$$

Solving this for w we obtain two conjugated solutions. Selecting the one with negative imaginary part and plugging it into Eq. (9) yields the *arcsine distribution*,

$$\mathrm{AS}(x) = \frac{1}{\pi}\frac{1}{\sqrt{x(2-x)}}, \quad x \in [0,2]. \tag{17}$$

This distribution gives us the level density of the suitably normalized sum of a random unitary matrix U and its adjoint $U^\dagger$. It describes the ensemble of quantum states obtained by reduction of a coherent combination of maximally entangled states [5] and will be used here to construct other distributions.

Before moving on to the cubic equation and more complicated cases, let us recall how to obtain in this way the general form of the Marchenko-Pastur distribution. It describes the asymptotic level density $\rho(x)$ of random states of $\rho = XX^\dagger/\mathrm{Tr}XX^\dagger$, where X is a rectangular complex Ginibre matrix of size $N \times M$. We choose the rectangularity parameter $c = M/N \leqslant 1$. The case $c > 1$ yields the same nonzero eigenvalues and, additionally, $N - M$ zero eigenvalues. Let us then start with the corresponding S transform, $S_c(w) = 1/(1+cw)$, which reduces to Eq. (4) for $c = 1$. Plugging this expression into (8) leads to a quadratic equation,

$$zw = (1+w)(1+cw). \tag{18}$$

Its solution with respect to w with a negative imaginary part together with Eq. (9) allows one to obtain

$$P_{1,c}(x) = \frac{1}{2\pi xc}\sqrt{(x-x_-)(x_+-x)}, \tag{19}$$

where $x \in [x_-,x_+]$, with the edges of the support at $x_\pm = 1 + c \pm 2\sqrt{c}$. In the case $c \to 0$, the Marchenko-Pastur distribution reduces to $\rho(\lambda) = \delta(\lambda - 1)$.

B. Cubic equation and Cardano solutions

Next, we are going to present solutions of problems motivated by ensembles of random matrices, for which Eq. (8) becomes a cubic polynomial in $w = w(z)$.

1. Fuss-Catalan distribution of order 2

To show the presented method in action, we rederive the Fuss-Catalan distribution $P_2(x) = [P_1(x)]^{\boxtimes 2}$, which describes ensemble Eq. (1) with X being a product of two independent square Ginibre matrices. As a starting point we thus take the square of the S transform of the MP distribution, $S_{FC_2}(w) = [S_{MP}(w)]^2 = (1+w)^{-2}$. Putting this form into Eq. (8) we get a cubic equation,

$$wz = (1+w)^3. \tag{20}$$

Calculating the Green's function Eq. (7) and making use of Eq. (9), one obtains the Fuss-Catalan distribution of order 2,

$$P_2(x) = \frac{\sqrt[3]{2}\sqrt{3}}{12\pi}\frac{[\sqrt[3]{2}(27+3\sqrt{81-12x})^{\frac{2}{3}} - 6\sqrt[3]{x}]}{x^{\frac{2}{3}}(27+3\sqrt{81-12x})^{\frac{1}{3}}}, \tag{21}$$

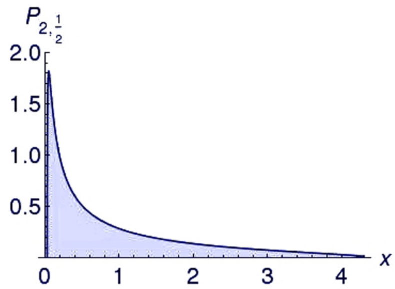

FIG. 1. (Color online) Generalized Fuss-Catalan distribution of order 2 $P_{2,c}(x)$ plotted for rectangularity parameter $c = 1/2$.

where $x \in [0,27/4]$. This result was first obtained in Ref. [17] in the context of the construction of generalized coherent states from combinatorial sequences and later used in Ref. [8] to describe the asymptotic level density of mixed quantum states related to certain graphs.

2. Generalized Fuss-Catalan distribution $P_{2,c}$

In an analogous way we can treat the case of a product of two independent rectangular Ginibre matrices characterized by an rectangularity parameter $c = M/N$. The corresponding S transform, $S_{2,c} = 1/(1 + cw)^2$, leads to a modified equation of the third order,

$$wz = (1 + w)(1 + cw)^2. \tag{22}$$

Solving it with respect to w and computing the corresponding Green's function Eq. (7) and its imaginary part one obtains a level density. A particular case of the generalized Fuss-Catalan distribution of order two obtained for $c = 1/2$ is shown in Fig. 1. This precise case was very recently studied in Ref. [16], where an explicit density was provided. Moments of the distributions $P_{s,c}$ can be expressed in terms of the Fuss-Narayana numbers, see Ref. [34].

3. Free square root of the Marchenko-Pastur distribution

To derive this distribution we consider the square root of the S transform of the MP distribution, $S_{1/2}(w) = [S_{\rm MP}(w)]^{1/2}$, which, used in Eq. (8), yields a Cardano cubic equation,

$$w^3 + (3 - z^2)w^2 + 3w + 1 = 0. \tag{23}$$

Writing down the Green's function Eq. (7), we use Eq. (9) to get an explicit form of the free multiplicative square root of the Marchenko-Pastur distribution, $P_{1/2}(x) := [P_1(x)]^{\boxtimes 1/2}$,

$$P_{1/2}(x) = x^{-1/3}\frac{(9+Y)^{1/3} - (9-Y)^{1/3}}{2^{4/3}3^{1/6}\pi} + x^{1/3}\frac{(9+Y)^{2/3} - (9-Y)^{2/3}}{2^{4/3}3^{5/6}\pi}, \tag{24}$$

where $Y(x) = \sqrt{81 - 12x^2}$ and x belongs to $[0,\sqrt{27/4}]$.

This distribution was derived in Ref. [19] using the inverse Mellin transform and the Meijer G functions. For the moment, we are not aware of any method to generate an ensemble of random matrices characterized asymptotically by the above level density.

4. Bures distribution

The Bures distribution describes the asymptotic level density of random mixed states distributed according to the measure [24] induced by the Bures metric [35]. As already mentioned in Sec. II, to generate random states with respect to this measure, it is sufficient [25] to take $X = (\mathbb{1} + U)G$, where U is a Haar random unitary matrix and G is a square random Ginibre matrix of the same size, and substitute it into Eq. (1). Using the Haagerup-Larsen theorem, we have demonstrated in Sec. II why the Bures distribution can be represented as the multiplicative free product of the positive arcsine law and the Marchenko-Pastur law: $B_1 = {\rm AS} \boxtimes {\rm MP}$. The free S transform of B_1 reads

$$S_{B_1}(w) = \frac{w+2}{2(w+1)^2} = \frac{w+2}{2w+2}\frac{1}{1+w}. \tag{25}$$

Observe that the first factor is the S transform of AS while the second one, $1/(1 + w)$, is the S transform of MP. This is the aforementioned law of free multiplication. The S transform Eq. (25) together with Eq. (8) leads to an equation of order 3, $wz(w + 2) = 2(1 + w)^3$, which can be explicitly solved with respect to the complex variable w. Making use of Eqs. (7) and (9), one arrives at the Bures density

$$B_1(x) = C\left[\left(\frac{a}{x} + \sqrt{\left(\frac{a}{x}\right)^2 - 1}\right)^{2/3} - \left(\frac{a}{x} - \sqrt{\left(\frac{a}{x}\right)^2 - 1}\right)^{2/3}\right], \tag{26}$$

where $C = 1/4\pi\sqrt{3}$ and $a = 3\sqrt{3}$. This distribution, first obtained in Ref. [24], is defined on a support larger than the standard MP distribution, $x \in [0,a]$, and it diverges for $x \to 0$ as $x^{-2/3}$.

5. Generalized Bures distributions

The generalized Bures distribution can be defined by a convolution of the arcsine and the Marchenko-Pastur distribution with rectangularity parameter c, namely $B_{1,c} = {\rm AS} \boxtimes P_{1,c}$. The corresponding ensemble of random matrices can be obtained writing $X = (1 + U)G$, where U stands for a random unitary matrix of size N generated according to the Haar measure on $U(N)$, while G denotes a rectangular non-Hermitian random Ginibre matrix of order $N \times K$ with $c = K/N$. Similar ensembles of random matrices were recently studied by Jarosz [7]. Getting the corresponding ensemble of density matrices, one may use superpositions of pure states of a four-party system followed by a projection on maximally entangled states and taking a partial trace [5].

Multiplying the corresponding S transforms, we get $S_{B_{1,c}}(w) = (w + 2)/[2(1 + w)(1 + cw)]$, which leads to the following cubic equation: $wz(w + 2) = 2(1 + cw)(1 + w)^2$. In the special case $c = 1/2$, the above equation simplifies to the quadratic one, $wz = (1 + w)^2$, corresponding to the Marchenko-Pastur distribution. The generalized Bures distribution $B_{1,c}(x)$ for $c \in [1/2,1]$ can be thus interpreted as an interpolation between the MP and Bures distributions. In the case $c \leqslant 1$ this distribution is absolutely continuous. In the case $c > 1$, presented in Figs. 2 and 3, the distribution consists

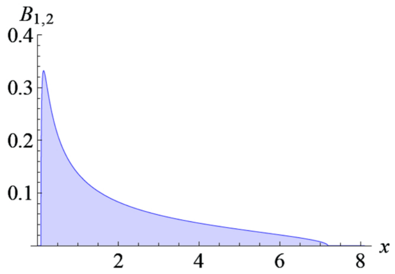

FIG. 2. (Color online) The continuous part of the generalized Bures distribution $B_{1,c}(x)$ plotted for rectangularity parameter $c = 2$, so the shaded area equals $1/2$.

of a Dirac delta, $\delta(x)$, with weight $(1 - 1/c)$ and a continuous part; see Th. 4.1 in Ref. [36].

We shall conclude this section emphasizing that the method discussed here is not limited to the cases presented. For instance, analyzing the free multiplicative square root of the arcsine distribution, $\mathrm{AS}^{\boxtimes 1/2}$, or its free square, $\mathrm{AS}^{\boxtimes 2}$, one arrives at similar cubic equations, $(w+2)w^2z^2 = 2(w+1)^3$ and $(w+2)^2wz = 4(w+1)^3$, respectively, which allow for the derivation of corresponding level densities.

C. Quartic equation and Ferrari solutions

The list of cases for which Eq. (8) forms a quartic equation contains, for instance, the third-order Fuss-Catalan distribution P_3, the third root of the Marchenko Pastur distribution $P_{1/3}$, and the higher-order Bures distribution.

1. *Fuss-Catalan distribution of order 3*

To find an analytical expression for the Fuss-Catalan distribution, $P_3 = [P_1(x)]^{\boxtimes 3}$, describing the asymptotic level density of the normalized Wishart matrix $XX^\dagger$, where X is a product of three independent Ginibre matrices, we start with the third power of the S transform corresponding to the Marchenko Pastur distribution, $S_3(w) = S_{\mathrm{MP}}^3 = 1/(1+w)^3$. Equation (8) leads then to the following quartic equation:

$$w^4 + 4w^3 + 6w^2 + w(4 - z) + 1 = 0. \qquad (27)$$

Making use of the standard Ferrari formulas, we obtain four explicit solutions of this equation given as square roots of expressions which contain polynomials of z in powers $1/3$ and $-1/3$. Analyzing the imaginary part of the corresponding Green's function Eq. (7), as discussed in Appendix, we arrive at an explicit expression for the Fuss-Catalan distribution of

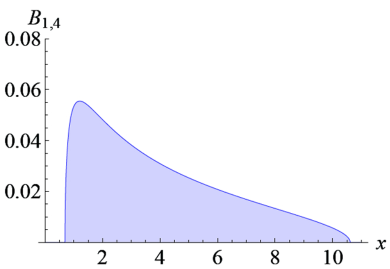

FIG. 3. (Color online) As in Fig. 2 for $c = 4$, so the area under the curve is $1/4$.

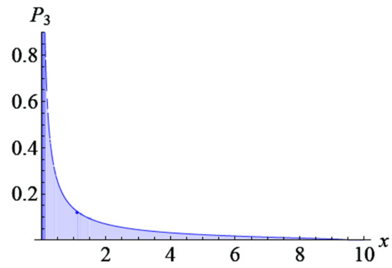

FIG. 4. (Color online) Fuss-Catalan distribution $P_3(x) = [P_1(x)]^{\boxtimes 3}$ given in Eq. (28).

order 3,

$$P_3(x) = \frac{x^{-3/4}}{2(3^{1/4}\pi)}\sqrt{4Y - \frac{3^{3/4}x^{1/4}}{\sqrt{Y}}}, \qquad (28)$$

where $Y(x) = \cos[\frac{1}{3}\arccos(\frac{3\sqrt{3}}{16}\sqrt{x})]$ and $x \in [0,256/27]$. Interestingly, the same distribution, shown in Fig. 4 was derived earlier in Ref. [18] and expressed in terms of combinations of hypergeometric functions ${}_3F_2(x)$, which in this specific case admits a more elementary representation.

Note that in an analogous way it is also possible to obtain expressions for the generalized Fuss-Catalan distributions of order 3, $P_{3,c}$, which correspond to the S transform $S_{2,c} = 1/(1+cw)^3$. This distribution, representing the asymptotic level density of the Wishart matrices obtained from a product of three independent rectangular Ginibre matrices with rectangularity parameter $c = N/K$, may in principle be further generalized for three different rectangularity parameters, so the S transform reads $S_{2,c} = 1/(1+c_1w)(1+c_2w)(1+c_3w)$; see also Ref. [16].

2. *Free third root of Marchenko-Pastur*

Consider the third root of the S transform corresponding to the MP distribution, $S_{1/3}(w) = [S_{\mathrm{MP}}(w)]^{1/3}$. This choice applied to Eq. (8) leads again to a quartic equation in terms of w,

$$w^4 + (4 - z^3)w^3 + 6w^2 + 4w + 1 = 0. \qquad (29)$$

Solving this equation analytically for w, evaluating the Green's function Eq. (7), and applying Eq. (9) we arrive at the following form of the third free multiplicative root of the Marchenko-Pastur distribution, $P_{1/3}(x) := [P_1(x)]^{\boxtimes 1/3}$,

$$P_{1/3}(x) = \frac{1}{2\pi x}\left[Y + 4x^3 - \frac{1}{2}x^6 + \left|\frac{x^3(24 - 12x^3 + x^6)}{4\sqrt{Y - 2x^3 + \frac{1}{4}x^6}}\right|\right]^{1/2}, \qquad (30)$$

where $Y(x) = (4/\sqrt{3})x^{3/2}\cos[\frac{1}{3}\arccos(\frac{3\sqrt{3}}{16}x^{3/2})]$ and $x \in [0,(256/27)^{1/3}]$.

3. *2-Bures distributions*

The higher-order s-Bures distribution can be defined as a free convolution of the arcsine and the s–Fuss-Catalan distribution, $B_s = \mathrm{AS} \boxtimes P_s$. It describes the asymptotic level density of the Wishart matrices $XX^\dagger$, where $X = (\mathbb{1} + U)G_1 \cdots G_s$. Here U denotes a random unitary matrix distributed according to the Haar measure while $G_1, \ldots G_s$ are independent square complex Ginibre matrices. In the case $s = 1$ the standard Bures ensemble [25] is retrieved. Note that these distributions coincide with $\mu[(s+2)/2,1/2]$ from Ref. [10], up to a dilation by 2. Indeed, the free S transform of s Bures is

$$S(w) = \frac{w+2}{2(w+1)^{s+1}}, \tag{31}$$

which can be compared with (4.11) in Ref. [10] for $p = (s+2)/2$ and $r = 1/2$.

Consider the case $s = 2$ for which the Cauchy function $S_{B_2}(w) = (w+2)/[2(1+w)^3]$ leads to the quartic equation

$$wz(w+2) = 2(1+w)^4.$$

Next, of four analytical Ferrari solutions select $w(z) = -1 + [z + i\sqrt{(z-8)z}]^{1/2}/2$, which can be rewritten as $w = -1 + \sqrt{8z}\exp[i\arccos(\sqrt{z/8})]$. Plugging this into Eq. (7) we get the Green's function, which, used in Eq. (9), yields the desired density, $B_{2,1}(x) = \sin[\frac{1}{2}\arccos(\sqrt{2x}/4)]/(2^{1/4}\pi x^{3/4})$. Making use of the known formula for the sine of the half angle, $\sin(x/2) = \sqrt{[1-\cos(x)]/2}$ we can get rid of the arccosine and arrive at the result

$$B_2(x) = \frac{1}{\pi\, 2^{5/4} x^{3/4}} \sqrt{2 - \sqrt{x/2}}, \tag{32}$$

for $x \in [0,8]$, see Refs. [19] and [37]. It is worth noting that other recent representations of the Fuss-Catalan, Raney, and related distributions [20,37–39] also contain sine functions, the argument of which is an inverse trigonometric function of the rescaled argument.

In a similar way one obtains results for the generalized 2-Bures distribution $B_{2,c}(x)$, corresponding to the product $X = (\mathbb{1} + U)G_1G_2$ with rectangular matrices G_1 and G_2. For any rectangularity parameter $c = N/M$, the corresponding quartic equation reads now $wz(w+2) = 2(1+cw)(1+w)^3$ and can be solved analytically. The corresponding level densities are too lengthy to be reproduced here. However, in the special case $c = 1/2$ this equation reduces to the case of Eq. (20), so the generalized 2-Bures distribution with rectangularity parameter $c = 1/2$ coincides with the Fuss-Catalan distribution cf. Eq. (21), $B_{2,1/2}(x) = P_2(x)$.

The list of other interesting cases that lead to quartic equations includes, for instance, the free multiplicative convolution of the arcsine and Bures, $\mathrm{AS}\boxtimes B = \mathrm{AS}^{\boxtimes 2}\boxtimes\mathrm{MP}$, or the free multiplicative square root of the Bures distribution, $B^{\boxtimes 1/2} = \mathrm{AS}^{\boxtimes 1/2}\boxtimes\mathrm{MP}^{\boxtimes 1/2}$. The corresponding level densities can be obtained by solving quartic equations $(w+2)^2wz = 4(w+1)^4$ and $(w+2)w^2z^2 = 2(w+1)^4$, respectively.

V. CONCLUDING REMARKS

Making use of the S transform and the Cauchy (Green's) function, it is possible to write down an explicit form for the probability measures defined by the free multiplicative convolution of the MP distribution P_1 and other probability measures with known S transform. For instance, the multiplicative convolution of the arcsine distribution and P_1 raised in the free multiplicative sense to an integer power leads to an algebraic equation for the argument of the S transform. We studied some relevant cases for which this algebraic equation is of the third or fourth order, so, based on the known Cardano and Ferrari solutions, one can analytically derive an explicit form of the required probability measures. This is the case, for instance, for the free multiplicative powers of the Marchenko-Pastur distribution, $[P_1(x)]^{\boxtimes s}$, with an exponent s equal to 2 and 3 and also 1/2 and 1/3, and for the convolution of P_1 and $P_2(x) = [P_1(x)]^{\boxtimes 2}$ with the arcsine distribution (AS). Among the results of this work, it is worth mentioning the new analytic expressions for the densities Eqs. (24), (28), and (30).

Several distributions derived in this paper are useful in the theory of random matrices and its numerous applications in physics. Integer multiplicative powers of the MP, called Fuss-Catalan distributions, describe the asymptotic level density of generalized Wishart random matrices, $W = XX^\dagger$, where X represents a product of s independent non-Hermitian random square Ginibre matrices, $X = X_1 \cdots X_s$. We obtained here an explicit expression for $P_3 = [P_1(x)]^{\boxtimes 3}$ in terms of elementary functions. We also analyzed the extension of the problem for the case of rectangular Ginibre matrices. Furthermore, the case of the multiplicative convolution of AS with P_1 and P_2 corresponds to the Bures distribution B_1, the generalized Bures distribution $B_{1,c}$, and the higher-order Bures distribution B_2. All of of these describe level distributions of generalized Wishart matrices $XX^\dagger$, where X is obtained by multiplying the sum of two random Haar matrices with a product of s random Ginibre matrices. These results are applicable in the description of the asymptotic level density of certain ensembles of random quantum states [5].

As a by-product of our analysis we derived explicit results for the probability measure corresponding to the free multiplicative square and cubic root of the Marchenko-Pastur distribution, written $P_{1/2} = [\mathrm{MP}]^{\boxtimes 1/2}$ and $P_{1/3} = [\mathrm{MP}]^{\boxtimes 1/3}$, respectively. Note that for $p < 1$ the distribution $[\mathrm{MP}]^{\boxtimes p}$ is not infinitely divisible with respect to the additive free convolution $\boxplus$, so the method of Cabanal-Duvillard [40] is not applicable. In fact, a stronger statement is true: If $p < 1$, then the additive free power $([\mathrm{MP}]^{\boxtimes p})^{\boxplus t}$ exists if and only if $t \geqslant 1$, see the recent result of Arizmendi and Hasebe [41]. It is thus unlikely that there exists a random matrix model which corresponds to the level density described, for instance, by the multiplicative free square root of the Marchenko-Pastur distribution.

Note added. We recently became aware of two works where related issues of Raney-type distributions have been addressed using either differential equations [42] or combinatorial analysis [37].

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APPENDIX: ON IMAGINARY PART OF SOLUTION OF A QUARTIC EQUATION

To further demonstrate the derivation of the spectral density, we treat in this Appendix an example corresponding to the Fuss-Catalan distribution of order 3, cf. Eq. (28). Writing down the Ferrari solutions of the quartic equation (27), we identify the one with an negative imaginary part, denoted by w_3, so the imaginary part of the corresponding Green's function Eq. (7) yields the desired spectral density Eq. (9).

The full expression for this solution consists of two terms, $w_3(z) = a_1 + a_2$. We may omit the real term a_1, as it does not contribute to the imaginary part of the Green's function. The relevant term then reads

$$a_2 = -\frac{6^{2/3}}{12}\sqrt{-A - B + \frac{12z}{\sqrt{A+B}}},$$

where z-dependent symbols $A=(\frac{8z}{3z^2+T/\sqrt{3}})^{1/3}$ and $B=(18z^2 + \sqrt{12}T)^{1/3}$ contain a square root $T = \sqrt{z^3(-256+27z)}$. Its argument is negative for $z \in [0,256/27]$, so T can be rewritten as $T = i\sqrt{z^3(256-27z)} = it$, where t is a real number. Let us now write the argument of the cubic root in B in polar form, $Z = re^{i\phi}$, with radius $r = 32\sqrt{3}z^{3/2}$ and phase $\phi = \arccos(3\sqrt{3}\sqrt{z}/16)$. Then the key term reads

$$a_2 = -\frac{6^{2/3}}{12}\sqrt{-\frac{8z}{(Z/6)^{1/3}} - Z^{1/3} + \frac{12z}{\sqrt{\frac{8z}{(Z/6)^{1/3}} + Z^{1/3}}}}.$$

We can take the third root of Z represented in polar form, $Z^{1/3} = r^{1/3}\exp(i\phi/3)$, group the terms $y\exp(i\phi/3)$ and $y\exp(-i\phi/3)$, and replace them with $2y\cos(\phi/3)$. Simplifying this expression, we arrive eventually at the final form of the Green's function Eq. (7) and, by taking its imaginary part Eq. (9), we arrive at the Fuss-Catalan distribution of order 3 Eq. (28), defined for $x \in [0,256/27]$.

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