

FUSS-CATALAN NUMBERS IN NONCOMMUTATIVE PROBABILITY

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ABSTRACT. We prove that if $p \geq 1$ and $0 \leq r \leq p$ then the Fuss-Catalan sequence $\binom{mp+r}{m} \frac{r}{mp+r}$ is positive definite. We study the family of the corresponding probability measures $\mu(p, r)$ on \mathbb{R} from the point of view of noncommutative probability. For example, we prove that if $0 \leq 2r \leq p$ and $r + 1 \leq p$ then $\mu(p, r)$ is \boxplus -infinitely divisible. As a by-product, we show that the sequence $\frac{m^m}{m!}$ is positive definite and the corresponding probability measure is \boxtimes -infinitely divisible.

1. INTRODUCTION

For natural p, r let $A_m(p, r)$ denote the number of all such sequences $(a_1, a_2, \dots, a_{mp+r})$ that: (1) $a_i \in \{1, 1-p\}$, (2) $a_1 + a_2 + \dots + a_s > 0$ for all s such that $1 \leq s \leq mp+r$ and (3) $a_1 + a_2 + \dots + a_{mp+r} = r$. It turns out that this quantity is given by the two-parameter Fuss-Catalan numbers (2.1) (see [5]). Note that the right hand side of (2.1) allows us to define $A_m(p, r)$ for all real parameters p and r . In particular, the *Catalan numbers* $A_m(2, 1)$ are known as moments of the Marchenko–Pastur distribution:

$$(1.1) \quad d\tilde{\pi}(x) = \frac{1}{\pi} \sqrt{\frac{4-x}{x}} dx \quad \text{on } [0, 4],$$

which in the free probability theory plays the role of the Poisson measure. In this paper we are going to study for which parameters $p, r \in \mathbb{R}$ the sequence $\{A_m(p, r)\}_{m=0}^\infty$ is positive definite, i.e. is the moment sequence of some probability measure, which we will denote $\mu(p, r)$. Recently T. Banica, S. T. Belinschi, M. Capitaine and B. Collins [1] showed that if $p > 1$ then $\{A_m(p, 1)\}_{m=0}^\infty$ is the moment sequence of a probability measure which can be expressed as the multiplicative free power $\tilde{\pi}^{\boxtimes p-1}$.

We are going to prove that if $p \geq 1$ and $0 \leq r \leq p$ then $\{A_m(p, r)\}_{m=0}^\infty$ is the moment sequence of a unique probability measure $\mu(p, r)$ which has a compact support contained in $[0, \infty)$. Moreover, if $0 \leq 2r \leq p$ and $r + 1 \leq p$ then $\mu(p, r)$ is infinitely divisible with respect to the free convolution \boxplus . In some particular cases we are able to determine the multiplicative free convolution, the boolean power and the monotonic convolution of the measures $\mu(p, r)$. We will also prove that if $0 \leq r \leq p-1$ then the sequence $\left\{\binom{mp+r}{m}\right\}_{m=0}^\infty$ is positive definite and the corresponding probability measure can be expressed as $\mu(p-r, 1)^{\uplus p} \triangleright \mu(p, r)$, where \uplus and \triangleright denote the boolean and the monotonic convolution, respectively.

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The paper is organized as follows. In Section 2 we prove three combinatorial identities. Then we use them to derive some formulas for the generating functions. In Section 4 we regard the numbers $A_m(p, r)$ as moments of a *probability quasi-measure* $\mu(p, r)$, by which we mean a linear functional $\mu : \mathbb{R}[x] \rightarrow \mathbb{R}$ satisfying $\mu(1) = 1$. On the class of probability quasi-measures one can introduce the free, boolean and monotonic convolutions in combinatorial way. The class of compactly supported probability measures on \mathbb{R} , regarded as a subclass of the former, is closed under these operations. We prove some formulas involving the probability quasi measures $\mu(p, r)$, for example we find the free R - and S -transforms (4.8), (4.11), the boolean power $\mu(p, 1)^{\natural t}$ (4.18) and, in special cases, the multiplicative free (4.12), (4.13), (4.14) and the monotonic convolution (4.20) of the measures $\mu(p, r)$.

In Section 5 we prove that if $p \geq 1$ and $0 \leq r \leq p$ then $\mu(p, r)$ is a measure (we conjecture that this condition is also necessary for $p, r > 0$). The proof involves the multiplicative free convolution. We also prove that if $0 \leq 2r \leq p$ and $r + 1 \leq p$ then $\mu(p, r)$ is \boxplus -infinitely divisible.

In the final part we extend our results to the dilations of the measures $\mu(p, r)$, with parameter $h > 0$. Taking the limit with $h \rightarrow 0$ we prove in particular that the sequence $\frac{m^m}{m!}$ is positive definite and the corresponding probability measure ν_0 is \boxtimes -infinitely divisible.

2. SOME COMBINATORIAL IDENTITIES

We will work with the two-parameter Fuss-Catalan numbers defined by $A_0(p, r) := 1$ and

$$(2.1) \quad A_m(p, r) := \frac{r}{m!} \prod_{i=1}^{m-1} (mp + r - i),$$

$m \geq 1$, where p, r are real parameters. Note that (2.1) can be written as $\binom{mp+r}{m} \frac{r}{mp+r}$, unless $mp + r = 0$. One can check that for $m \geq 0$

$$(2.2) \quad A_m(p, r) = A_m(p, r - 1) + A_{m-1}(p, p + r - 1),$$

where by convention $A_{-1}(p, r) := 0$, and

$$(2.3) \quad A_m(p, p) = A_{m+1}(p, 1).$$

It is also known (see [10]) that

$$(2.4) \quad \sum_{k=0}^m A_k(p, r) A_{m-k}(p, s) = A_m(p, r + s).$$

Now we are going to prove three identities, which will be needed later on.

Proposition 2.1.

$$(2.5) \quad \sum_{k=0}^m A_k(p - r, c) A_{m-k}(p, kr + d) = A_m(p, c + d).$$

Proof. It is straightforward to check that the formula is true for $m = 0$ and $m = 1$. Denoting the left hand side by $S_m(p, r, c, d)$ we have from (2.2):

$$\begin{aligned}
S_m(p, r, c, d) &= \sum_{k=0}^m A_k(p-r, c) A_{m-k}(p, kr+d) \\
&= \sum_{k=0}^m [A_k(p-r, c-1) + A_{k-1}(p-r, p-r+c-1)] A_{m-k}(p, kr+d) \\
&= \sum_{k=0}^m A_k(p-r, c-1) A_{m-k}(p, kr+d) \\
&\quad + \sum_{k=1}^m A_{k-1}(p-r, p-r+c-1) A_{m-k}(p, kr+d) \\
&= S_m(p, r, c-1, d) + \sum_{k=0}^{m-1} A_k(p-r, p-r+c-1) A_{m-1-k}(p, kr+r+d) \\
&= S_m(p, r, c-1, d) + S_{m-1}(p, r, p-r+c-1, r+d),
\end{aligned}$$

so that

$$S_m(p, r, c, d) = S_m(p, r, c-1, d) + S_{m-1}(p, r, p-r+c-1, r+d).$$

Now we fix m and assume that the formula holds for $m-1$. We will prove that for m it holds for every natural c . Indeed, it holds for $c=0$ and if it holds for $c-1$ then, by assumption,

$$\begin{aligned}
S_m(p, r, c, d) &= S_m(p, r, c-1, d) + S_{m-1}(p, r, p-r+c-1, r+d) \\
&= A_m(p, c+d-1) + A_{m-1}(p, p+c+d-1) = A_m(p, c+d),
\end{aligned}$$

which proves that the statement holds for c . Therefore it holds for all natural c . Now we note that both sides of (2.5) are polynomials on c of order m , therefore the formula holds for all c . \square

Proposition 2.2.

$$\begin{aligned}
(2.6) \quad &(1-t) \sum_{l=0}^m A_l(p, 1) \sum_{j=0}^{m-l} A_{m-l-j}(p, j(p-1)+r) t^j \\
&\quad + t \sum_{j=0}^m A_{m-j}(p, j(p-1)+r) t^j = A_m(p, r+1).
\end{aligned}$$

Proof. Using first (2.4) and then (2.2) we obtain:

$$\begin{aligned}
& t \sum_{j=0}^m A_{m-j}(p, j(p-1) + r) t^j \\
& \quad + (1-t) \sum_{l=0}^m A_l(p, 1) \sum_{j=0}^{m-l} A_{m-l-j}(p, j(p-1) + r) t^j \\
&= t \sum_{j=0}^m A_{m-j}(p, j(p-1) + r) t^j \\
& \quad + (1-t) \sum_{j=0}^m \sum_{l=0}^{m-j} A_l(p, 1) A_{m-j-l}(p, j(p-1) + r) t^j \\
&= t \sum_{j=0}^m A_{m-j}(p, j(p-1) + r) t^j \\
& \quad + (1-t) \sum_{j=0}^m A_{m-j}(p, j(p-1) + r + 1) t^j \\
&= \sum_{j=0}^m A_{m-j}(p, j(p-1) + r + 1) t^j - \sum_{j=0}^{m-1} A_{m-j-1}(p, j(p-1) + r + p) t^{j+1} \\
&= A_m(p, r + 1). \quad \square
\end{aligned}$$

Proposition 2.3.

$$(2.7) \quad \sum_{k=0}^m A_{m-k}(p, k(p-1) + r) p^k = \binom{mp+r}{m}.$$

Proof. Denoting the left hand side by $T_m(p, r)$ we have

$$\begin{aligned}
T_m(p, r) &= \sum_{k=0}^m A_{m-k}(p, k(p-1) + r) p^k \\
&= \sum_{k=0}^m [A_{m-k}(p, k(p-1) + r - 1) + A_{m-1-k}(p, k(p-1) + p + r - 1)] p^k \\
&= T_m(p, r - 1) + T_{m-1}(p, p + r - 1).
\end{aligned}$$

Now we proceed as before, using the binomial identity

$$\binom{mp+r-1}{m} + \binom{mp+r-1}{m-1} = \binom{mp+r}{m}. \quad \square$$

3. GENERATING FUNCTIONS

In this part we are going to study the generating functions

$$(3.1) \quad \mathcal{B}_p(z) := \sum_{m=0}^{\infty} A_m(p, 1) z^m,$$

which are convergent in some neighborhood of 0. From (2.4) and (2.3) we have

$$(3.2) \quad \mathcal{B}_p(z)^r = \sum_{m=0}^{\infty} A_m(p, r) z^m$$

and

$$(3.3) \quad \mathcal{B}_p(z) = 1 + z\mathcal{B}_p(z)^p.$$

Now we are going to interpret formulas (2.5), (2.6), (2.7) in terms of these generating functions.

Proposition 3.1. *For any real parameters p, r we have*

$$(3.4) \quad \mathcal{B}_{p-r}(z\mathcal{B}_p(z)^r) = \mathcal{B}_p(z).$$

Proof. First we note that if $A(z) = \sum_{m=0}^{\infty} a_m z^m$, $B(z) = \sum_{n=1}^{\infty} b_n z^n$ then

$$(3.5) \quad A(B(z)) = a_0 + \sum_{m=1}^{\infty} z^m \sum_{k=1}^m a_k \sum_{\substack{i_1, i_2, \dots, i_k \geq 1 \\ i_1 + i_2 + \dots + i_k = m}} b_{i_1} b_{i_2} \dots b_{i_k}.$$

Putting $b_i := A_{i-1}(p, r)$ for fixed k, m we have:

$$\begin{aligned} \sum_{\substack{i_1, i_2, \dots, i_k \geq 1 \\ i_1 + i_2 + \dots + i_k = m}} b_{i_1} b_{i_2} \dots b_{i_k} &= \sum_{\substack{j_1, j_2, \dots, j_k \geq 0 \\ j_1 + j_2 + \dots + j_k = m-k}} A_{j_1}(p, r) A_{j_2}(p, r) \dots A_{j_k}(p, r) \\ &= A_{m-k}(p, kr), \end{aligned}$$

the coefficient of $\mathcal{B}_p(z)^{kr}$ at z^{m-k} . Now we put $a_k := A_k(p-r, 1)$ and applying (2.5), with $c = 1$, $d = 0$, we get

$$(3.6) \quad \sum_{k=1}^m a_k \sum_{\substack{i_1, i_2, \dots, i_k \geq 1 \\ i_1 + i_2 + \dots + i_k = m}} b_{i_1} b_{i_2} \dots b_{i_k} = A_m(p, 1),$$

which completes the proof. \square

Note that in the proof we applied (2.5) only with $c = 1$ and $d = 0$.

For $p, r, t \in \mathbb{R}$ we denote

$$(3.7) \quad \mathcal{D}_{p,r,t}(z) := \frac{\mathcal{B}_p(z)^{1+r}}{(1-t)\mathcal{B}_p(z) + t}.$$

Proposition 3.2. *For $p, r, t \in \mathbb{R}$ we have*

$$(3.8) \quad \mathcal{D}_{p,r,t}(z) = \sum_{m=0}^{\infty} z^m \sum_{k=0}^m A_{m-k}(p, k(p-1) + r) t^k,$$

in particular:

$$(3.9) \quad \mathcal{D}_{p,r,p}(z) = \sum_{m=0}^{\infty} \binom{mp+r}{m} z^m.$$

Moreover

$$(3.10) \quad \mathcal{D}_{p-r,s,t}(z\mathcal{B}_p(z)^r) \mathcal{B}_p(z)^r = \mathcal{D}_{p,r+s,t}(z).$$

Proof. The formulas are consequences of (2.6), (2.7) and (3.4). \square

Proposition 3.3. *In some neighborhood of 0 we have*

$$(3.11) \quad \mathcal{B}_p(z(1+z)^{-p}) = 1+z,$$

and more generally, for $r \neq 0$ we have

$$(3.12) \quad \mathcal{B}_p\left(\left((1+z)^{\frac{1}{r}}-1\right)(1+z)^{\frac{-p}{r}}\right)^r = 1+z.$$

Proof. Since $\mathcal{B}_p(0) = 1$ and $\mathcal{B}'(0) = 1$ there is a function f_p defined on a neighborhood of 0 such that $f_p(0) = 0$ and $\mathcal{B}(f_p(z)) = 1+z$. Substituting $z \mapsto f_p(z)$ in (3.3) we obtain $f_p(z) = z(1+z)^{-p}$. Now we substitute $z \mapsto (1+w)^{1/r} - 1$ to (3.11) and taking the r -th power we obtain (3.12). \square

Remark. Note that (3.11) leads to an analytic proof of (3.4). Namely, substituting in (3.4) $z \mapsto z(1+z)^{-p}$ we get

$$\begin{aligned} \mathcal{B}_{p-r}\left(z(1+z)^{-p}\mathcal{B}_p\left(z(1+z)^{-p}\right)^r\right) &= \mathcal{B}_{p-r}\left(z(1+z)^{-p}(1+z)^r\right) \\ &= 1+z = \mathcal{B}_p\left(z(1+z)^{-p}\right). \end{aligned}$$

Finally we note a symmetry possessed by the generating functions.

Proposition 3.4. *For $p, r, t \in \mathbb{R}$ we have*

$$(3.13) \quad \mathcal{B}_p(-z)^r = \mathcal{B}_{1-p}(z)^{-r},$$

$$(3.14) \quad \mathcal{D}_{p,r,t}(-z) = \mathcal{D}_{1-p,-1-r,1-t}(z).$$

Proof. One can check that $(-1)^m A_m(p, r) = A_m(1-p, -r)$, which proves (3.13), and (3.13) leads to (3.14). \square

4. RELATIONS WITH NONCOMMUTATIVE PROBABILITY

By a *probability quasi-measure* we will mean a linear functional μ on the set $\mathbb{R}[x]$ of polynomials with real coefficients, satisfying $\mu(1) = 1$. In case when μ is given by $\mu(P) = \int P(t) d\tilde{\mu}(t)$ for some probability measure $\tilde{\mu}$ on \mathbb{R} we will identify μ with $\tilde{\mu}$ and say that μ is *proper* or is just a *probability measure*. A probability quasi-measure μ is uniquely determined by its *moment sequence* $\{\mu(x^m)\}_{m=0}^{\infty}$. It is proper if and only if its moment sequence is *positive definite*, i.e. if

$$\sum_{i,j=0}^{\infty} \mu(x^{i+j})\alpha_i\alpha_j \geq 0$$

holds for every sequence α_i of real numbers, with finitely many nonzero entries. All probability measures encountered in this paper are compactly supported and therefore uniquely determined by their moment sequences. For a probability quasi-measure μ we define its *generating function*, which is the (at least formal) power series

$$M_\mu(z) := \sum_{m=0}^{\infty} \mu(x^m)z^m$$

and its *reflection* $\hat{\mu}$ by $\hat{\mu}(x^m) := (-1)^m \mu(x^m)$ or, equivalently, by $M_{\hat{\mu}}(z) := M_\mu(-z)$. If μ is a probability measure then so is $\hat{\mu}$ and then we have $\hat{\mu}(X) = \mu(-X)$ for every borel subset of \mathbb{R} .

For $p, r, t \in \mathbb{R}$ we define probability quasi-measures $\mu(p, r)$ and $\nu(p, r, t)$ by

$$(4.1) \quad \mu(p, r)(x^m) := A_m(p, r),$$

$$(4.2) \quad \nu(p, r, t)(x^m) := \sum_{k=0}^m A_{m-k}(p, k(p-1) + r)t^k,$$

in particular

$$(4.3) \quad \nu(p, r, p)(x^m) = \binom{mp+r}{m}.$$

For example, $\mu(1, 1) = \nu(1, 0, 1) = \delta_1$ and for every $p \in \mathbb{R}$ we have $\mu(p, 0) = \nu(0, 0, 0) = \delta_0$. Note that $\nu(p, r, 0) = \mu(p, r)$ so that the class of probability quasi-measures $\mu(p, r)$ is contained in that of $\nu(p, r, t)$, we will be interested however mainly in the former.

First we note that Proposition 3.3 leads to the following formulas for the reflections.

Proposition 4.1.

$$(4.4) \quad \widehat{\mu(p, r)} = \mu(1-p, -r),$$

$$(4.5) \quad \widehat{\nu(p, r, t)} = \nu(1-p, -1-r, 1-t). \quad \square$$

There are several convolutions of probability quasi-measures, apart from the classical one: $(\mu * \nu)(x^n) := \sum_{k=0}^n \binom{n}{k} \mu(x^k) \nu(x^{n-k})$, which are related to various notions of independence (namely, the free, boolean and the monotonic independence) in noncommutative probability.

1. *Free convolution* (see [2, 12, 8]) is defined in the following way. For a probability quasi-measure μ we define its *free R -transform* (or the *additive free transform*) $R_\mu(z)$ by the formula:

$$(4.6) \quad M_\mu(z) = R_\mu(zM_\mu(z)) + 1.$$

If $R_\mu(z) = \sum_{k=1}^{\infty} r_k(\mu)z^k$ then $r_k(\mu)$ are called *free cumulants* of μ (combinatorial relation between moments and free cumulants is described in [8] and [4]). Then the free convolution $\mu \boxplus \nu$ can be defined as the unique probability quasi-measure which satisfies

$$(4.7) \quad R_{\mu \boxplus \nu}(z) = R_\mu(z) + R_\nu(z).$$

We also define *free power* $\mu^{\boxplus t}$ by $R_{\mu^{\boxplus t}}(z) := tR_\mu(z)$.

As a consequence of (4.6) and (3.4) we obtain:

Proposition 4.2. *For the free additive transform of $\mu(p, r)$ we have*

$$(4.8) \quad R_{\mu(p, r)}(z) = \mathcal{B}_{p-r}(z)^r - 1$$

so that the free cumulants of $\mu(p, r)$ are equal to $A_m(p-r, r)$, $m \geq 1$. \square

The *free S -transform* (or the *free multiplicative transform*) of a quasi-measure μ , with $\mu(x^1) \neq 0$, is defined by the relation

$$(4.9) \quad R_\mu(zS_\mu(z)) = z \quad \text{or} \quad M_\mu(z(1+z)^{-1}S_\mu(z)) = 1+z.$$

Then the *multiplicative free convolution* $\mu_1 \boxtimes \mu_2$ and the *multiplicative free power* $\mu^{\boxtimes t}$ are defined by

$$(4.10) \quad S_{\mu_1 \boxtimes \mu_2}(z) := S_{\mu_1}(z)S_{\mu_2}(z) \quad \text{and} \quad S_{\mu^{\boxtimes t}}(z) := S_\mu(z)^t.$$

Proposition 4.3. For $r \neq 0$ the S -transform of the measure $\mu(p, r)$ is equal to

$$(4.11) \quad S_{\mu(p,r)}(z) = \frac{(1+z)^{\frac{1}{r}} - 1}{z} (1+z)^{\frac{r-p}{r}}.$$

Consequently

$$(4.12) \quad \mu(1+p_1, 1) \boxtimes \mu(1+p_2, 1) = \mu(1+p_1+p_2, 1),$$

and more generally

$$(4.13) \quad \mu(p_1, r) \boxtimes \mu(1+p_2, 1) = \mu(p_1+rp_2, r).$$

We have also

$$(4.14) \quad \mu(1+p, 1)^{\boxtimes t} = \mu(1+tp, 1).$$

Proof. Formula (4.11) is a consequence of (3.12). In particular

$$(4.15) \quad S_{\mu(1+p,1)}(z) = (1+z)^{-p}$$

which leads to (4.12), (4.13) and (4.14). \square

2. The *boolean convolution* $\mu_1 \uplus \mu_2$ and the *boolean power* $\mu^{\uplus t}$ (see [11, 3]) can be defined by putting

$$(4.16) \quad \frac{1}{M_{\mu_1 \uplus \mu_2}(z)} = \frac{1}{M_{\mu_1}(z)} + \frac{1}{M_{\mu_2}(z)} - 1,$$

$$(4.17) \quad M_{\mu^{\uplus t}}(z) = \frac{M_{\mu}(z)}{(1-t)M_{\mu}(z) + t}.$$

Then definition (3.7) leads to:

Proposition 4.4. For $p, t \in \mathbb{R}$ we have

$$(4.18) \quad \mu(p, 1)^{\uplus t} = \nu(p, 0, t). \quad \square$$

3. *Monotonic convolution* (see [7]) is an associative, noncommuting operation $\mu_1 \triangleright \mu_2 = \mu$ which is defined by

$$(4.19) \quad M_{\mu}(z) = M_{\mu_1}(zM_{\mu_2}(z)) \cdot M_{\mu_2}(z).$$

Then (3.4) leads to:

Proposition 4.5. For any parameters $a, b, r, t \in \mathbb{R}$ we have

$$(4.20) \quad \mu(a, b) \triangleright \mu(a+r, r) = \mu(a+r, b+r),$$

$$(4.21) \quad \nu(a, b, t) \triangleright \mu(a+r, r) = \nu(a+r, b+r, t). \quad \square$$

In the next section we are going to study which of the probability quasi-measures $\mu(p, r)$ and $\nu(p, r, t)$ are actually probability measures. For this purpose we will use some of the the following facts, which are available in [12, 8, 11, 7]: The class of all compactly supported probability measures on \mathbb{R} is closed under the free, boolean, and monotonic convolution and also under taking the powers $\mu^{\boxplus s}$, and $\mu^{\uplus t}$, for $s, t \in \mathbb{R}$, $s \geq 1$, $t > 0$. Moreover, the class of probability measures with compact support contained in $[0, \infty)$ is closed under the free, multiplicative free and boolean convolution and under taking the powers $\mu^{\boxplus s}$, $\mu^{\boxtimes s}$ and $\mu^{\uplus t}$ for $s, t \in \mathbb{R}$, $s \geq 1$ and $t > 0$.

A probability measure μ on \mathbb{R} (resp. on $[0, \infty)$) is called \boxplus -infinitely divisible (resp. \boxtimes -infinitely divisible) if $\mu^{\boxplus t}$ (resp. $\mu^{\boxtimes t}$) is a probability measure for every $t > 0$. If μ

has compact support and r_m are its free cumulants then μ is \boxplus -infinitely divisible if and only if the sequence $\{r_{m+2}\}_{m=0}^{\infty}$ is positive definite.

5. POSITIVITY

The aim of this section is to study which of the quasi measures $\mu(p, r)$ and $\nu(p, r, t)$ are actually measures, i.e. for which parameters $p, r \in \mathbb{R}$ the sequence $\{A_m(p, r)\}_{m=0}^{\infty}$ is positive definite.

Theorem 5.1. *If $p \geq 1$, $0 \leq r \leq p$ then $\{A_m(p, r)\}_{m=0}^{\infty}$ is the moment sequence of a probability measure $\mu(p, r)$ with a compact support contained in $[0, \infty)$. If $p \leq 0$, $p-1 \leq r \leq 0$ then $\mu(p, r)$ is a probability measure which is the reflection of $\mu(1-p, -r)$.*

Proof. We know already that $\tilde{\pi} = \mu(2, 1)$ is the free Poisson law with parameter 1. Then, as was noted in [1], $\tilde{\pi}$ is \boxtimes -infinitely divisible and for $s > 0$ we have $\pi^{\boxtimes s} = \mu(1+s, 1)$. Hence if $p \geq 1$ then $\mu(p, 1)$ is a probability measure with a compact support contained in $[0, \infty)$. It implies that the sequence $A_m(p, p) = A_{m+1}(p, 1)$ is also positive definite, namely we have

$$\int_{\mathbb{R}} f(x) d\mu(p, p)(x) = \int_{\mathbb{R}} f(x)x d\mu(p, 1)(x).$$

Hence $\mu(p, p)$, $p \geq 1$, is a probability measure with a compact support contained in $[0, \infty)$. For $1 \leq r \leq p$ we apply (4.13) to obtain:

$$\mu(p, r) = \mu(r, r) \boxtimes \mu(p/r, 1).$$

For $r \in (0, 1)$ the measure $\mu(1, r)$ is related to the Euler beta function

$$(5.1) \quad B(a, b) := \int_0^1 x^{a-1}(1-x)^{b-1} dx.$$

We will use its properties: $B(a, 1-a) = \frac{\pi}{\sin a\pi}$ and $B(a, b) = \frac{a-1}{a+b-1}B(a-1, b)$. If we define

$$(5.2) \quad \mu_r := \frac{\sin \pi r}{\pi} x^{r-1}(1-x)^{-r} dx$$

on $[0, 1]$ then we have

$$\int_{\mathbb{R}} x^m d\mu_r(x) = \frac{\sin \pi r}{\pi} B(m+r, 1-r) = \prod_{k=1}^m \frac{r+k-1}{k} = A_m(1, r).$$

which means that $\mu(1, r) = \mu_r$. Now for $s \geq 0$ we have

$$\mu(1+rs, r) = \mu(1, r) \boxtimes \mu(1+s, 1),$$

which ends the proof of the first part. The second one is a consequence of (4.4). \square

We conjecture that the last theorem fully characterizes the set of parameters $p, r \in \mathbb{R}$ for which $\mu(p, r)$ is a measure (apart from the trivial case $\mu(p, 0) = \delta_0$). Note that $A_0(p, r)A_2(p, r) - A_1(p, r)^2 = r(2p-1-r)/2$ so a necessary condition for positive definiteness of the sequence $A_m(p, r)$ is that $r(2p-1-r) \geq 0$.

Remark. According to Penson and Solomon [9]:

$$(5.3) \quad \mu(3, 1) = \frac{\sqrt[6]{108}(2^{1/3}(27 + 3\sqrt{81 - 12x})^{2/3} - 6x^{1/3})}{12\pi x^{2/3}(27 + 3\sqrt{81 - 12x})^{1/3}} dx$$

on $[0, 27/4]$.

Corollary 5.1. *If either $0 \leq 2r \leq p$, $r + 1 \leq p$ or $p \leq 2r + 1$, $p \leq r \leq 0$ then $\mu(p, r)$ is \boxplus -infinitely divisible.*

Proof. By Theorem 13.16 in [8] a compactly supported probability measure μ , with $R_\mu(z) = \sum_{m=1}^{\infty} r_m z^m$, is \boxplus -infinitely divisible if and only if the sequence $\{r_{m+2}\}_{m=0}^{\infty}$ is positive definite. Then it is sufficient to refer to (4.8) and to note that $\{A_{m+2}(p - r, r)\}_{m=0}^{\infty}$ is the moment sequence of the measure $x^2 d\mu(p - r, r)(x)$. \square

Corollary 5.2. *If either $0 \leq r \leq p - 1$, $t > 0$ or $p \leq 1 + r \leq 0$, $t < 1$ then $\nu(p, r, t)$ is a probability measure. In particular, if either $0 \leq r \leq p - 1$ or $p \leq 1 + r \leq 0$ then the sequence $\left\{\binom{mp+r}{m}\right\}_{m=0}^{\infty}$ is positive definite.*

Proof. For $0 \leq r \leq p - 1$, $t > 0$ we apply (4.21) and (4.18):

$$\nu(p, r, t) = \nu(p - r, 0, t) \triangleright \mu(p, r) = \mu(p - r, 1)^{\boxplus t} \triangleright \mu(p, r)$$

and Theorem 5.1. Then we use (4.5). \square

A measure ν on \mathbb{R} is called *symmetric* if $\widehat{\nu} = \nu$. For a probability quasi-measure μ define its *symmetrization* μ^s by $M_{\mu^s}(z) := M_\mu(z^2)$. If μ is a probability measure with support contained in $[0, \infty)$ then μ^s is a symmetric measure on \mathbb{R} , which satisfies $\int_{\mathbb{R}} f(t^2) d\mu^s(t) = \int_{\mathbb{R}} f(t) d\mu(t)$ for every compactly supported continuous function on \mathbb{R} . Denote by $\mu^s(p, r)$ and $\nu^s(p, r, t)$ the symmetrization of $\mu(p, r)$ and $\nu(p, r, t)$. Then, by (3.4) and (4.9), for the free additive transform we have

$$(5.4) \quad R_{\mu^s(p, r)}(z) = \mathcal{B}_{p-2r}(z^2)^r - 1.$$

In the same way as Corollary 5.2 one can prove

Corollary 5.3. *If $p \geq 1$, $0 \leq r \leq p$ then $\mu^s(p, r)$ is a symmetric probability measure on \mathbb{R} . Moreover, if $p - 2r \geq 1$ and $0 \leq 3r \leq p$ then $\mu^s(p, r)$ is \boxplus -infinitely divisible. \square*

Note also that

$$(5.5) \quad \mu^s(p, 1)^{\boxplus t} = \nu^s(p, 0, t),$$

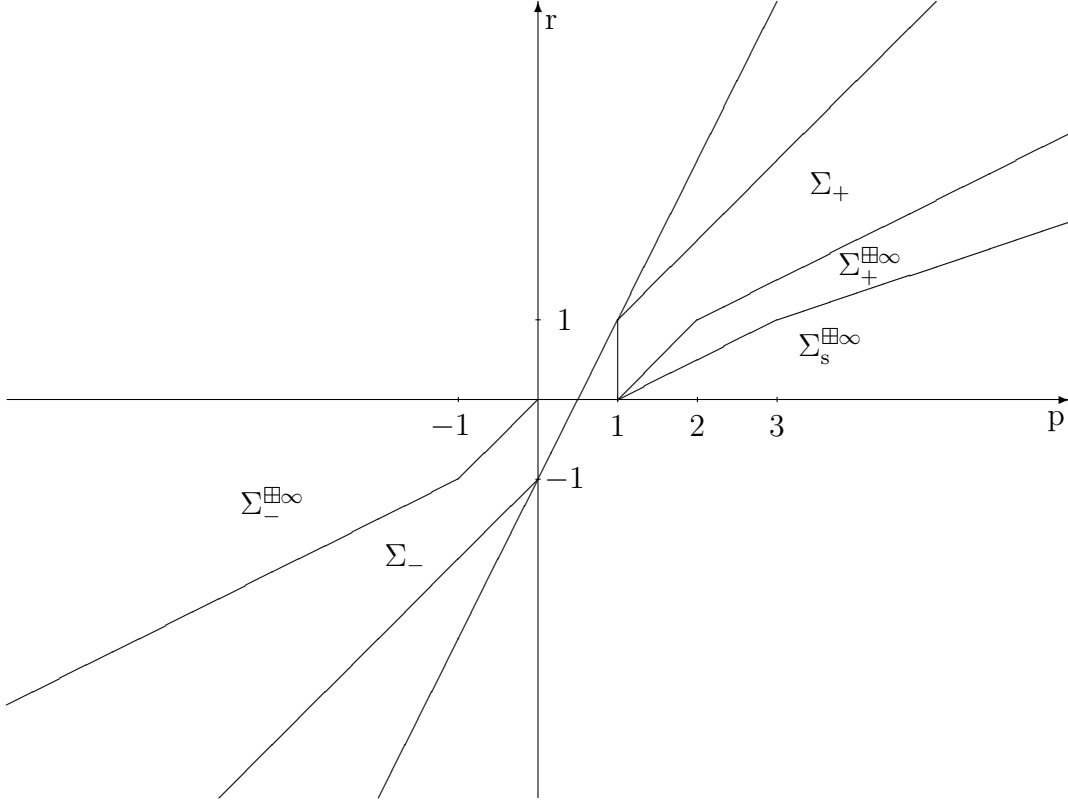
$$(5.6) \quad \mu^s(a, b) \triangleright \mu^s(a + 2r, r) = \mu^s(a + 2r, b + r),$$

$$(5.7) \quad \nu^s(a, b, t) \triangleright \mu^s(a + 2r, r) = \nu^s(a + 2r, b + r, t).$$

Corollary 5.4. *Assume that $0 \leq 2r \leq p - 1$ and $t > 0$. Then $\nu^s(p, r, t)$ is a symmetric probability measure on \mathbb{R} . Equivalently, the support of the measure $\nu(p, r, t)$ is contained in $[0, \infty)$.*

Proof. Using (5.7) and (5.5) we have that

$$\nu^s(p, r, t) = \nu^s(p - 2r, 0, t) \triangleright \mu^s(p, r) = \mu^s(p - 2r, 1)^{\boxplus t} \triangleright \mu^s(p, r). \quad \square$$

5.1. Picture for $\mu(p, r)$.


Here we illustrate the main results concerning the measures $\mu(p, r)$.

- (1) If $\mu(p, r)$ is a probability measure then $r(2p - 1 - r) \geq 0$ (the right-top and left-bottom sector between the p -axis and the line $r = 2p - 1$),
- (2) Σ_+ (including $\Sigma_+^{\boxplus\infty}$ and $\Sigma_s^{\boxplus\infty}$): $\mu(p, r)$ is a probability measure with a compact support contained in $[0, \infty)$,
- (3) Σ_- (including $\Sigma_-^{\boxplus\infty}$): $\mu(p, r)$ is a probability measure, the reflection of $\mu(1 - p, -r)$,
- (4) $\Sigma_+^{\boxplus\infty} \cup \Sigma_-^{\boxplus\infty}$ (including $\Sigma_s^{\boxplus\infty}$): $\mu(p, r)$ is \boxplus -infinitely divisible,
- (5) $\Sigma_s^{\boxplus\infty}$: the symmetrization of $\mu(p, r)$ is \boxplus -infinitely divisible.

6. DILATIONS

For a probability quasi-measure μ we define its *dilation with parameter* $c > 0$ by $(D_c\mu)(x^m) := c^m\mu(x^m)$. Then for the moment generating function we have: $M_{D_c\mu}(z) = M_\mu(cz)$ and similarly for the free R -transform: $R_{D_c\mu}(z) = R_\mu(cz)$, while for the S -transform we have $S_{D_c\mu}(z) = \frac{1}{c}S_\mu(z)$. If μ is proper then we have $(D_c\mu)(X) = \mu(\frac{1}{c}X)$ for every borel subset X of \mathbb{R} . In this part we are going to study dilations of the measures $\mu(p, r)$ and $\nu(p, r, t)$ and their limits as the parameter goes to 0.

For parameters $h \geq 0$ and $a, p, r \in \mathbb{R}$ define sequences

$$(6.1) \quad \binom{a}{m}_h := \frac{1}{m!} \prod_{i=0}^{m-1} (a - ih),$$

$$(6.2) \quad A_m(p, r, h) := \frac{r}{m!} \prod_{i=1}^{m-1} (mp + r - ih),$$

with $A_0(p, r, h) := 1$. In particular $A_m(p, r, h) = \frac{r}{mp+r} \binom{mp+r}{m}_h$ whenever $mp + r \neq 0$. Then, for $h \geq 0$ and $p, r, t \in \mathbb{R}$ define probability quasi-measures:

$$(6.3) \quad \mu(p, r, h)(x^m) := A_m(p, r, h),$$

$$(6.4) \quad \nu(p, r, t, h)(x^m) := \sum_{k=0}^m A_{m-k}(p, k(p-h) + r, h)t^k.$$

and their moment generating functions $\mathcal{B}_{p,r,h}(z)$ and $\mathcal{D}_{p,r,t,h}(z)$ respectively.

Note that if h is strictly positive then $A_m(p, r, h) = h^m A_m(p/h, r/h)$ and hence these measures can be represented as dilations:

$$(6.5) \quad \mu(p, r, h) = D_h \mu(p/h, r/h),$$

$$(6.6) \quad \nu(p, r, t, h) = D_h \nu(p/h, r/h, t/h).$$

Therefore the corresponding moment generating functions are

$$(6.7) \quad \mathcal{B}_{p,r,h}(z) = \mathcal{B}_{p/h}(hz)^{r/h},$$

$$(6.8) \quad \mathcal{D}_{p,r,t,h}(z) = \mathcal{D}_{p/h,r/h,t/h}(hz) = \frac{h\mathcal{B}_{p,h+r,h}(z)}{(h-t)\mathcal{B}_{p,h,h}(z) + t}.$$

These formulas allow us to derive properties of the probability quasi-measures $\mu(p, r, h)$ and $\nu(p, r, t, h)$ directly from our previous results when $h > 0$, and after taking the limit with $h \rightarrow 0$ if $h = 0$.

Proposition 6.1. *For $h > 0$ and $p, r, t \in \mathbb{R}$*

$$(6.9) \quad \mathcal{B}_{p,h,h}(z) = 1 + zh\mathcal{B}_{p,p,h}(z),$$

$$(6.10) \quad \log(\mathcal{B}_{p,1,0}(z)) = z\mathcal{B}_{p,p,0}(z),$$

$$(6.11) \quad \mathcal{D}_{p,r,t,0}(z) = \frac{\mathcal{B}_{p,r,0}(z)}{1 - zt\mathcal{B}_{p,p,0}(z)}.$$

Proof. First formula is a consequence of (3.3) and (6.7). Then we have

$$\frac{\mathcal{B}_{p,1,h}(z)^h - 1}{h} = \frac{\mathcal{B}_{p,h,h}(z) - 1}{h} = z\mathcal{B}_{p,p,h}(z).$$

Taking the limit with $h \rightarrow 0$ we obtain (6.10).

For (6.11) we write use (6.8) and (6.9) to get

$$\frac{1}{h} [(h-t)\mathcal{B}_{p,h,h}(z) + t] = 1 - (t-h) \frac{\mathcal{B}_{p,h,h}(z) - 1}{h} = 1 - (t-h)z\mathcal{B}_{p,p,h}(z)$$

and then we take limit with $h \rightarrow 0$. □

Proposition 6.2. *For $h \geq 0$ and $p, r, s \in \mathbb{R}$ we have*

$$(6.12) \quad \mathcal{B}_{p-r,s,h}(z\mathcal{B}_{p,r,h}(z)) = \mathcal{B}_{p,s,h}(z). \quad \square$$

Proposition 6.3. For $h \geq 0$ and $p, r \in \mathbb{R}$ we have

$$(6.13) \quad \nu(p, r, p, h)(x^m) = \binom{mp+r}{m}_h.$$

Proof. For $h > 0$ it is a consequence of (6.6). Then we take limit with $h \rightarrow 0$. \square

Proposition 6.4. For $h \geq 0$ and $p, r, t \in \mathbb{R}$ we have

$$(6.14) \quad \widehat{\mu(p, r, h)} = \mu(h-p, -r, h),$$

$$(6.15) \quad \widehat{\nu(p, r, t, h)} = \nu(h-p, -h-r, h-t, h).$$

Proof. It holds because $A_m(p, r, h)(-1)^m = A_m(h-p, -r, h)$. \square

Proposition 6.5. For the free transforms we have

$$(6.16) \quad R_{\mu(p, r, h)}(z) = \mathcal{B}_{p-r, r, h}(z) - 1$$

$$(6.17) \quad S_{\mu(p, r, h)}(z) = \frac{(1+z)^{h/r} - 1}{hz} (1+z)^{(r-p)/r} \quad \text{for } h > 0,$$

$$(6.18) \quad S_{\mu(p, r, 0)}(z) = \frac{\log(1+z)}{rz} (1+z)^{(r-p)/r},$$

$$(6.19) \quad S_{\nu(p, 0, t, 0)}(z) = \frac{1}{t} e^{\frac{-pz}{t(1+z)}}.$$

In particular $\nu(p, 0, t, 0) = D_t(\nu(1, 0, 1, 0)^{\boxtimes p/t})$.

Proof. Formulas (6.16, 6.17) are consequences of (6.7, 4.11 and 6.12). Therefore, for $h > 0$ we have

$$(6.20) \quad \mathcal{B}_{p, r, h} \left(\frac{(1+z)^{h/r} - 1}{h} (1+z)^{-p/r} \right) = 1+z,$$

which leads to

$$(6.21) \quad \mathcal{B}_{p, r, 0} \left(\frac{\log(1+z)}{r(1+z)^{p/r}} \right) = 1+z$$

and to (6.18). In particular, substituting $(1+z) \mapsto e^{\frac{pz}{t(1+z)}}$, we have

$$(6.22) \quad \mathcal{B}_{p, p, 0} \left(\frac{z}{t(1+z)} e^{\frac{-pz}{t(1+z)}} \right) = e^{\frac{pz}{t(1+z)}}$$

which, combined with (6.11) gives

$$(6.23) \quad \mathcal{D}_{p, 0, t, 0} \left(\frac{z}{t(1+z)} e^{\frac{-pz}{t(1+z)}} \right) = \frac{1}{1 - \frac{z}{1+z}} = 1+z. \quad \square$$

Proposition 6.6. For $h > 0$ and $p, t \in \mathbb{R}$ we have

$$(6.24) \quad \mu(p, h, h)^{\boxplus t} = \nu(p, 0, th, h),$$

$$(6.25) \quad \nu(p, 0, 1, 0)^{\boxplus t} = \nu(p, 0, t, 0).$$

Proof. (6.25) is a consequence of (6.11). \square

Proposition 6.7. For $h \geq 0, t > 0, a, b \in \mathbb{R}$ we have

$$(6.26) \quad \mu(a, b, h) \triangleright \mu(a+r, r, h) = \mu(a+r, b+r, h),$$

$$(6.27) \quad \nu(a, b, t, h) \triangleright \mu(a+r, r, h) = \nu(a+r, b+r, t, h). \quad \square$$

Proposition 6.8. *Assume that $h \geq 0$.*

1. *If $p \geq h$ and $0 \leq r \leq p$ then $\mu(p, r, h)$ is a probability measure with support contained in $[0, \infty)$. If $p \leq 0$, $p - h \leq r \leq 0$ then $\mu(p, r)$ is a probability measure which is the reflection of $\mu(h - p, -r)$.*

2. *If either $0 \leq 2r \leq p$, $r + h \leq p$ or $p \leq 2r + h$, $p \leq r \leq 0$ then $\mu(p, r, h)$ is \boxplus -infinitely divisible.*

3. *If either $0 \leq r \leq p - h$, $t > 0$ or $p \leq h + r \leq 0$, $t < h$ then $\nu(p, r, t, h)$ is a probability measure. In particular, if either $0 \leq r \leq p - h$ or $p \leq h + r \leq 0$ then the sequence $\left(\frac{m^{p+r}}{m}\right)_h$, $m \geq 0$, is positive definite. \square*

We conclude with remarks on the probability measure $\nu_0 := \nu(1, 0, 1, 0)$ such that $\nu_0(x^m) = \binom{m}{0}_0 = \frac{m^m}{m!}$. From (4.9), (6.19) we have

$$(6.28) \quad S_{\nu_0}(z) = e^{\frac{-z}{1+z}},$$

$$(6.29) \quad R_{\nu_0}(ze^{\frac{-z}{1+z}}) = z,$$

$$(6.30) \quad M_{\nu_0}\left(\frac{z}{1+z}e^{\frac{-z}{1+z}}\right) = 1 + z.$$

Theorem 6.1. *The sequence $\left\{\frac{m^m}{m!}\right\}_{m=0}^{\infty}$ is positive definite and the corresponding probability measure ν_0 has compact support contained in $[0, e]$. Moreover, μ_0 is \boxtimes -infinitely divisible.*

Proof. Since $\frac{m^m}{m!} = \lim_{h \rightarrow 0} \binom{m}{m}_h$ holds for every m so both the sequences $\left\{\frac{m^m}{m!}\right\}_{m=0}^{\infty}$, $\left\{\frac{m^m}{m!}\right\}_{m=1}^{\infty}$ are positive definite. Moreover, $\lim_{m \rightarrow \infty} \sqrt[m]{\frac{m^m}{m!}} = e$, so the corresponding probability measure has support contained in $[0, e]$.

For the \boxtimes -infinite divisibility we recall (see Theorem 3.7.3 in [2]) that a probability measure μ with support contained in $[0, \infty)$ is \boxtimes -infinite divisible if and only if the function $\Sigma_{\mu}(z) := S_{\mu}(z(1-z)^{-1})$, can be expressed as $\Sigma_{\mu}(z) = e^{v(z)}$, where $v : \mathbb{C} \setminus [0, \infty) \mapsto \mathbb{C}$ is analytic, satisfies $v(\bar{z}) = \overline{v(z)}$ and maps the upper half-plane \mathbb{C}^+ into the lower half-plane \mathbb{C}^- . In our case we have $\Sigma_{\nu_0}(z) = e^{-z}$ and the function $v(z) = -z$ does satisfy these assumptions. \square

Let us briefly reconstruct the way we have obtained the measure ν_0 . Put $\tilde{\pi} := \mu(2, 1, 1)$, the free Poisson measure. Then $\mu(p, h, h) = D_h \mu(p/h, 1, 1) = D_h(\tilde{\pi}^{\boxtimes \frac{p}{h}-1})$ so putting $h = 1/n$ and $p = 1$ we have

$$(6.31) \quad \left(D_{\frac{1}{n}}(\tilde{\pi}^{\boxtimes n-1})\right)^{\uplus n} \longrightarrow \nu_0, \quad \text{with } n \rightarrow \infty,$$

where the convergence here means that the m -th moment of $\left(D_{\frac{1}{n}}(\tilde{\pi}^{\boxtimes n-1})\right)^{\uplus n}$ tends to $\frac{m^m}{m!}$. Note also that from (6.29) one can calculate free cumulants of ν_0 : $r_1 = 1$, $r_2 = 1$, $r_3 = \frac{1}{2}$, $r_4 = \frac{-1}{3}$. Since $r_4 < 0$ the measure ν_0 is not \boxplus -infinitely divisible.

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