Irreducible Representations of the Free Product of Groups

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ABSTRACT. We study some properties, such as uniform boundedness, unitarity and irreducibility, of a class of representations of the free product of groups. In particular we show that the spherical functions on the free product of two groups, introduced by Cartwright and Soardi, are coefficients of irreducible representations.

INTRODUCTION

There have been several attempts to construct representations of free product of groups or algebras. One family of such constructions is related to the commutative convolution algebra of radial functions on the free group $*_{i\in I}\mathbb{Z}$, see [8, 9, 14, 15, 19, 29, 33], or on the free product of cyclic groups of the same order, see [13, 36]. On the other hand there are developed methods to produce a representation of $G = *_{i\in I} G_i$ (or, more generally, of a unital free product $\mathcal{A} = *_{i\in I}\mathcal{A}_i$ of $*$-algebras $\mathcal{A}_i$) from those of G_i's (or $\mathcal{A}_i$'s), see [1, 2, 4, 5, 22–25, 34, 35].

Much effort has been devoted to study irreducibility of such representations. For those related to radial functions the method is to study the projection on a cyclic vector, see [8, 9, 13, 19, 29, 31]. Interesting constructions of irreducible representations of the free group are due to Kuhn, Steger [17, 18] and Paschke [27, 28]. Let us also mention papers by Młotkowski [22], Kuhn and Steger [16]. The former proves irreducibility for a family of representations on the free product of *infinite* groups, the latter proves that for a family of representations of the *infinitely generated* free group.

The starting point of this paper is the observation that for the free product group $G = *_{i\in I} G_i$ there is an associated unital, noncommutative (unless $|I| = 1$) $*$-algebra $\mathcal{A}(\tau)$ depending only on the parameters $\tau_i := (|G_i| - 1)^{-1}$ (cf. [21]).

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If all G_i's are finite, then $\mathcal{A}(\tau)$ can be identified with the convolution algebra of finitely supported type dependent functions on G. For a representation π_0 of $\mathcal{A}(\tau)$, acting on a Hilbert space, we construct in Section 5 a representation π of G, acting on a larger Hilbert space. We say that π is *induced* from π_0. We prove that π inherits many properties of π_0, namely:

(1) If π_0 is a $*$-representation, then π is unitary (Theorem 5.5 (iv)).
(2) If π_0 satisfies some version of uniform boundedness, then π is uniformly bounded (Theorem 5.5 (v)).
(3) If π_0 and σ_0 are representations of $\mathcal{A}(\tau)$ which are disjoint from the regular representation of $\mathcal{A}(\tau)$ and if π_0 and σ_0 are not equivalent (resp. π_0 and σ_0 are disjoint), then so are the induced representations (Theorem 5.10).
(4) If a $*$-representation π_0 of $\mathcal{A}(\tau)$ is weakly contained in a $*$-representation σ_0, then the same holds for the induced unitary representations (Theorem 5.11). In particular, if π_0 is weakly contained in the regular representation of $\mathcal{A}(\tau)$ (defined in Section 2), then the induced representation is weakly contained in the regular representation of G. In Section 3 we adapt Haagerup's [11] method to characterize those representations of $\mathcal{A}(\tau)$ which are weakly contained in the regular representation.
(5) Finally we prove that if π_0 is a $*$-representation of $\mathcal{A}(\tau)$ which is irreducible, then so is π, *unless* π_0 is contained in the regular representation of $\mathcal{A}(\tau)$ (Theorem 6.4). We will see in Section 2 that there are at most two irreducible subrepresentations of the regular representations of $\mathcal{A}(\tau)$, and they are one-dimensional.

In the nonunitary case, we have managed to prove that if π_0 is a *finitely dimensional* irreducible representation of $\mathcal{A}(\tau)$, not equivalent to a $*$-representation, then π is fully irreducible (Theorem 6.8).

In the last section we apply our results to the free product of two groups $G = G_+ * G_-$, with $|G_+| = r$, $|G_-| = s$, $r > s \geq 2$ (an important example is $\mathbb{Z}_3 * \mathbb{Z}_2 \cong \mathrm{PSL}(2,\mathbb{Z})$). Cartwright and Soardi [6] studied a family of spherical functions φ_λ on such group, with $\lambda \in \mathbb{C}$, $\lambda \neq (r+s-4)/2$. Here we prove that every φ_λ is a coefficient of a fully irreducible (unless $\lambda = -2$ or $\lambda = s-2$) representation of G. Moreover, this representation is unitary if and only if $\lambda \in [-2,\ s-2] \cup [r-2,\ r+s-2]$.

1. THE ALGEBRA $\mathcal{A}(\tau)$

If X is a set, then $\mathcal{F}(X)$ (resp. $\mathcal{F}_0(X)$) will stand for the class of finitely supported complex functions on X (which satisfy $\sum_{x\in X} f(x) = 0$). For two complex functions f, g on X we put $\langle f, g\rangle := \sum_{x\in X} f(x)g(x)$ whenever the sum is well defined.

We recall some notions which were studied in [21]. Let I be a nonempty fixed set and let $S(I)$ denote the set of all formal words of the form

$$u = i_1 i_2 \ldots i_m, \quad \text{with } m \geq 0,\ i_k \in I \text{ and } i_k \neq i_{k+1}, \text{ for all } k < m, \tag{1.1}$$

including the empty word e. The length m will be denoted by $|u|$. For $i \in I$ we define

$$S_i := \{i_1 i_2 \dots i_m \in S(I) : m = 0 \text{ or } i_1 \neq i\}. \tag{1.2}$$

Let τ be a function $I \to [0, +\infty)$. We extend τ to $S(I)$ by putting

$$\tau(u) := \tau_{i_1} \tau_{i_2} \cdots \tau_{i_m} \tag{1.3}$$

for u as in (1.1), in particular $\tau(e) = 1$. Then we define τ-*convolution* as an associative operation on $\mathcal{F}(S(I))$, such that δ_e is the unit and for each u as in (1.1), with $|u| \geq 1$,

$$\delta_i *_\tau \delta_u = \begin{cases} (1-\tau_i)\delta_u + \tau_i \delta_{u'} & \text{if } i = i_1, \\ \delta_{iu} & \text{if } i \neq i_1, \end{cases} \tag{1.4}$$

where $u' = i_2 \cdots i_m$ and $iu = i i_1 \cdots i_m$ (δ_x denotes the characteristic function of the point set $\{x\}$). We also define involution by $f^*(u) := \overline{f(u^*)}$, where $u^* := i_m \cdots i_2 i_1$ for u as in (1.1). In this way $\mathcal{F}(S(I))$ becomes a unital $*$-algebra, which we will denote by $\mathcal{A}(\tau)$. Note that $\mathcal{A}(\tau)$ can be defined as the unital free $*$-algebra on I, subject to the relations $i^2 = (1-\tau_i)i + \tau_i e$ and $i^* = i$ for $i \in I$, and is also an example of generic algebra (see [12]).

Observe that the *characters* of $\mathcal{A}(\tau)$, i.e., functions φ on $S(I)$ such that $\langle \varphi, f *_\tau g \rangle = \langle \varphi, f \rangle \langle \varphi, g \rangle$ holds for all f, $g \in \mathcal{F}(S(I))$, are of the form $\varphi(u) = \varphi(i_1) \cdots \varphi(i_m)$ for u as in (1.1), where $\varphi(i) \in \{1, -\tau_i\}$. In particular, taking $\varphi \equiv 1$ we see that $\mathcal{F}_0(S(I)) = \ker \varphi$ is an ideal in $\mathcal{A}(\tau)$.

For two functions σ, $\tau : I \to [0, +\infty)$ we define a $*$-isomorphism $H_{\tau\sigma} : \mathcal{A}(\sigma) \to \mathcal{A}(\tau)$ putting

$$H_{\tau\sigma}(\delta_i) := \frac{1+\sigma_i}{1+\tau_i}\delta_i + \frac{\tau_i - \sigma_i}{1+\tau_i}\delta_e$$

for each $i \in I$. Indeed, one can check that

$$H_{\tau\sigma}(\delta_i) *_\tau H_{\tau\sigma}(\delta_i) = (1-\sigma_i) H_{\tau\sigma}(\delta_i) + \sigma_i \delta_e.$$

Observe also that for $f, g \in \mathcal{F}(S(I))$

$$(f *_\tau g^*)(e) = (g^* *_\tau f)(e) = \sum_{u \in S(I)} f(u)\overline{g(u)}\tau(u). \tag{1.5}$$

Definition 1.1. A function $\varphi : S(I) \to \mathbb{C}$ is said to be τ-*positive definite* (resp. τ-*negative definite*) if $\langle \varphi, f^* *_\tau f \rangle \geq 0$ (resp. $\langle \varphi, f_0^* *_\tau f_0 \rangle \leq 0$) holds for every $f \in \mathcal{F}(S(I))$ (resp. for every $f_0 \in \mathcal{F}_0(S(I))$). We will denote by $\mathcal{P}(\tau)$ and $\mathcal{N}(\tau)$ the class of τ-positive and τ-negative definite functions on $S(I)$ respectively, and by $B(\tau)$ the class of linear combinations of τ-positive definite functions on $S(I)$, i.e., the τ-*Fourier-Stielties algebra*.

In particular, the characters of $\mathcal{A}(\tau)$ are τ-positive definite.

Proposition 1.2. *Suppose that for every $i \in I$ we have $0 \le \sigma_i \le \tau_i$. Then*

$$\mathcal{P}(\sigma) \subseteq \mathcal{P}(\tau) \quad \text{and} \quad \mathcal{N}(\sigma) \subseteq \mathcal{N}(\tau).$$

Proof. It is sufficient to show that for every $f \in \mathcal{F}(S(I))$ we have

$$f^* *_\tau f = f^* *_\sigma f + R,$$

where R is a finite sum of terms of the form $f_0^* *_\sigma f_0$, with $f_0 \in \mathcal{F}_0(S(I))$. We proceed by induction on n, the maximal length of words in the support of f. Decompose f as

$$f = f(e)\delta_e + \sum_{i\in I} \delta_i * f_i$$

(we write simply "$*$" whenever "$*_\tau$" can be replaced by "$*_\sigma$") with supp $f_i \subseteq S_i$, see (1.2). Then, by induction, we can write $f_i^* *_\tau f_i = f_i^* *_\sigma f_i + R(i)$. Now we have

$$f^* *_\tau f - f^* *_\sigma f = \sum_{i\in I} \left\{ (\delta_i * f_i)^* *_\tau (\delta_i * f_i) - (\delta_i * f_i)^* *_\sigma (\delta_i * f_i) \right\}$$

and

$$\begin{aligned}
&(\delta_i * f_i)^* *_\tau (\delta_i * f_i) \\
&\qquad = (1-\tau_i) f_i^* * \delta_i * f_i + \tau_i f_i^* *_\tau f_i \\
&\qquad = f_i^* *_\sigma \big((1-\tau_i)\delta_i + \tau_i \delta_e\big) *_\sigma f_i + \tau_i R(i) \\
&\qquad = (\delta_i * f_i)^* *_\sigma (\delta_i * f_i) + (\tau_i - \sigma_i) f_i^* *_\sigma (\delta_e - \delta_i) *_\sigma f_i + \tau_i R(i).
\end{aligned}$$

To conclude we note that

$$\delta_e - \delta_i = \frac{1}{1+\sigma_i} (\delta_e - \delta_i) *_\sigma (\delta_e - \delta_i). \qquad \square$$

We will identify, by $\varphi(u) := \varphi(\delta_u)$, the dual space $\mathcal{F}(S(I))'$ with the space of complex functions on $S(I)$. Denote by $T_{\sigma\tau}$ the dual map to $H_{\tau\sigma}$, i.e., $T_{\sigma\tau}(\varphi) = \varphi \circ H_{\tau\sigma}$.

Proposition 1.3. *$T_{\sigma\tau}$ maps $\mathcal{P}(\tau)$ onto $\mathcal{P}(\sigma)$, $\mathcal{N}(\tau)$ onto $\mathcal{N}(\sigma)$ and $B(\tau)$ onto $B(\sigma)$.*

Proof. For a τ-positive definite function ψ and for $f \in \mathcal{F}(S(I))$ we have

$$\langle T_{\sigma\tau}\psi, f^* *_\sigma f \rangle = \langle \psi, H_{\tau\sigma}(f^* *_\sigma f) \rangle = \langle \psi, H_{\tau\sigma}(f)^* *_\tau H_{\tau\sigma}(f) \rangle \ge 0. \qquad \square$$

Proposition 1.4. *Assume that* $\{P_i\}_{i\in I}$ *is a family of projections on a Hilbert space* $\mathcal{H}_0$, ζ, $\eta \in \mathcal{H}_0$ *and let* $A_i = (1+\tau_i)P_i - \tau_i \operatorname{Id}$, $B_i = (1+\sigma_i)P_i - \sigma_i \operatorname{Id}$. *Define* $\varphi(u) = [A_{i_1}A_{i_2}\cdots A_{i_m}\zeta, \eta]$ *and* $\psi(u) = [B_{i_1}B_{i_2}\cdots B_{i_m}\zeta, \eta]$ *for* $u = i_1 i_2 \cdots i_m \in S(I)$. *Then* $\psi = T_{\sigma\tau}\varphi$.

Proof. Let $A : \mathcal{A}(\tau) \to \mathcal{B}(\mathcal{H}_0)$ and $B : \mathcal{A}(\sigma) \to \mathcal{B}(\mathcal{H}_0)$ be the unique homomorphisms which satisfy $A(\delta_i) = A_i$ and $B(\delta_i) = B_i$. Since

$$B_i = \frac{1+\sigma_i}{1+\tau_i}A_i + \frac{\tau_i - \sigma_i}{1+\tau_i}\operatorname{Id},$$

we have $B = A \circ H_{\tau\sigma}$. Now define $\Phi : \mathcal{B}(\mathcal{H}_0) \to \mathbb{C}$ putting $\Phi(T) := [T\zeta, \eta]$. Then

$$\psi = \Phi \circ B = \Phi \circ A \circ H_{\tau\sigma} = \varphi \circ H_{\tau\sigma} = T_{\sigma\tau}(\varphi).$$ □

Now we prove that τ-positive definiteness admits the standard GNS construction:

Proposition 1.5. *Let* φ *be a complex function on* $S(I)$ *and let* τ *be a function* $I \to [0, +\infty)$. *Then* φ *is* τ*-positive definite* (*resp. belongs to* $B(\tau)$) *if and only if there exists a Hilbert space* $\mathcal{H}_0$, *a vector* $\xi \in \mathcal{H}_0$ (*resp. vectors* ζ, $\eta \in \mathcal{H}_0$) *and a family* $\{P_i\}_{i\in I}$ *of orthogonal projections on* $\mathcal{H}_0$ *that for every* $u = i_1 i_2 \cdots i_m \in S(I)$

$$\varphi(u) = [A_{i_1}A_{i_2}\cdots A_{i_m}\xi, \xi] \quad (\text{resp. } \varphi(u) = [A_{i_1}A_{i_2}\cdots A_{i_m}\zeta, \eta]),$$

where $A_i = (1+\tau_i)P_i - \tau_i \operatorname{Id}$.

Proof. First assume that $\tau \equiv 1$. Then τ-positive definiteness coincides with the usual one on the free product group $*_{i\in I}\mathbb{Z}_2$ and our assertion is well known in this case. The general case follows from Propositions 1.3 and 1.4. □

It turns out that one part of the last proposition can be generalized.

Lemma 1.6. *Assume that* $\{A_i\}_{i\in I}$ *is a family of operators on a Hilbert space* $\mathcal{H}_0$ *such that* $-\tau_i \operatorname{Id} \le A_i \le \operatorname{Id}$ *for every* $i \in I$. *Then for* $\xi \in \mathcal{H}_0$ *the function*

$$\varphi(i_1 \cdots i_m) := [A_{i_1} \cdots A_{i_m}\xi, \xi]$$

is τ*-positive definite on* $S(I)$.

Proof. Define a linear function $\Phi : \mathcal{F}(S(I)) \to \mathcal{B}(\mathcal{H}_0)$ by putting $\Phi(\delta_u) := \Phi(u) := A_{i_1} \cdots A_{i_m}$ for $u = i_1 \cdots i_m$ as in (1.1). We will prove that for every finitely supported function $\eta : S(I) \to \mathcal{H}_0$ we have

$$\sum_{u,v\in S(I)} [\Phi(\delta_{v^*} *_\tau \delta_u)\eta(u), \eta(v)] \ge \Big\| \sum_{u\in S(I)} \Phi(u)\eta(u)\Big\|^2.$$

We proceed by induction on $n := \max\{|u| : u \in \mathrm{supp}(\eta)\}$. For $i \in I$ and $u = i_1 \cdots i_m \in S(I)$ define

$$\eta_i(u) := \begin{cases} \eta(iu) & \text{if } u \in S_i, \\ 0 & \text{otherwise.} \end{cases}$$

Then, by induction,

$$R(i) := \sum_{u,v \in S(I)} [\Phi(\delta_{v^*} *_\tau \delta_u)\eta_i(u), \eta_i(v)] - \Big\| \sum_{u \in S(I)} \Phi(u)\eta_i(u) \Big\|^2 \geq 0.$$

Note that $\Phi(v^*u) = \Phi^*(v)\Phi(u)$ holds if $u = e$ or $v = e$ or if $|u|, |v| \geq 1$ and u, v start with different letters. Therefore

$$\begin{aligned}
&\sum_{u,v \in S(I)} [\Phi(\delta_{v^*} *_\tau \delta_u)\eta(u), \eta(v)] - \sum_{u,v \in S(I)} [\Phi(u)\eta(u), \Phi(v)\eta(v)] \\
&\quad = \sum_{i \in I} \sum_{u,v \in S_i} \Big\{ [\Phi(\delta_{v^*} *_\tau (\delta_i *_\tau \delta_i) *_\tau \delta_u)\eta(iu), \eta(iv)] \\
&\qquad\qquad - [A_i^2 \Phi(u)\eta(iu), \Phi(v)\eta(iv)] \Big\} \\
&\quad = \sum_{i \in I} \sum_{u,v \in S_i} \Big\{ [\Phi((1 - \tau_i)\delta_{v^*iu} + \tau_i \delta_{v^*} *_\tau \delta_u)\eta_i(u), \eta_i(v)] \\
&\qquad\qquad - [A_i^2 \Phi(u)\eta_i(u), \Phi(v)\eta_i(v)] \Big\} \\
&\quad = \sum_{i \in I} \{\tau_i R(i) + [((1 - \tau_i)A_i - A_i^2 + \tau_i \,\mathrm{Id})\xi_i, \xi_i]\} \geq 0,
\end{aligned}$$

where $\xi_i := \sum_{u \in S(I)} \Phi(u)\eta_i(u)$, because

$$(1 - \tau_i)A_i - A_i^2 + \tau_i \,\mathrm{Id} = (\mathrm{Id} - A_i)(\tau_i \,\mathrm{Id} + A_i)$$

is a nonnegative operator. Now, for $f \in \mathcal{F}(S(I))$ we have

$$\begin{aligned}
\langle \varphi, f^* *_\tau f \rangle &= [\Phi(f^* *_\tau f)\xi, \xi] \\
&= \sum_{u,v \in S(I)} [\langle \Phi, \delta_{v^*} *_\tau \delta_u \rangle f(u)\xi, f(v)\xi] \geq 0.
\end{aligned}$$ □

Taking $\mathcal{H}_0 = \mathbb{C}$ we obtain a family of τ-positive definite functions:

Corollary 1.7. *Assume that $0 \leq r \leq 1$. Then the function $u \mapsto r^{|u|}$ is τ-positive definite on $S(I)$ for every $\tau : I \to [0, \infty)$.*

Now we can prove a version of the Schur theorem:

Corollary 1.8. *Assume that φ_1 and φ_2 is $\tau^{(1)}$- and $\tau^{(2)}$-positive definite on $S(I)$, respectively, where the functions $\tau^{(1)}, \tau^{(2)} : I \to [0, \infty)$ are such that $\tau_i^{(1)} \cdot \tau_i^{(2)} \leq 1$ for every $i \in I$. Then the product $\varphi_1 \cdot \varphi_2$ is σ-positive definite, where $\sigma_i := \max\{\tau_i^{(1)}, \tau_i^{(2)}\}$.*

In particular, if $\tau : I \to [0, 1]$, then $\mathcal{P}(\tau)$ is closed under pointwise multiplication.

Proof. For $k = 1, 2$, let $(\mathcal{H}_k, \pi_k, \xi_k)$ be the GNS triple for φ_k, and let

$$A_k(i) := \pi_k(\delta_i) = (1 + \tau_i^{(k)})P_k(i) - \tau_i^{(k)} \operatorname{Id}_k$$

for a selfadjoint projection $P_k(i)$ on $\mathcal{H}_k$. Defining the operator $A(i) := A_1(i) \otimes A_2(i)$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$ we have

$$\begin{aligned} A(i) = P_1(i) \otimes P_2(i) - \tau_i^{(2)} P_1(i) \otimes (\operatorname{Id}_2 - P_2(i)) - \tau_i^{(1)} (\operatorname{Id}_1 - P_1(i)) \otimes P_2(i) \\ + \tau_i^{(1)} \cdot \tau_i^{(2)} (\operatorname{Id}_1 - P_1(i)) \otimes (\operatorname{Id}_2 - P_2(i)), \end{aligned}$$

so that $\mathcal{H}_1 \otimes \mathcal{H}_2$ is decomposed into orthogonal direct sum of four subspaces and $A(i)$ acts on each of them by multiplying by 1, $-\tau_i^{(2)}$, $-\tau_i^{(1)}$ and $\tau_i^{(1)} \cdot \tau_i^{(2)}$ respectively, so that $-\sigma_i \operatorname{Id}_1 \otimes \operatorname{Id}_2 \leq A(i) \leq \operatorname{Id}_1 \otimes \operatorname{Id}_2$. We also have

$$\varphi_1(u)\varphi_2(u) = [A(i_1) \cdots A(i_m)\xi_1 \otimes \xi_2, \xi_1 \otimes \xi_2]$$

for $u = i_1 \cdots i_m$, which, by Lemma 1.6, concludes the proof. ◻

2. The Regular Representation of $\mathcal{A}(\tau)$

Let us fix $\tau : I \to [0, \infty)$ and denote $I^{\text{fin}} := \{i \in I : \tau_i > 0\}$ (this notation will be justified in Section 6). Let $\mathcal{F}^{\text{fin}}(S(I))$ denote the class of those $f \in \mathcal{F}(S(I)$ for which $\operatorname{supp} f \subseteq S(I^{\text{fin}})$. We will work on the Hilbert space $\ell^2(\tau)$ of complex functions f on $S(I)$, with support in $S(I^{\text{fin}})$, satisfying

$$||f||_2^2 := \sum_{u \in S(I)} |f(u)|^2 \tau(u) < \infty,$$

where $\tau(u)$ was defined in (1.3), with the scalar product

$$[f, g] := \sum_{u \in S(I)} f(u)\overline{g(u)}\tau(u).$$

According to (1.5), we have

$$[f, g] = (f *_\tau g^*)(e) = (g^* *_\tau f)(e)$$

for all f, $g \in \mathcal{F}^{\text{fin}}(S(I))$ and in view of Lemma 2.2 it remains true for all f, $g \in \ell^2(\tau)$.

There are two natural $*$-representations of $\mathcal{A}(\tau)$ acting on $\ell^2(\tau)$, namely the left and the right regular one:

$$\lambda_0(a)f := (a *_\tau f) \cdot \chi, \quad \rho_0(b)f := (f *_\tau b^*) \cdot \chi$$

for $a, b \in \mathcal{A}(\tau)$, $f \in \ell^2(\tau)$, where χ stands for the characteristic function of the set $S(I^{\text{fin}})$. In particular, $\lambda_0(\delta_i) = \rho_0(\delta_i) = 0$ whenever $i \in I \setminus I^{\text{fin}}$.

We will study two corresponding $*$-subalgebras of $\mathcal{B}(\ell^2(\tau))$, namely

$$\mathcal{L} := \lambda_0(\mathcal{A}(\tau)) \quad \text{and} \quad \mathcal{R} := \rho_0(\mathcal{A}(\tau)),$$

and the von Neumann algebras which are their commutants:

$$\begin{aligned} \mathcal{S} := \mathcal{R}' &= \{A \in \mathcal{B}(\ell^2(\tau)) : AB = BA \text{ for every } B \in \mathcal{R}\}, \\ \mathcal{T} := \mathcal{L}' &= \{B \in \mathcal{B}(\ell^2(\tau)) : AB = BA \text{ for every } A \in \mathcal{L}\}. \end{aligned}$$

The aim of this section is to show that every minimal $\mathcal{L}$-invariant closed subspace of $\ell^2(\tau)$ is one-dimensional.

Lemma 2.1. *If $f, g \in \ell^2(\tau)$, then the function $f *_\tau g$ is well defined. Moreover, for every $u \in S(I)$ there is a constant $C(u)$ such that*

$$|(f *_\tau g)(u)| \le C(u)\|f\|_2 \cdot \|g\|_2$$

for every $f, g \in \ell^2(\tau)$.

Proof. For $u \in S(I) \setminus S(I^{\text{fin}})$ we can put $C(u) = 0$. Fix $u = i_1 \cdots i_m \in S(I^{\text{fin}})$. Then

$$\begin{aligned} (f *_\tau g)(u) = &\sum_{k=0}^{m} \sum_{j_1\cdots j_n \in S(I^{\text{fin}}),\ j_n \neq i_k, i_{k+1}} f(i_1 \cdots i_k j_n \cdots j_1) g(j_1 \cdots j_n i_{k+1} \cdots i_m) \tau(j_1 \cdots j_n) \\ &+ \sum_{k=1}^{m} \sum_{j_1\cdots j_n \in S(I^{\text{fin}}),\ j_n \neq i_k} f(i_1 \cdots i_k j_n \cdots j_1) g(j_1 \cdots j_n i_k i_{k+1} \cdots i_m) \\ &\qquad \times \tau(j_1 \cdots j_n)(1 - \tau_{i_k}). \end{aligned}$$

Therefore, putting

$$c_1(u) = \max\{|1 - \tau_{i_k}|\tau_{i_k}^{-1/2} : k = 1, \ldots, m\},$$

we get

$$
\begin{aligned}
&\big|(f *_\tau g)(u)\big| \\
&\le \frac{1}{\sqrt{\tau(u)}} \sum_{k=0}^{m} \sum_{j_1\cdots j_n \in S(I^{\text{fin}}),\ j_n \neq i_k, i_{k+1}} \Big| f(i_1 \cdots i_k j_n \cdots j_1)\sqrt{\tau(i_1 \cdots i_k j_n \cdots j_1)} \\
&\qquad\qquad \times g(j_1 \cdots j_n i_{k+1} \cdots i_m)\sqrt{\tau(j_1 \cdots j_n i_{k+1} \cdots i_m)}\Big| \\
&\quad + \frac{1}{\sqrt{\tau(u)}} \sum_{k=1}^{m} \sum_{j_1\cdots j_n \in S(I^{\text{fin}}),\ j_n \neq i_k} \frac{|1-\tau_{i_k}|}{\sqrt{\tau_{i_k}}} \\
&\qquad\qquad \times \Big| f(i_1 \cdots i_k j_n \cdots j_1)\sqrt{\tau(i_1 \cdots i_k j_n \cdots j_1)} \\
&\qquad\qquad \times g(j_1 \cdots j_n i_k i_{k+1} \cdots i_m)\sqrt{\tau(j_1 \cdots j_n i_k i_{k+1} \cdots i_m)}\Big| \\
&\le \frac{1}{\sqrt{\tau(u)}}\|f\|_2 \cdot \|g\|_2 + \frac{c_1(u)}{\sqrt{\tau(u)}}\|f\|_2 \cdot \|g\|_2,
\end{aligned}
$$

which concludes the proof. □

Knowing that the map $f \mapsto \tilde{f} = \sum_{u \in S(I)} f(u)\mu_u$ (formula (4.2)) of $\mathcal{A}(\tau)$ onto $\mathcal{F}_t(G)$ preserves the ℓ^2-norm: $\|\tilde{f}\|_{\ell^2(\tau)} = \|f\|_{\ell^2(G)}$, we note that if $\tau_i \in \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$ for every $i \in I^{\text{fin}}$, then one can take $C(u) = \tau^{-1}(u)$ for $u \in S(I^{\text{fin}})$.

We will use the following facts:

Lemma 2.2.

(1) *If $f, f_n, g, g_n \in \ell^2(\tau)$ and if $\|f_n - f\|_2 \to 0$, $\|g_n - g\|_2 \to 0$, then*

$$(f_n *_\tau g_n)(u) \to (f *_\tau g)(u)$$

for every $u \in S(I)$.

(2) *If $f, g, h \in \ell^2(\tau)$ and if one of them has finite support, then*

$$(f *_\tau g) *_\tau h = f *_\tau (g *_\tau h),$$

in particular

$$[f *_\tau g, h^*] = [f, h^* *_\tau g^*] = [g, f^* *_\tau h^*].$$

Proof. For (1) we observe that

$$
\begin{aligned}
&|f_n *_\tau g_n(u) - f *_\tau g(u)| \\
&\qquad \le |(f_n *_\tau (g_n - g))(u)| + ((f_n - f) *_\tau g)(u)| \\
&\qquad \le C(u)\|f_n\|_2 \cdot \|g_n - g\|_2 + C(u)\|f_n - f\|_2 \cdot \|g\|_2.
\end{aligned}
$$

To prove (2) assume that $h \in \mathcal{F}^{\text{fin}}(S(I))$ and take sequences $f_n, g_n \in \mathcal{F}^{\text{fin}}(S(I)) \cap \ell^2(\tau)$ such that $\|f_n - f\|_2 \to 0$, $\|g_n - g\|_2 \to 0$. Then for fixed $u \in S(I)$ we have

$$((f_n *_\tau g_n) *_\tau h)(u) = (f_n *_\tau (g_n *_\tau h))(u).$$

Since $(f_n *_\tau g_n)(v) \to (f *_\tau g)(v)$ for every $v \in S(I)$ and h has finite support, we have

$$((f_n *_\tau g_n) *_\tau h)(u) \to ((f *_\tau g) *_\tau h)(u).$$

We also have $\|g_n *_\tau h - g *_\tau h\|_2 \to 0$ so that part (1) implies that

$$(f_n *_\tau (g_n *_\tau h))(u) \to (f *_\tau (g *_\tau h))(u).$$

The other cases can be proved similarly. ◻

Lemma 2.3. *For $a \in \ell^2(\tau)$ define operators $\lambda_0(a)$ and $\rho_0(a)$, with domains $\mathcal{D}(\lambda_0(a))$ and $\mathcal{D}(\rho_0(a))$, as the closure of the maps*

$$\mathcal{F}^{\text{fin}}(S(I)) \ni f \mapsto a *_\tau f \quad \text{and} \quad \mathcal{F}^{\text{fin}}(S(I)) \ni f \mapsto f *_\tau a^*$$

respectively. Then $\lambda_0(a^*) \subseteq \lambda_0(a)^*$ and $\rho_0(a^*) \subseteq \rho_0(a)^*$. Moreover, if $g \in \mathcal{D}(\lambda_0(a)^*)$ (resp. $g_1 \in \mathcal{D}(\rho_0(a)^*)$), then $\lambda_0(a)^* g = a^* *_\tau g$ (resp. $\rho_0(a)^* g_1 = g_1 *_\tau a$).

Proof. First we note that in view of Lemma 2.1 the maps $\mathcal{F}^{\text{fin}}(S(I)) \ni f \mapsto a *_\tau f$ and $\mathcal{F}^{\text{fin}}(S(I)) \ni f \mapsto f *_\tau a^*$ are closable.

Assume that $(g,h) \in \lambda_0(a^*)$. Then there is a sequence $g_n \in \mathcal{F}^{\text{fin}}(S(I))$ such that $g_n \to g$ and $a^* *_\tau g_n \to h$ in $\ell^2(\tau)$. Then, by Lemma 2.2, for $f \in \mathcal{F}^{\text{fin}}(S(I))$ we have

$$[\lambda_0(a)f, g] = \lim_{n\to\infty} [a *_\tau f, g_n] = \lim_{n\to\infty} [f, a^* *_\tau g_n] = [f, h],$$

which means that $(g,h) \in \lambda_0(a)^*$.

Now, if $g \in \mathcal{D}(\lambda_0(a)^*)$ and $\lambda_0(a)^* g = h$, then, by definition of adjoint and by Lemma 2.2,

$$[\delta_u, h] = [\lambda_0(a)\delta_u, g] = [a *_\tau \delta_u, g] = [\delta_u, a^* *_\tau g],$$

which means that $h = a^* *_\tau g$. ◻

Proposition 2.4.

(1) *If $A \in \mathcal{S}$, then there exists a function $a \in \ell^2(\tau)$ such that $A(f) = a *_\tau f$ for $f \in \ell^2(\tau)$. Similarly, if $B \in \mathcal{T}$, then there exists a function $b \in \ell^2(\tau)$ such that $B(f) = f *_\tau b^*$ for $f \in \ell^2(\tau)$.*

(2) *$\mathcal{S}$ is the weak closure of $\mathcal{L}$ and $\mathcal{T}$ is the weak closure of $\mathcal{R}$.*
(3) *The map $A \mapsto \mathrm{Tr}(A) := [A\delta_e, \delta_e]$ is a faithful tracial state on $\mathcal{S}$ (and on $\mathcal{T}$).*

Proof.
(1) Fix $A \in \mathcal{S}$ and put $a := A(\delta_e)$. If $f \in \mathcal{F}^{\mathrm{fin}}(S(I))$, then by the definition of $\mathcal{S}$

$$\begin{aligned} A(f) &= A(\delta_e *_\tau f) = A(\rho_0(f^*)(\delta_e)) \\ &= \rho_0(f^*)(A(\delta_e)) = (\rho_0(f^*))(a) = a *_\tau f. \end{aligned}$$

In order to prove this equality for all $f \in \ell^2(\tau)$ we define functionals $\varphi_u(f) := (Af)(u)$, $u \in S(I^{\mathrm{fin}})$. Since $\tau(u)|f(u)|^2 \leq \|f\|_2^2$, we have $\|\varphi_u\| \leq \|A\|/\sqrt{\tau(u)}$. Hence if $f_n \in \mathcal{F}^{\mathrm{fin}}(S(I))$ and $\|f_n - f\|_2 \to 0$, then

$$(Af)(u) = \lim(Af_n)(u) = \lim(a *_\tau f_n)(u),$$

so $a *_\tau f$ is a well defined function equal to Af.

(2) Note that $\mathcal{S}$ is weakly closed and $\mathcal{L} \subseteq \mathcal{S}$, hence the weak closure of $\mathcal{L}$ is contained in $\mathcal{S}$. On the other hand, if $A \in \mathcal{S}$, $B \in \mathcal{T}$, then $AB = BA$ in view of the previous point and Lemma 2.2. Therefore $\mathcal{S} \subseteq \mathcal{T}' = \mathcal{L}''$, which is equal to the weak closure of $\mathcal{L}$ by the von Neumann theorem.

(3) Take $A, B \in \mathcal{S}$ and put $a = A(\delta_e)$, $b = B(\delta_e) \in \ell^2(\tau)$. Then

$$\begin{aligned} \mathrm{Tr}(AB) &= [AB\delta_e, \delta_e] = [a *_\tau (b *_\tau \delta_e), \delta_e] \\ &= (a *_\tau b)(e) = (b *_\tau a)(e) = [BA\delta_e, \delta_e] = \mathrm{Tr}(BA). \end{aligned}$$

If $0 \leq A \in \mathcal{S}$, then $A = C^*C$ for some $C \in \mathcal{S}$, hence putting $c = C(\delta_e)$ we have

$$\mathrm{Tr}(A) = [C^*C\delta_e, \delta_e] = (c^* *_\tau c)(e) = ||c||_2^2,$$

which concludes the proof. ◻

From now on we fix a minimal nontrivial closed $\mathcal{L}$-invariant subspace $V \subseteq \ell^2(\tau)$ and a minimal biinvariant (i.e., both $\mathcal{L}$- and $\mathcal{R}$-invariant) closed subspace W containing V.

Lemma 2.5. *The subspace W can be decomposed into the orthogonal sum $W = \bigoplus_\alpha V_\alpha$, with V as one of the summands and where each V_α is closed, $\mathcal{L}$-invariant and the restriction of λ_0 to V_α is equivalent to the restriction of λ_0 to V.*

Proof. Let W_0 be a maximal orthogonal sum of the form $\bigoplus_\alpha V_\alpha$, such that V is one of the summands, $V_\alpha \subseteq W$ is λ_0-invariant and the restriction of λ_0 to V_α is equivalent to the restriction of λ_0 to V. Denote by P the orthogonal projection of $\ell^2(\tau)$ onto $W_0^\perp$. Since this subspace is $\mathcal{L}$-invariant, we have $AP = PA$ for every $A \in \mathcal{L}$.

Assume that W_0 is not $\mathcal{R}$ invariant, i.e., there is $b \in \mathcal{A}(\tau)$ and an index β such that $V_\beta *_\tau b \not\subseteq W_0$. Define an operator $B : V_\beta \to W_0^\perp$ by $B(f) := P(f *_\tau b)$. Note that $f *_\tau b$, and hence $B(f)$, belongs to W and that $AB(f) = BA(f)$ for $f \in V_\beta$, $A \in \mathcal{L}$. Indeed, if $A = \lambda_0(a)$, then

$$AB(f) = AP(f *_\tau b) = PA(f *_\tau b) = P(a *_\tau f *_\tau b) = BA(f).$$

We claim that B is a multiple of an isometry $V_\beta \to V' := B(V_\beta)$. To see this, take the polar decomposition $B = UD$. By definition $D = \sqrt{B^*B}$. For $a \in \mathcal{A}(\tau)$ and $\xi \in V_\beta$

$$\lambda_0(a)B^*B\xi = B^*\lambda_0(a)B\xi = B^*B\lambda_0(a)\xi,$$

so by the Schur lemma B^*B, and hence D, is a nonzero scalar multiple of the identity. Now U, defined by $UD\xi := B\xi$ for $\xi \in V_\beta$, is obviously a unitary operator $V_\beta \to V'$, satisfying $U\lambda_0(a)\xi = \lambda_0(a)U\xi$ for $a \in \mathcal{A}(\tau)$, $\xi \in V_\beta$. Therefore V' can be added to the sum $\bigoplus_\alpha V_\alpha$, which is a contradiction. ◻

Lemma 2.6. *The subspace V has finite dimension.*

Proof. Let

$$\mathcal{L}_W = \{A|_W : A \in \mathcal{L}\}, \quad \mathcal{S}_W = \{A|_W : A \in \mathcal{S}\},$$
$$\mathcal{L}_V = \{A|_V : A \in \mathcal{L}\}, \quad \mathcal{S}_V = \{A|_V : A \in \mathcal{S}\}.$$

By the previous lemma $\mathcal{L}_W \cong \mathcal{L}_V$, $\mathcal{S}_W \cong \mathcal{S}_V$ as $*$-algebras (cf. A20 in the Appendix of [7]) and by the Schur lemma combined with the von Neumann theorem we have $\mathcal{S}_V = \mathcal{B}(V)$.

For $X \in \mathcal{S}_W$ we define an operator $\widetilde{X}$ on $\ell^2(\tau)$ by $\widetilde{X} := XQ$, where Q denotes the orthogonal projection of $\ell^2(\tau)$ onto W. Note that $B\widetilde{X} = \widetilde{X}B$ for any $B \in \mathcal{R}$. Indeed, if $X = \lambda_0(x)|_W$, $B = \rho_0(b)$, with $x, b \in \mathcal{A}(\tau)$, then for $f = f_1 + f_2 \in \ell^2(\tau)$, with $f_1 = Qf$, we have

$$B\widetilde{X}f = BXf_1 = B(x *_\tau f_1) = x *_\tau f_1 *_\tau b^*,$$

$$\widetilde{X}Bf = XQBf = XQBf = XBf_1 = X(f_1 *_\tau b^*) = x *_\tau f_1 *_\tau b^*$$

because W is bi-invariant. Moreover, $X \mapsto \widetilde{X}$ is a $*$-homomorphism of $\mathcal{S}_W$ into $\mathcal{S}$. Therefore we can define a tracial state on $\mathcal{S}_W \cong \mathcal{B}(V)$ by $\widetilde{\mathrm{Tr}}(X) := \mathrm{Tr}(\widetilde{X})$ (Tr was defined in Proposition 2.4). But if $\dim V$ is infinite, then there is no tracial state on $\mathcal{B}(V)$ because we can take an orthogonal decomposition $V = V_1 \oplus V_2$ and partial isometries C_1, C_2 of V onto V_1, V_2 respectively such that $C_1C_1^* + C_2C_2^* = \mathrm{Id}$ and $C_1^*C_1 + C_2^*C_2 = 2\,\mathrm{Id}$, which excludes existence of a tracial state on $\mathcal{B}(V)$ for infinite dimensional V. ◻

Lemma 2.7. *For a function $a \in \ell^2(\tau)$ the following conditions are equivalent:*

(1) $\lambda_0(a)$ *is a bounded operator,*
(2) $\lambda_0(a^*)$ *is a bounded operator,*
(3) $\rho_0(a)$ *is a bounded operator,*
(4) $\rho_0(a^*)$ *is a bounded operator,*
(5) $a = A(\delta_e)$ *for some* $A \in S$,
(6) $a^* = B(\delta_e)$ *for some* $B \in \mathcal{T}$.

Proof. In view of Lemma 2.3 we have (1) $\iff$ (2) and (3) $\iff$ (4) and by the first part of Proposition 2.4 we have (1) $\iff$ (5) and (3) $\iff$ (6). To conclude, we note that $\|a *_\tau f\|_2 = \|f^* *_\tau a^*\|_2$ for $f \in \ell^2(\tau)$, which means that (1) $\iff$ (3). $\square$

Definition 2.8. A function $a \in \ell^2(\tau)$ is said to be *moderated* if satisfies conditions of the previous lemma. Note that moderated functions constitute a $*$-algebra and if a, b are moderated, then $\lambda_0(a *_\tau b) = \lambda_0(a)\lambda_0(b)$ and $\lambda_0(a)^* = \lambda_0(a^*)$.

Lemma 2.9. *Every function* $f \in V$ *is moderated.*

Proof. Let P denote the orthogonal projection of $\ell^2(\tau)$ onto V. Since $P \in \mathcal{L}' = \mathcal{T}$, we have $P(g) = g *_\tau k$ for some moderated function k such that $k = k *_\tau k = k^*$. Hence if g is a moderated function, then so is $P(g) = g *_\tau k$.

Moderated functions form a dense linear subspace $\mathcal{M}$ of $\ell^2(\tau)$, hence $\mathcal{M} \cap V = P(\mathcal{M})$ is a dense subspace of V. But V has finite dimension, which implies that $\mathcal{M} \cap V = V$. $\square$

Lemma 2.10. *Suppose that* $f, g \in V$ *and set* $\varphi(u) = [\lambda_0(\delta_u)f, g]$. *Then there is* $f_0 \in \ell^2(\tau)$ *such that* $\varphi(u) = f_0(u)\tau(u)$ *for every* $u \in S(I)$.

Proof. By Lemma 2.2 we have

$$\varphi(u) = [\lambda_0(\delta_u)f, g] = (\delta_u *_\tau (f *_\tau g^*))(e) = (f *_\tau g^*)(u^*) \cdot \tau(u).$$

By the previous lemma, f and g are moderated, hence so is $f *_\tau g^*$, therefore $f *_\tau g^* \in \ell^2(\tau)$. $\square$

Now we are able to prove the main result of this section.

Theorem 2.11. *If* V *is an* $\mathcal{L}$*-invariant minimal nontrivial subspace of* $\ell^2(\tau)$, *then* $\dim V = 1$.

Proof. We know already from Lemma 2.6 that $\dim V < \infty$. It is sufficient to prove that all the operators $\lambda_0(\delta_i)|_V$, $i \in I$, commute.

Assume that $|I^{\text{fin}}| \geq 2$ and fix $I_0 \subseteq I^{\text{fin}}$ with $|I_0| = 2$, say $I_0 = \{1, 2\}$. Denote by $\mathcal{A}_0$ the unital $*$-subalgebra of $\mathcal{A}(\tau)$ generated by $\{\delta_1, \delta_2\}$. Set also $\mathcal{L}_0 := \{\lambda_0(a) : a \in \mathcal{A}_0\}$. Now decompose V into an orthogonal direct sum of minimal $\mathcal{L}_0$-invariant subspaces:

$$V = V_1 \oplus V_2 \oplus \cdots \oplus V_s.$$

The $*$-algebra $\mathcal{A}_0$ is isomorphic to the convolution $*$-algebra of finitely supported functions on the dihedral group $\mathbf{Z}_2 * \mathbf{Z}_2$. This group can be represented as the semidirect product $\mathbb{Z}_2 \ltimes \mathbb{Z}$, which implies that $\dim V_r \le 2$ for $r \le s$ (see Lemma 7.1). We only need to show that all V_r have dimension one.

Suppose that $\dim V_r = 2$ for some $r \le s$ and denote $A_i := \lambda_0(\delta_i)|_{V_r}$, $i = 1, 2$. Then we have $A_i = (1+\tau_i)P_i - \tau_i \operatorname{Id}$, where P_i is an orthogonal projection on V_r. Due to minimality of V_r both projections have one-dimensional images. Therefore $\det A_i = -\tau_i$ and $\det(A_1A_2) = \tau_1\tau_2$. Hence the operator A_1A_2 has an eigenvalue γ_0 satisfying $|\gamma_0| \ge \sqrt{\tau_1\tau_2}$. Now take the corresponding unit eigenvector $\xi_0 \in V_r$ and consider the function φ on $S(I)$ given by: $\varphi(u) = [\lambda_0(\delta_u)\xi_0, \xi_0]$. Then for $u = 1212\cdots12$, with $|u| = 2m$, we have

$$\varphi(u) = \left|\,[(A_1A_2)^m\xi_0, \xi_0]\,\right| = |\gamma_0^m| \ge \left(\sqrt{\tau_1\tau_2}\right)^m,$$

which implies that the function $S(I_0) \ni u \mapsto \varphi(u)/\tau(u)$ does not belong to $\ell^2(S(I_0), \tau)$. This contradicts the last lemma and therefore proves that $\dim V_r = 1$ for every $r \le s$. Therefore the operators $\lambda_0(\delta_1)|_V$ and $\lambda_0(\delta_2)|_V$ do commute. ◻

Let us now describe one dimensional λ_0-invariant subspaces of $\ell^2(\tau)$ and the corresponding τ-positive definite functions.

For $i \in I$ we have the partition $S(I) = S_i \dot\cup iS_i$, where S_i was defined in (1.2), and the orthogonal decomposition $\ell^2(\tau) = M_i \oplus N_i$, where

$$M_i = \{f_1 \in \ell^2(\tau) : f_1(u) = \tau_i f_1(iu) \text{ for every } u \in S_i\},$$
$$N_i = \{f_2 \in \ell^2(\tau) : f_2(u) + f_2(iu) = 0 \text{ for every } u \in S_i\}.$$

Indeed, for $f_1 \in M_i$, $f_2 \in N_i$ we have

$$\begin{aligned}[f_1, f_2] &= \sum_{u\in S(I)} f_1(u)\overline{f_2(u)}\tau(u)\\ &= \sum_{u\in S_i}\left(f_1(u)\overline{f_2(u)}\tau(u) + f_1(iu)\overline{f_2(iu)}\tau(iu)\right) = 0.\end{aligned}$$

On the other hand, every $f \in \ell^2(\tau)$ can be decomposed as $f = f_1 + f_2$, $f_1 \in M_i$, $f_2 \in N_i$ where for $u \in S_i$

$$\begin{aligned}f_1(u) &= \frac{\tau_i}{1+\tau_i}(f(u) + f(iu)),\\ f_1(iu) &= \frac{1}{1+\tau_i}(f(u) + f(iu)),\\ f_2(u) &= \frac{1}{1+\tau_i}(f(u) - \tau_i f(iu)),\\ f_2(iu) &= \frac{1}{1+\tau_i}(\tau_i f(iu) - f(u)).\end{aligned}$$

Consider the operator $A_i := \lambda_0(\delta_i)$ on $\ell^2(\tau)$. For $f = \sum_{u\in S(I)} f(u)\delta_u$ we have

$$\delta_i *_\tau f = \sum_{u\in S_i} \Big(f(u)\delta_{iu} + (1-\tau_i)f(iu)\delta_{iu} + \tau_i f(iu)\delta_u\Big).$$

Therefore, if $f \in M_i$, then $A_i f = f$ while for $f \in N_i$ we have $A_i f = -\tau_i f$.

If a function $f : S(I) \to \mathbb{C}$, supported on $S(I^{\text{fin}})$, is an eigenfunction for every A_i, $i \in I$, then, up to a constant, $f(i_1 i_2 \cdots i_n) = f(i_1)f(i_2)\cdots f(i_n)$, where $f(i) \in \{\tau_i^{-1}, -1\}$ for $i \in I^{\text{fin}}$. In view of Lemma 1.5 in [10], such a function belongs to $\ell^2(\tau)$ if and only if

$$\sum_{i\in I_1} \frac{1}{1+\tau_i} + \sum_{i\in I_2} \frac{\tau_i}{1+\tau_i} < 1,$$

where $I_1 := \{i \in I^{\text{fin}} \mid f(i) = \tau_i^{-1}\}$, $I_2 := \{i \in I^{\text{fin}} \mid f(i) = -1\}$. If this holds, then the corresponding τ-positive definite function

$$\varphi(i_1 i_2 \cdots i_m) = [A_{i_1} A_{i_2} \cdots A_{i_m} f, f]$$

is, up to a constant, the character of $\mathcal{A}(\tau)$ given by

$$\varphi(i_1 i_2 \cdots i_m) = \varphi(i_1)\varphi(i_2)\cdots\varphi(i_m),$$

where

$$\varphi(i) = \begin{cases} 1 & \text{if } f(i) = \tau_i^{-1}, \\ -\tau_i & \text{if } f(i) = -1. \end{cases}$$

For example, the trivial representation of $\mathcal{A}(\tau)$, corresponding to the character $\varphi \equiv 1$, is contained in the regular representation if and only if

$$\sum_{i\in I^{\text{fin}}} \frac{1}{1+\tau_i} < 1\,.$$

Now we make the following elementary observation:

Lemma 2.12. *Suppose that for every $i \in I$ we have $0 \le \alpha_i^0 \le \alpha_i^1 \le 1$, $\alpha_i^0 + \alpha_i^1 = 1$ and that $\sum_{i\in I} \alpha_i^{\varepsilon_i} < 1$ for some $\varepsilon : I \to \{0,1\}$. Then either $\varepsilon_i = 0$ for all $i \in I$ or there is $j \in I$ such that $\alpha_j^0 > \alpha_i^0$ for every $i \in I \setminus \{j\}$, $\varepsilon_j = 1$ and $\varepsilon_i = 0$ for all $i \ne j$.*

Putting $\alpha_i^0 := \min\{1/(1+\tau_i),\ \tau_i/(1+\tau_i)\}$, we obtain the following result.

Corollary 2.13. *The space $\ell^2(\tau)$ contains at most two one-dimensional $\mathcal{L}$-invariant subspaces.*

The next result will be needed later on.

Theorem 2.14. *Assume that $a \in \ell^2(\tau)$ is such that the function $u \mapsto a(u)\tau(u)$ is τ-positive definite. Then there is $c = c^* \in \ell^2(\tau)$ such that $a = c *_\tau c$ and $c \cdot \tau$ is τ-positive definite. Moreover, the* GNS *representation related to $a \cdot \tau$ is contained in λ_0.*

The proof, similarly as for groups (see [7]), is based on the following lemmas.

Lemma 2.15. *Assume that a is a moderated function. Then the operator $\lambda_0(a)$ is nonnegative if and only if the function $u \mapsto a(u)\tau(u)$ is τ-positive definite.*

Proof. For $f \in \mathcal{F}^{\text{fin}}(S(I))$ we have

$$\begin{aligned} [\lambda_0(a)f, f] &= (a *_\tau (f *_\tau f^*))(e) \\ &= \sum_{u \in S(I)} a(u)(f *_\tau f^*)(u)\tau(u) \\ &= \langle a \cdot \tau, f *_\tau f^* \rangle. \end{aligned}$$ □

Lemma 2.16. *Assume that a, b are moderated functions such that $a *_\tau b = b *_\tau a$ and $0 \le \lambda_0(a) \le \lambda_0(b)$. Then*

$$||b - a||_2^2 \le ||b||_2^2 - ||a||_2^2.$$

Proof. Put $c = b - a$. Since the operator $\lambda_0(a *_\tau c) = \lambda_0(a)\lambda_0(c)$ is nonnegative, we have $[a, c] = (a *_\tau c)(e) \ge 0$, so $[a, a] \le [a, a] + [a, c] = [a, b]$ which gives

$$||b - a||_2^2 = ||b||_2^2 + ||a||_2^2 - 2[a, b] \le ||b||_2^2 - ||a||_2^2.$$ □

Lemma 2.17. *Assume that a_1, a_2, ... are pairwise commuting moderated functions such that*

$$0 \le \lambda_0(a_1) \le \lambda_0(a_2) \le \cdots$$

and $\sup \|a_n\|_2 < \infty$. *Then there is $a \in \ell^2(\tau)$ such that $\|a - a_n\|_2 \to 0$.*

Proof. In view of the previous lemma, the sequence $\|a_n\|_2$ is increasing and a_n is a Cauchy sequence in $\ell^2(\tau)$. □

Proof of Theorem 2.14. If a is moderated, then the proof goes as in Theorem 13.8.6 in [7]. For the general case let T denote the Friedrichs extension of the nonnegative operator $\rho_0(a) = \rho_0(a^*)$. Since the operators

$$U_i := \lambda_0 \left(\frac{2}{1 + \tau_i} \delta_i - \frac{1 - \tau_i}{1 + \tau_i} \delta_e \right)$$

are unitary, we note that (cf. Lemma 13.8.3 in [7]):

(i) $U_i T = T U_i$ for $i \in I$ and therefore $\lambda_0(\delta_u)T = T\lambda_0(\delta_u)$ for $u \in S(I)$,
(ii) $Th = h *_\tau a$ for every h in the domain of T.

(The second statement holds by Lemma 2.3 as $T \subseteq \lambda_0(a)^*$.) Take the spectral resolution $T = \int_0^\infty \zeta \, dE_\zeta$ of T. Then the projections E_ζ commute with all $\lambda_0(\delta_u)$, $u \in S(I)$. Put $a_\zeta := E_\zeta a \in \ell^2(\tau)$. For $g \in \mathcal{A}(\tau)$, $u \in S(I)$ we have

$$\begin{aligned}[g *_\tau a_\zeta^*,\ \delta_u] &= [g,\ \delta_u *_\tau a_\zeta] = [g,\ \delta_u *_\tau E_\zeta a] \\ &= [g,\ E_\zeta(\delta_u *_\tau a)] = [E_\zeta g,\ \delta_u *_\tau a] = [E_\zeta g *_\tau a^*,\ \delta_u].\end{aligned}$$

This means that

$$g *_\tau a_\zeta^* = (E_\zeta g) *_\tau a^* = (E_\zeta g) *_\tau a = T E_\zeta g,$$

which implies that a_ζ^* is moderated, $a_\zeta^* = a_\zeta$, and $a \cdot \tau$ is τ-positive definite. The rest of the proof goes as for 13.8.6 in [7].

For the second part we write

$$\begin{aligned}[\lambda_0(\delta_u)c,\ c] &= ((\delta_u *_\tau c) *_\tau c)(e) = (\delta_u *_\tau (c *_\tau c))(e) \\ &= (\delta_u *_\tau a)(e) = a(u) \cdot \tau(u).\end{aligned}$$ ◻

We conclude this section with two propositions which will not be used in the sequel.

Proposition 2.18. *Assume that $a, b, c \in \ell^2(\tau)$.*

(1) *If a is moderated and $b *_\tau c \in \ell^2(\tau)$, then $a *_\tau (b *_\tau c) = (a *_\tau b) *_\tau c$.*
(2) *If b is moderated, then $a *_\tau (b *_\tau c) = (a *_\tau b) *_\tau c$.*
(3) *If c is moderated and $a *_\tau b \in \ell^2(\tau)$, then $a *_\tau (b *_\tau c) = (a *_\tau b) *_\tau c$.*

Proof. Assume that a is moderated and $b *_\tau c \in \ell^2(\tau)$. Then there is a sequence $a_m \in \mathcal{F}^{\text{fin}}(S(I))$ such that $\|\lambda_0(a_m)\| \le \|\lambda_0(a)\|$ and $\lambda_0(a_m) \to \lambda_0(a)$ in the weak operator topology. In view of the Mazur-Orlicz theorem we can assume that $\lambda_0(a_m)$ converges to $\lambda_0(a)$ in the *strong* operator topology. Therefore

$$\|a_m *_\tau (b *_\tau c) - a *_\tau (b *_\tau c)\|_2 \to 0,$$

which implies the pointwise convergence. We have also $\|a_m *_\tau b - a *_\tau b\|_2 \to 0$, which, by Lemma 2.2 (1), implies $((a_m *_\tau b) *_\tau c)(u) \to ((a *_\tau b) *_\tau c)(u)$ for every $u \in S(I)$. Since, by Lemma 2.2 (2), $(a_m *_\tau b) *_\tau c = a_m *_\tau (b *_\tau c)$, we get the first statement. The third one can be proved in a similar way.

Now assume that b is moderated. Similarly as before we can take a sequence $b_n \in \mathcal{F}^{\text{fin}}(S(I))$ such that $\lambda_0(b_n) \to \lambda_0(b)$ in the strong operator topology. Appealing to the Mazur-Orlicz theorem again, we can also assume that $\rho_0(b_n) \to \rho_0(b)$ in the strong operator topology. Therefore $\|a *_\tau b_n - a *_\tau b\|_2 \to 0$ and $\|b_n *_\tau c - b *_\tau c\|_2 \to 0$. Now we can conclude by applying Lemma 2.2. ◻

Proposition 2.19. *Assume that $a, c \in \ell^2(\tau)$.*
If a is moderated, then

$$\lambda_0(a)\rho_0(c) \subseteq \rho_0(c)\lambda_0(a) \quad \text{and} \quad \lambda_0(a)\rho_0(c)^* \subseteq \rho_0(c)^*\lambda_0(a).$$

If c is moderated, then

$$\rho_0(c)\lambda_0(a) \subseteq \lambda_0(a)\rho_0(c) \quad \text{and} \quad \lambda_0(a)\rho_0(c)^* \subseteq \rho_0(c)^*\lambda_0(a).$$

Proof. Suppose that a is moderated and $b \in \mathcal{D}(\rho_0(c))$. Then $\rho_0(c)b = b *_\tau c^*$, $\lambda_0(a)\rho_0(c)b = a *_\tau (b *_\tau c^*)$ (Lemma 2.3) and, by the definition of $\rho_0(c)$, there is a sequence $b_n \in \mathcal{F}^{\text{fin}}(S(I))$ such that $\|b_n - b\|_2 \to 0$ and $\|b_n *_\tau c - b *_\tau c\|_2 \to 0$. Take a sequence $a_m \in \mathcal{F}^{\text{fin}}(S(I))$ as in the proof of Proposition 2.18. Then

$$a_m *_\tau b_n \to a_m *_\tau b,$$

as $n \to \infty$, in $\ell^2(\tau)$ for every m and

$$(a_m *_\tau b_n) *_\tau c = a_m *_\tau (b_n *_\tau c) \to a_m *_\tau (b *_\tau c) = (a_m *_\tau b) *_\tau c$$

in $\ell^2(\tau)$, which means that $a_m *_\tau b \in \mathcal{D}(\rho_0(c))$ and $\rho_0(c)(a_m *_\tau b) = (a_m *_\tau b) *_\tau c$ for every m. Since $\lambda_0(a_m) \to \lambda_0(a)$ in the strong operator topology we have $\|a_m *_\tau b - a *_\tau b\|_2 \to 0$ and, by Proposition 2.18,

$$(a_m *_\tau b) *_\tau c = a_m *_\tau (b *_\tau c) \to a *_\tau (b *_\tau c) = (a *_\tau b) *_\tau c$$

in $\ell^2(\tau)$. Therefore $a *_\tau b \in \mathcal{D}(\rho_0(c))$ and

$$\rho_0(c)\lambda_0(a)b = (a *_\tau b) *_\tau c = a *_\tau (b *_\tau c) = \lambda_0(a)\rho_0(c)b.$$

For the second inclusion we note that $\lambda_0(a)$, as an element of the von Neumann algebra S, the weak closure of $\lambda_0(\mathcal{A}(\tau))$, can be expressed as $\lambda_0(a) = \sum_{k=1}^{4} \alpha_k U_k$, where U_k are unitary elements of S. Then $U_k = \lambda_0(u_k)$ for moderated $u_k \in \ell^2(\tau)$ and hence $U_k\rho_0(c) \subseteq \rho_0(c)U_k$. Assume that $f \in \mathcal{D}(\rho_0(c)U_k)$, which means that $g := U_k f \in \mathcal{D}(\rho_0(c))$. Since $U_k^{-1} = \lambda_0(u_k^*)$ we also have $U_k^{-1}\rho_0(c) \subseteq \rho_0(c)U_k^{-1}$, which implies $U_k^{-1}g = f \in \mathcal{D}(\rho_0(c))$. Therefore $U_k\rho_0(c)^* = \rho_0(c)^*U_k$. Multiplying both sides by α_k and taking the sum $\sum_{k=1}^{4}$ leaves unchanged the domain on the left side, but can enlarge that on the right side. ◻

3. REPRESENTATIONS WEAKLY CONTAINED IN THE REGULAR ONE

Let π, σ be $*$-representations of a $*$-algebra $\mathcal{A}$. Then π is said to be *weakly contained* in σ if $\|\pi(a)\| \leq \|\sigma(a)\|$ holds for every $a \in \mathcal{A}$. This is equivalent to say that $\ker(\sigma) \subseteq \ker(\pi)$, where $\ker(\pi)$ denotes the kernel of the extension of π to the enveloping C^*-algebra of $\mathcal{A}$.

Following ideas of Haagerup [11] we are now going to describe those $*$-representations of $\mathcal{A}(\tau)$ which are weakly contained in the regular representation λ_0.

Theorem 3.1. *Let $(\pi_0, \mathcal{H}_0)$ be a $*$-representation of $\mathcal{A}(\tau)$.*

(1) *If, for every $\xi \in \mathcal{H}_0$ and $0 < r < 1$, the function*

$$S(I) \ni u \mapsto [\pi_0(\delta_u)\xi, \xi] \cdot r^{|u|}$$

can be expressed as $f_0 \cdot \tau$ for some $f_0 \in \ell^2(\tau)$, then π_0 is weakly contained in λ_0.

(2) *Assume that there are constants $0 < c_1 \le c_2$ such that $c_1 \le \tau_i \le c_2$ for every $i \in I^{\text{fin}}$. If π_0 is weakly contained in the regular representation of $\mathcal{A}(\tau)$, then for every $\zeta, \eta \in \mathcal{H}_0$ and $0 < r < 1$ the function*

$$S(I) \ni u \mapsto [\pi_0(\delta_u)\zeta, \eta] \cdot r^{|u|}$$

can be expressed as $f_0 \cdot \tau$ for some $f_0 \in \ell^2(\tau)$.

Set $E_m := \{u \in S(I) : |u| = m\}$ and let χ_m denote the characteristic function of E_m. First we prove two lemmas.

Lemma 3.2. *Let f and g be two functions in $\mathcal{F}^{\text{fin}}(S(I))$ with support in E_k and E_l respectively. Then*

$$\|(f *_\tau g) \cdot \chi_m\|_2 \le \|f\|_2 \cdot \|g\|_2$$

if $|k - l| \le m \le k + l$ and $k + l - m$ is even. Also,

$$\|(f *_\tau g) \cdot \chi_m\|_2 \le \sqrt{C}\|f\|_2 \cdot \|g\|_2$$

if $|k-l| \le m \le k+l$ and $k+l-m$ is odd, where $C := \sup\{|1-\tau_i|^2 \cdot \tau_i^{-1} : i \in I^{\text{fin}}\}$. We have $\|(f *_\tau g) \cdot \chi_m\|_2 = 0$ for all other values of m.

Note that the constant C is finite if and only if $\inf\{\tau_i : i \in I^{\text{fin}}\} > 0$ and $\sup\{\tau_i : i \in I^{\text{fin}}\} < \infty$.

Proof. Writing $uv \in S(I)$ we will mean that the concatenation of u and v is in $S(I)$.

(1) If $m = k + l$, then

$$\begin{aligned} ||(f *_\tau g) \cdot \chi_m||_2^2 &= \sum_{|u_1|=k,\ |u_2|=l,\ u_1u_2 \in S(I)} |f(u_1)|^2 \cdot |g(u_2)|^2 \tau(u_1)\tau(u_2) \\ &\le \sum_{|u_1|=k,\ |u_2|=l} |f(u_1)|^2 \tau(u_1) |g(u_2)|^2 \tau(u_2) = ||f||_2^2 \cdot ||g||_2^2. \end{aligned}$$

(2) Now assume that $m = k + l - 2p$ and define two auxiliary functions:

$$f'(w) := \Big(\sum_{|v|=p,\ wv\in S(I)} |f(wv)|^2\tau(v)\Big)^{1/2}$$

if $|w| = k - p$, and $f'(w) := 0$ otherwise,

$$g'(w) := \Big(\sum_{|z|=p,\ zw\in S(I)} |g(zw)|^2\tau(z)\Big)^{1/2}$$

if $|w| = l - p$, and $g'(w) := 0$ otherwise. Then we have

$$\begin{aligned} \|f'\|_2^2 &= \sum_{|w|=k-p} \Big(\sum_{|v|=p,\ wv\in S(I)} |f(wv)|^2\tau(v)\Big)\tau(w) \\ &= \sum_{|z|=k} |f(z)|^2\tau(z) = \|f\|_2^2 \end{aligned}$$

and similarly $\|g'\|_2 = \|g\|_2$.

Now fix $u = i_1 \cdots i_m$ and put $u_1 := i_1 \cdots i_{k-p}$, $u_2 := i_{k-p+1} \cdots i_m$. Using the Cauchy inequality we get

$$\begin{aligned} |(f *_\tau g)(u)| &\le \sum_{|v|=p,\ u_1v\in S(I),\ v^*u_2\in S(I)} |f(u_1v)g(v^*u_2)\tau(v)| \\ &\le f'(u_1)g'(u_2) = (f' *_\tau g')(u). \end{aligned}$$

Therefore, applying the previous point to f' and g',

$$\|(f *_\tau g)\cdot\chi_m\|_2 \le \|(f' *_\tau g')\cdot\chi_m\|_2 \le \|f'\|_2\cdot\|g'\|_2 = \|f\|_2\cdot\|g\|_2.$$

(3) Assume that $m = k + l - 1$. If $|u| = m$, then we can write $u = u_1 i u_2$, with $|u_1| = k-1$, $|u_2| = l-1$ and we have $(f *_\tau g)(u) = f(u_1 i)g(iu_2)(1-\tau_i)$. Therefore

$$\begin{aligned} &\|(f *_\tau g)\cdot\chi_m\|_2^2 \\ &\quad= \sum_{i_1\cdots i_m\in S(I)} |f(i_1\cdots i_k)|^2\,|g(i_k\cdots i_m)|^2(1-\tau_{i_k})^2\tau(i_1\cdots i_m) \\ &\quad= \sum_{i_1\cdots i_m\in S(I)} |f(i_1\cdots i_k)|^2\tau(i_1\cdots i_k)|g(i_k\cdots i_m)|^2\tau(i_k\cdots i_m)\frac{(1-\tau_{i_k})^2}{\tau_{i_k}} \\ &\quad\le C\|f\|_2^2\cdot\|g\|_2^2. \end{aligned}$$

(4) Finally let $m = k + l + 1 - 2p$, $p \geq 1$. Define

$$f'(w) := \Big(\sum_{|v|=p,\ wv \in S(I)} |f(wv)|^2 \tau(v) \Big)^{1/2}$$

if $|w| = k + 1 - p$ and $f'(w) = 0$ otherwise, and similarly

$$g'(w) := \Big(\sum_{|z|=p,\ zw \in S(I)} |g(zw)|^2 \tau(z) \Big)^{1/2}$$

if $|w| = l + 1 - p$ and $g'(w) = 0$ otherwise. Then we have

$$\|f'\|_2^2 = \sum_{|w|=k+1-p} \Big(\sum_{|v|=p,\ wv \in S(I)} |f(wv)|^2 \tau(v) \Big) \tau(w) = \|f\|_2^2$$

and similarly $\|g'\|_2 = \|g\|_2$.

Now fix $u = u_1 i u_2 \in S(I)$, with $|u_1| = k - p$, $|u_2| = l - p$. Then

$$\begin{aligned}
|(f *_\tau g)(u)| &\leq \sum_{|v|=p,\ u_1 iv \in S(I),\ v^* iu_2 \in S(I)} |f(u_1 iv) g(v^* iu_2) \tau(v)(1 - \tau_i)| \\
&\leq \Big(\sum_{|v|=p,\ u_1 iv \in S(I)} |f(u_1 iv)|^2 \tau(v) \Big)^{1/2} \Big(\sum_{|v|=p,\ v^* iu_2 \in S(I)} |g(v^* iu_2)|^2 \tau(v) \Big)^{1/2} |1 - \tau_i| \\
&= |f'(u_1 i) g'(iu_2)(1 - \tau_i)| \\
&= |(f' *_\tau g')(u)|.
\end{aligned}$$

Now from the previous point we get

$$\begin{aligned}
\|(f *_\tau g) \cdot \chi_m\|_2 &\leq \|(f' *_\tau g') \cdot \chi_m\|_2 \\
&\leq \sqrt{C} \|f\|_2 \cdot \|g\|_2.
\end{aligned}$$

□

Lemma 3.3. *Let f be a function supported on E_k. Then*

$$\|\lambda_0(f)\|_2 \leq \sqrt{C + 1}(2k + 1)\|f\|_2,$$

where, as before, $C = \sup\{|1 - \tau_i|^2 \cdot \tau_i^{-1} : i \in I^{\text{fin}}\}$.

Proof. Fix $g \in \ell^2(\tau)$ and put $g_l := g \cdot \chi_l$. Then, using the previous lemma and Cauchy's inequality, we have

$$\begin{aligned}
&\|(f *_\tau g) \cdot \chi_m\|_2 \\
&\qquad \le \sum_{l=0}^{\infty} \|(f *_\tau g_l) \cdot \chi_m\|_2 \\
&\qquad \le \|f\|_2 \Big(\sum_{r=0}^{\min\{m,k\}} \|g_{m+k-2r}\|_2 + \sqrt{C} \sum_{r=1}^{\min\{m,k\}} \|g_{m+k+1-2r}\|_2 \Big) \\
&\qquad \le \|f\|_2 \Big(\sum_{r=0}^{2\min\{m,k\}} ||g_{m+k-r}||_2^2 \Big)^{1/2} \Big(\sum_{r=0}^{\min\{m,k\}} 1 + \sum_{r=1}^{\min\{m,k\}} C \Big)^{1/2} \\
&\qquad \le ((k+1)(C+1))^{1/2} \|f\|_2 \Big(\sum_{r=0}^{2\min\{m,k\}} ||g_{m+k-r}||_2^2 \Big)^{1/2}.
\end{aligned}$$

Therefore

$$\begin{aligned}
||f *_\tau g||_2^2 \le \sum_{m=0}^{\infty} ||(f *_\tau g) \cdot \chi_m||_2^2 \\
&\le (C+1)(k+1)||f||_2^2 \sum_{m=0}^{\infty} \Big(\sum_{r=0}^{2\min\{m,k\}} ||g_{m+k-r}||_2^2 \Big) \\
&\le (C+1)(k+1)||f||_2^2 \cdot (2k+1) \sum_{s=0}^{\infty} ||g_s||_2^2 \\
&= (C+1)(k+1)(2k+1)||f||_2^2 \cdot ||g||_2^2,
\end{aligned}$$

and this concludes the proof. ◻

Proof of Theorem 3.1. Suppose that for every $\xi \in \mathcal{H}_0$ and $0 < r < 1$ there is $f_r \in \ell^2(\tau)$ such that

$$\varphi_r(u) := [\pi_0(\delta_u)\xi, \xi] \cdot r^{|u|} = f_r(u) \cdot \tau(u)$$

then, by Corollaries 1.7 and 1.8, φ_r is a τ-positive definite function (as pointwise product of a τ- and a 0-positive definite function) and, by Theorem 2.14, is a coefficient of the regular representation λ_0. Therefore, letting $r \to 1$, we see that the function $u \mapsto [\pi_0(\delta_u)\xi, \xi]$ is a pointwise limit of coefficients of λ_0 which proves that π_0 is weakly contained in λ_0.

Now assume that $(\pi_0, \mathcal{H}_0)$ is weakly contained in λ_0. Then in particular $\pi_0(\delta_i) = 0$ whenever $i \in I \setminus I^{\text{fin}}$, so we can assume that $I = I^{\text{fin}}$. Fix $\zeta, \eta \in \mathcal{H}_0$ and put $g(u) := [\pi_0(\delta_u)\zeta, \eta]$. For a function $f \in \ell^2(\tau)$ supported on E_k we have

$$[\pi_0(f)\zeta, \eta] = \sum_{|u|=k} f(u)g(u) = \sum_{|u|=k} f(u)\frac{g(u)}{\tau(u)}\tau(u),$$

and, by Lemma 3.2,

$$|\,[\pi_0(f)\zeta,\eta]\,| \le \|\lambda_0(f)\|\;\|\zeta\|\;\|\eta\| \le \sqrt{C+1}\,(2k+1)\|f\|_2\;\|\zeta\|\;\|\eta\|.$$

This means that

$$\sum_{|u|=k}\left|\frac{g(u)}{\tau(u)}\right|^2\tau(u) \le (C+1)(2k+1)^2\|\zeta\|^2\;\|\eta\|^2.$$

Hence, for every $0<r<1$ we have

$$\sum_{u\in S(I)}\left|\frac{g(u)}{\tau(u)}\right|^2 r^{2|u|}\tau(u) < \infty. \qquad \square$$

4. Type-dependent Functions on the Free Product Group

Let $\{G_i\}_{i\in I}$ be a family of discrete nontrivial groups and let $G = *_{i\in I}G_i$ be their free product (see [30]). Every element x of G can be uniquely represented as a *reduced word*:

$$\text{(4.1)}\qquad \begin{aligned} &x = g_1g_2\cdots g_m, && m\ge 0,\; g_k\in G_{i_k}\setminus\{e\},\\ &i_k\ne i_{k+1}, && \text{for } k<m. \end{aligned}$$

For such an element we define its *length* $|x| := m$ and *type* as the formal word $t(x) := i_1i_2\cdots i_m \in S(I)$. We are particularly interested in *type-dependent* functions on G, i.e., those which are compositions of the form $\varphi\circ t$.

If all G_i's are finite, then the family $\mathcal{F}_t(G)$ of finitely supported type dependent functions is isomorphic to $\mathcal{A}(\tau)$, with $\tau_i = (|G_i|-1)^{-1}$. Indeed, for $u\in S(I)$ put

$$\text{(4.2)}\qquad \mu_u(x) := \begin{cases}\tau(u) & \text{if } t(x)=u,\\ 0 & \text{otherwise.}\end{cases}$$

Then the map $f\mapsto \sum_{u\in S(I)} f(u)\mu_u$ is an isomorphism of $\mathcal{A}(\tau)$ onto $\mathcal{F}_t(G)$, which can be extended to an isometric embedding of $\ell^2(\tau)$ into $\ell^2(G)$.

Now we present an alternative proof of Theorem 3.2 in [21], which characterizes the class of positive definite type-dependent functions.

Theorem 4.1. *Let $\{G_i\}_{i\in I}$ be a family of groups, $G = *_{i\in I}G_i$, and let φ be a complex function on $S(I)$. The type-dependent function $\varphi\circ t$ is positive definite on G (resp. lies in the Fourier-Stielties algebra $B(G)$) if and only if φ is τ-positive definite on $S(I)$ (resp. φ lies in $B(\tau)$), where $\tau_i = (|G_i|-1)^{-1}$.*

If all G_i's are finite, then the part concerning the Fourier-Stielties algebra could be derived from [21, Theorem 3.2 and Corollary 3.3].

Proof. During the proof we will regard $S(I)$ as a unital $*$-subgroup generated by elements $i \in I$ which satisfy $ii = i^* = i$.

Suppose that $\varphi \circ t \in B(G)$, then $\varphi \circ t$ can be represented as $\varphi(t(x)) = [\pi(x)\zeta, \eta]$ for a unitary representation $(\pi, \mathcal{H})$ of G and vectors $\zeta, \eta \in \mathcal{H}$ (with $\zeta = \eta$ when $\varphi \circ t$ is positive definite). We may assume that the vectors ζ and η are both cyclic (cf. Proposition 1 in [31]). If G_i is finite, then we put

$$P_i = \frac{1}{|G_i|} \sum_{g \in G_i} \pi(g) \quad \text{and} \quad A_i = \frac{1}{|G_i| - 1} \sum_{g \in G_i \setminus \{e\}} \pi(g).$$

Then we have $P_i P_i = P_i^* = P_i$ and $A_i = (1 + \tau_i) P_i - \tau_i \,\mathrm{Id}$.

Now, suppose that G_i is infinite and let $\{g_{n,i}\}_{n=1}^{\infty}$ be a fixed sequence of distinct elements of $G_i \setminus \{e\}$. For any natural number N we define an operator $T_{N,i}$ on $\mathcal{H}$ by

$$T_{N,i} = \frac{1}{N} \sum_{n=1}^{N} \pi(g_{n,i}).$$

For fixed $x, y \in G$ we have $t(y^{-1} g_{n,i} x) = t(y)^* i t(x)$ (product in the semigroup $S(I)$) for all but at most two n's. Therefore

$$\begin{aligned} [T_{N,i}\pi(x)\zeta, \pi(y)\eta] &= \frac{1}{N} \sum_{n=1}^{N} [\pi(y^{-1} g_{n,i} x)\zeta, \eta] \\ &= \frac{1}{N} \sum_{n=1}^{N} \varphi(t(y^{-1} g_{n,i} x)) \to \varphi(t(y)^* i t(x)) \end{aligned}$$

as $N \to \infty$. Since the vectors ζ, η are both cyclic and $T_{N,i}$'s are all contractions, there exists a weak limit A_i of the sequence $T_{N,i}$ satisfying

$$[A_i \pi(x)\zeta, \pi(y)\eta] = \varphi(t(y)^* i t(x)), \quad x, y \in G.$$

Note that $A_i^* = A_i$. Indeed

$$\lim_{N\to\infty} [T_{N,i}^* \pi(x)\zeta, \pi(y)\eta] = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \varphi(t(y^{-1} g_{n,i}^{-1} x)) = \varphi(t(y)^* i t(x)).$$

Next we show that $A_i A_i = A_i$. Consider the operator $A_i T_{N,i}$. For any $x, y \in G$

$$\begin{aligned} [A_i T_{N,i} \pi(x)\zeta, \pi(y)\eta] &= \frac{1}{N} \sum_{n=1}^{N} [A_i \pi(g_{n,i} x)\zeta, \pi(y)\eta] \\ &= \frac{1}{N} \sum_{n=1}^{N} \varphi(t(y)^* i t(g_{n,i} x)) = \varphi(t(y)^* i t(x)) \end{aligned}$$

(because $it(gx) = it(x)$ if $g \in G_i$) hence $A_i T_{N,i} = A_i$, which implies $A_i A_i = A_i$. In this way we have defined the operator A_i for every $i \in I$.

Now let $u = i_1 i_2 \cdots i_m$ be a fixed element in $S(I)$. For any $g_2 \in G_{i_2} \setminus \{e\}, \ldots, g_m \in G_{i_m} \setminus \{e\}$ we have

$$\begin{aligned}\varphi(u) &= \left[\frac{1}{|G_{i_1}| - 1} \sum_{g \in G_{i_1} \setminus \{e\}} \pi(g)\pi(g_2 g_3 \cdots g_m)\zeta, \eta\right] \\ &= [A_{i_1} \pi(g_2 g_3 \cdots g_m)\zeta,\ \eta] = [\pi(g_2 g_3 \cdots g_m)\zeta,\ A_{i_1}\eta]\end{aligned}$$

if G_{i_1} is finite and

$$\begin{aligned}\varphi(u) &= [T_{N,i_1} \pi(g_2 g_3 \cdots g_m)\zeta,\ \eta] \\ &= [A_{i_1} \pi(g_2 g_3 \cdots g_m)\zeta,\ \eta] = [\pi(g_2 g_3 \cdots g_m)\zeta,\ A_{i_1}\eta],\end{aligned}$$

for arbitrary N, if G_{i_1} is infinite. Continuing in this fashion we finally obtain

$$\varphi(u) = [\zeta,\ A_{i_m} \cdots A_{i_2} A_{i_1}\eta] = [A_{i_1} A_{i_2} \cdots A_{i_m}\zeta,\ \eta]$$

and, in view of Proposition 1.5, one part of the theorem is proved.

Now assume that φ is of the form

$$\varphi(i_1 i_2 \cdots i_m) = [A_{i_1} A_{i_2} \cdots A_{i_m}\zeta, \eta],$$

where $A_i = (1 + \tau_i)P_i - \tau_i \,\mathrm{Id}$, $\{P_i\}_{i \in I}$ is a family of orthogonal projections on a Hilbert space $\mathcal{H}_0$, and $\zeta, \eta \in \mathcal{H}_0$.

Fix $i \in I$. We show that the operator-valued function

$$U_i(g) = \begin{cases} \mathrm{Id} & \text{if } g = e, \\ A_i & \text{otherwise,} \end{cases}$$

is positive definite on the group G_i. Let $f : G_i \to \mathcal{H}_0$ be a finitely supported function. We can decompose f as $f = f_1 + f_2$ in such a way that $f_1 : G_i \to \mathrm{Im}\, P_i$, $f_2 : G_i \to \ker P_i$. Then by the definition of τ_i we have

$$\begin{aligned}&\sum_{g,h \in G_i} [U_i(h^{-1}g) f(g), f(h)] \\ &\quad = \sum_{g,h \in G_i} [f_1(g), f_1(h)] + \sum_{g \in G_i} [f_2(g), f_2(g)] - \tau_i \sum_{g \neq h} [f_2(g), f_2(h)] \\ &\quad = \Big\| \sum_{g \in G_i} f_1(g) \Big\|^2 + \begin{cases} \displaystyle\sum_{g \in G_i} \|f_2(g)\|^2 & \text{if } G_i \text{ is infinite,} \\ \displaystyle\frac{\tau_i}{2} \sum_{g \neq h} \|f_2(g) - f_2(h)\|^2 & \text{otherwise.} \end{cases}\end{aligned}$$

Hence U_i is positive definite on G_i. Now let us consider the function $U : G \to \mathcal{B}(\mathcal{H}_0)$:

$$U(x) = A_{i_1}A_{i_2}\cdots A_{i_m} \quad \text{if } t(x) = i_1 i_2 \cdots i_m.$$

By Theorem 7.1 in [3] (which will be proved again in the next section) U is positive definite on $G = *_{i\in I}G_i$, which implies that there exists a unitary representation $\pi : G \to \mathcal{B}(\mathcal{H})$ such that $\mathcal{H}_0 \subseteq \mathcal{H}$ and for any $x \in G$ we have $U(x) = P_0\pi(x)|_{\mathcal{H}_0}$, where P_0 denotes the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_0$. This implies that $\varphi(t(x)) = [U(x)\zeta, \eta] = [\pi(x)\zeta, \eta]$, which concludes the proof. ◻

5. The Induced Representation of the Free Product Group

Let $\mathcal{H}_0$ be a fixed Hilbert space, $G = *_{i\in I}G_i$ be the free product of groups and assume that for every $i \in I$ we are given a representation $\pi_i : G_i \to \mathcal{B}(\mathcal{H}_0 \oplus \mathcal{H}_i)$ (we do not require that π_i are unitary). We are going to construct a representation π of G, acting on a Hilbert space $\mathcal{H}$ which contains all spaces $\mathcal{H}_i$, $i \in I \cup \{0\}$, such that for every $i \in I$ and $g \in G_i$ the restriction of $\pi(g)$ to $\mathcal{H}_0 \oplus \mathcal{H}_i$ coincides with $\pi_i(g)$.

Constructions of such kind were studied by Avitzour [1], Voiculescu [34, 35], another one was due to Bożejko [2]. Then Bożejko, Leinert and Speicher [4, 5] generalized the previous ones by introducing the *conditionally free product of representations*. In all these constructions the common subspace $\mathcal{H}_0$ was one-dimensional. Further generalizations can be found in [23–25].

First we define for each $w \in G$ a Hilbert space $\mathcal{H}_w$ by putting

$$\mathcal{H}_e = \mathcal{H}_0 \oplus \Big(\bigoplus_{i\in I} \mathcal{H}_i\Big)$$

and, for $w \neq e$,

$$\mathcal{H}_w = \bigoplus_{j\in I\setminus\{i(w)\}} \mathcal{H}_j,$$

where for an element $x \neq e$ as in (4.1) we put $i(x) := i_m$, the type of the last letter of x. Now we define

$$\begin{aligned}\mathcal{H} &= \bigoplus_{w\in G} \mathcal{H}_w \\ &= \Big\{f : G \to \mathcal{H}_e \mid f(w) \in \mathcal{H}_w \text{ for } w \in G \text{ and } \sum_{w\in G} \|f(w)\|^2 < \infty\Big\}.\end{aligned}$$

For every $w \in G$ and $\xi \in \mathcal{H}_w$ we denote by (w, ξ) the function in $\mathcal{H}$ which has the value ξ at w and 0 elsewhere (i.e., $(w, \xi) := \delta_w \otimes \xi$). Then $\delta_w \otimes \mathcal{H}_w$ can be identified with the space of all functions in $\mathcal{H}$ vanishing outside $\{w\}$. We shall also identify $\mathcal{H}_0$ and $\mathcal{H}_i$ with the appropriate subspaces of $\delta_e \otimes \mathcal{H}_e$. If $i \in I \cup \{0\}$, then Q_i will stand for the orthogonal projection of $\mathcal{H}_e$ onto $\mathcal{H}_i$.

Fix $i \in I$. First we define π only on G_i putting $\pi(e) = \mathrm{Id}$ and for $g \in G_i \setminus \{e\}$

$$(5.1a)\qquad (\pi(g)f)(w) = \begin{cases} f(g^{-1}) + \pi_i(g)(Q_0 + Q_i)f(e) & \text{if } w = e, \\ (\mathrm{Id} - Q_0 - Q_i)f(e) & \text{if } w = g, \\ f(g^{-1}w) & \text{otherwise,} \end{cases}$$

or, in terms of the vectors (w, ξ),

$$(5.1b)\ \pi(g)(w,\xi) = \begin{cases} (e, \pi_i(g)(Q_0 + Q_i)\xi) + (g, (\mathrm{Id} - Q_0 - Q_i)\xi) & \text{if } w = e, \\ (gw, \xi) & \text{otherwise.} \end{cases}$$

Then π is a representation of G_i which coincides with π_i on $\mathcal{H}_0 \oplus \mathcal{H}_i$ and is a multiple of the left regular representation on the orthogonal complement of $\mathcal{H}_0 \oplus \mathcal{H}_i$. In particular, if π_i is unitary, then so is π.

Having defined $\pi(g)$ for all $g \in G_i$ and $i \in I$ we extend this to a representation all of G by putting

$$(5.1c)\qquad \pi(x) = \pi(g_1)\pi(g_2)\cdots\pi(g_m)$$

for x as in (4.1). Thus we obtained a representation $(\pi, \mathcal{H})$ of $G = *_{i\in I}G_i$ which we denote by $*_{i\in I}\pi_i$. Note that if all π_i's are unitary, then so is π.

Lemma 5.1. *Suppose that $x \neq e$ is as in (4.1) and $\xi \in \mathcal{H}_e$. Then*

$$\begin{aligned} &\pi(x)(e,\xi) \\ &= \Big(e,\ \pi_{i_1}(g_1)Q_0\pi_{i_2}(g_2)Q_0\cdots Q_0\pi_{i_{m-1}}(g_{m-1})Q_0\pi_{i_m}(g_m)(Q_0 + Q_{i_m})\xi\Big) \\ &\quad + \sum_{k=1}^{m-1}\Big(g_1g_2\cdots g_k, Q_{i_{k+1}}\pi_{i_{k+1}}(g_{k+1})Q_0\pi_{i_{k+2}}(g_{k+2})Q_0\cdots Q_0\pi_{i_m}(g_m) \\ &\qquad\qquad \times (Q_0 + Q_{i_m})\xi\Big) \\ &\quad + (g_1g_2\cdots g_m, (\mathrm{Id} - Q_0 - Q_{i_m})\xi). \end{aligned}$$

Proof. We apply induction on the length of x. If $|x| = 1$, then the formula is a consequence of the definition. Assume that it holds for x as in (4.1) and take a word of the form g_0x, where $g_0 \in G_{i_0} \setminus \{e\}$, $i_0 \neq i_1$. Using the fact that if $\eta \in \mathcal{H}_0 \oplus \mathcal{H}_{i_1}$, then $(Q_0 + Q_{i_0})\eta = Q_0\eta$ and $(\mathrm{Id} - Q_0 - Q_{i_0})\eta = Q_{i_1}\eta$ one can easily prove that the formula holds for g_0x. ◻

Let us denote by P_0 the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_0$.

Theorem 5.2. *Suppose that* $\pi = *_{i\in I}\pi_i$ *is the representation of* $G = *_{i\in I}G_i$ *defined above. Then:*

(i) *If all* π_i*'s are unitary, then so is* π.

(ii) *If* $\xi_0 \in \mathcal{H}_0$ *then, for* x *as in* (4.1),

$$P_0\pi(x)\xi_0 = Q_0\pi_{i_1}(g_1)Q_0\pi_{i_2}(g_2)\cdots Q_0\pi_{i_m}(g_m)\xi_0 .$$

(iii) *Assume that there are constants* $a_1, a_2, \ldots$ *such that*

$$\|\pi_{j_1}(h_1)Q_0\pi_{j_2}(h_2)\cdots Q_0\pi_{j_n}(h_n)\| \le a_n$$

(*the operator norm of a map* $\mathcal{H}_0\oplus\mathcal{H}_{j_n} \to \mathcal{H}_0\oplus\mathcal{H}_{j_1}$) *holds for every* $j_1j_2\cdots j_n \in S(I)$ *and every* $h_1 \in G_{j_1}\setminus\{e\}, \ldots, h_n \in G_{j_n}\setminus\{e\}$. *Then for every* $x \in G$,

$$\|\pi(x)\| \le 1 + a_1 + \cdots + a_m,$$

where $m = |x|$. *In particular, if* $\sum a_n < \infty$, *then* π *is uniformly bounded.*

(iv) *Assume that for each* $i \in I$ *the set* $\{\pi_i(g)\xi : g \in G_i,\ \xi \in \mathcal{H}_0\}$ *is linearly dense in* $\mathcal{H}_0 \oplus \mathcal{H}_i$. *Then the family* $\{\pi(x)(e,\xi) : x \in G,\ \xi \in \mathcal{H}_0\}$ *is linearly dense in* $\mathcal{H}$.

Proof. We have already noticed that unitarity of π_i's implies that of π, and part (ii) is a consequence of the last lemma. Now we will prove (iii).

Fix $x = g_1g_2\cdots g_m \ne e$ as in (4.1). Define $w_r = (g_{r+1}g_{r+2}\cdots g_m)^{-1}$, for $1 \le r \le m$. By the previous lemma we have

$$\begin{aligned}
&\pi(x)(w_r,\xi) = \pi(g_1g_2\cdots g_r)(e,\xi)\\
&= \Big(e, \pi_{i_1}(g_1)Q_0\pi_{i_2}(g_2)Q_0\cdots Q_0\pi_{i_{r-1}}(g_{r-1})Q_0\pi_{i_r}(g_r)(Q_0+Q_{i_r})\xi\Big)\\
&\quad + \sum_{k=1}^{r-1}\Big(g_1g_2\cdots g_k, Q_{i_{k+1}}\pi_{i_{k+1}}(g_{k+1})Q_0\pi_{i_{k+2}}Q_0\cdots Q_0\pi_{i_r}(g_r)(Q_0+Q_{i_r})\xi\Big)\\
&\quad + (g_1g_2\cdots g_r, (\mathrm{Id}-Q_0-Q_{i_r})\xi).
\end{aligned}$$

On the other hand if w is none of w_r, $1 \le r \le m$, then $\pi(x)(w,\xi) = (xw,\xi)$.

Consider the following operators on $\mathcal{H}$:

$$T_0(w,\xi) = \begin{cases} (xw_r, (\mathrm{Id}-Q_0-Q_{i_r})\xi) & \text{if } w = w_r,\ 1 \le r \le m,\\ (xw,\xi) & \text{otherwise,}\end{cases}$$

and for $1 \le s \le m$ we define T_s putting

$$\begin{aligned}
T_s(w_s,\xi) &= (e, \pi_{i_1}(g_1)Q_0\pi_{i_2}(g_2)Q_0\cdots Q_0\pi_{i_s}(g_s)(Q_0+Q_{i_s})\xi),\\
T_s(w_r,\xi) &= (g_1\cdots g_{r-s}, Q_{i_{r-s+1}}\pi_{i_{r-s+1}}(g_{r-s+1})Q_0\cdots Q_0\pi_{i_r}(g_r)(Q_0+Q_{i_r})\xi)
\end{aligned}$$

if $s < r \leq m$, and $T_s(w, \xi) = 0$ if w is not one of w_r for $s \leq r \leq m$. Then, putting $a_0 = 1$, we have $\|T_s\| \leq a_s$ and $\pi(x) = T_0 + T_1 + \cdots + T_m$, which proves (iii).

To prove (iv) denote by M the closure of the linear hull of the set $\{\pi(x)(e, \xi) : x \in G,\ \xi \in \mathcal{H}_0\}$. Then M is G-invariant and $\mathcal{H}_0 \subseteq M$. By the assumption, $\delta_e \otimes \mathcal{H}_i \subseteq M$ for every $i \in I$. Therefore $\delta_x \otimes \mathcal{H}_i = \pi(x)(\delta_e \otimes \mathcal{H}_i) \subseteq M$ for every $x \in G$ and $i \neq i(x)$, if $x \neq e$. But the family of such subspaces is linearly dense in $\mathcal{H}$, which concludes the proof. □

As a corollary, we will prove again Bożejko's result (Theorem 7.1 in [3]).

Corollary 5.3. *Let $\mathcal{H}_0$ be a Hilbert space and assume that for each $i \in I$ we have an operator-valued positive definite function $U_i : G_i \to \mathcal{B}(\mathcal{H}_0)$ satisfying $U_i(e) = \mathrm{Id}$. Define $U : G \to \mathcal{B}(\mathcal{H}_0)$ putting*

$$U(x) = U_{i_1}(g_1)U_{i_2}(g_2)\cdots U_{i_m}(g_m)$$

for $x = g_1 g_2 \cdots g_m$ as in (4.1). Then U is a positive definite function on G.

Proof. By [26, Theorem 7.1], for each $i \in I$ there is a Hilbert space $\mathcal{H}_i$ and a unitary representation $\pi_i : G_i \to \mathcal{B}(\mathcal{H}_0 \oplus \mathcal{H}_i)$ such that if $\xi \in \mathcal{H}_0$, $g \in G_i$, then $U_i(g)\xi = Q_0\pi_i(g)\xi$, where Q_0 denotes the orthogonal projection of $\mathcal{H}_0 \oplus \mathcal{H}_i$ onto $\mathcal{H}_0$. Take the representation $\pi = *_{i\in I}\pi_i$ of G constructed as above. Then π is unitary and if $\xi \in \mathcal{H}_0$, $x \in G$, then $P_0\pi(x)\xi = U(x)\xi$ which implies positive definiteness of U. □

Now assume that $\{P_i\}_{i\in I}$ is a family of (not necessarily orthogonal) projections on a Hilbert space $\mathcal{H}_0$. With every $i \in I$ we associate a space $\mathcal{H}_i$ and a representation $\pi_i : G_i \to \mathcal{B}(\mathcal{H}_0 \oplus \mathcal{H}_i)$ in the following way.

First assume that G_i is infinite. Then we set

$$\mathcal{H}_i := \Big\{ f : G_i \setminus \{e\} \to \ker P_i \mid \sum_{g\in G_i\setminus\{e\}} \|f(g)\|^2 < \infty \Big\},$$

so that

$$\mathcal{H}_0 \oplus \mathcal{H}_i = \Big\{ f : G_i \to \mathcal{H}_0 \mid f(g) \in \ker P_i \text{ for } g \neq e \text{ and } \sum_{g\in G_i} \|f(g)\|^2 < \infty \Big\}.$$

Now we define $\pi_i(e) = \mathrm{Id}$, and for $g \in G_i \setminus \{e\}$

$$(\pi_i(g)f)(h) = \begin{cases} P_i f(e) + f(g^{-1}) & \text{if } h = e, \\ (\mathrm{Id} - P_i)f(e) & \text{if } h = g, \\ f(g^{-1}h) & \text{otherwise.} \end{cases} \tag{5.2a}$$

Note that π_i acts trivially on $\operatorname{Im} P_i$, and as a multiple of the regular representation λ_i of G_i on $\ker P_i \oplus \mathcal{H}_i$.

The case when G_i is finite is a little bit more involved. Put

$$\mathcal{H}_i = \Big\{ f : G_i \setminus \{e\} \to \ker P_i \mid \sum_{g \in G_i \setminus \{e\}} f(g) = 0 \Big\}$$

so that

$$\mathcal{H}_0 \oplus \mathcal{H}_i = \Big\{ f : G_i \to \mathcal{H}_0 \mid \text{if } g \neq e, \text{ then } f(g) \in \ker P_i \text{ and } \textstyle\sum_{g \in G_i \setminus \{e\}} f(g) = 0 \Big\}.$$

Now we note that $\ker P_i \oplus \mathcal{H}_i$ can be identified with

$$\mathcal{H}_i' := \Big\{ F : G_i \to \ker P_i \mid \sum_{g \in G_i} F(g) = 0 \Big\}$$

by an isometry $T_i : \mathcal{H}_i' \to \ker P_i \oplus \mathcal{H}_i$:

$$T_i F(h) = \begin{cases} \sqrt{1+\tau_i} \cdot F(e) & \text{if } h = e, \\ F(h) + \tau_i \cdot F(e) & \text{otherwise,} \end{cases}$$

where $\tau_i = (|G_i| - 1)^{-1}$. Observe that $T_i^{-1}(\ker P_i)$ consists of functions of the form $e \mapsto \xi$, $g \mapsto -\tau_i \cdot \xi$ for $g \neq e$, with $\xi \in \ker P_i$. The natural representation π_i' of G_i acting on $\mathcal{H}_i'$ is a multiple of the semiregular representation:

$$(\pi_i'(g)F)(h) := F(g^{-1}h).$$

Now we define a representation $\pi_i : G_i \to \mathcal{B}(\mathcal{H}_0 \oplus \mathcal{H}_i)$ putting

$$(\pi_i(g)f) := \widetilde{P_i} f + (T_i \pi_i'(g) T_i^{-1})(\mathrm{Id} - \widetilde{P_i}) f,$$

where $\widetilde{P_i}$ is a projection on $\mathcal{H}_0 \oplus \mathcal{H}_i$ given by

$$\widetilde{P_i} f(h) = \begin{cases} P_i f(e) & \text{if } h = e, \\ 0 & \text{otherwise.} \end{cases}$$

As a result, for $g \in G_i \setminus \{e\}$ we obtain the following formula

(5.2b) $(\pi_i(g)f)(h)$

$$= \begin{cases} P_i f(e) + \sqrt{1+\tau_i} \cdot f(g^{-1}) - \tau_i \cdot (\mathrm{Id} - P_i) f(e) & \text{if } h = e, \\ \tau_i \cdot f(g^{-1}) + (1 - \tau_i)\sqrt{1+\tau_i} \cdot (\mathrm{Id} - P_i) f(e) & \text{if } h = g, \\ f(g^{-1}h) + \tau_i \cdot f(g^{-1}) - \tau_i \sqrt{1+\tau_i} \cdot (\mathrm{Id} - P_i) f(e) & \text{otherwise.} \end{cases}$$

Now we are going to define the main object of this paper.

Definition 5.4. Assume that $\{G_i\}_{i\in I}$ is a family of discrete groups and put $\tau_i := (|G_i| - 1)^{-1}$. Let $\pi_0 : \mathcal{A}(\tau) \to \mathcal{B}(\mathcal{H}_0)$ be a representation of $\mathcal{A}(\tau)$ (not necessarily a $*$-representation) so that $\pi_0(\delta_i) = (1+\tau_i)P_i - \tau_i \operatorname{Id}$ for a (not necessarily orthogonal) projection P_i on $\mathcal{H}_0$. For every $i \in I$ define a representation π_i of G_i by (5.2) and then define a representation π of $G := *_{i\in I} G_i$ using formulas (5.1). Then we will say that π is *induced from the representation* π_0 *of* $\mathcal{A}(\tau)$.

Now we observe that π inherits some properties of π_0.

Theorem 5.5. *Assume that* $G = *_{i\in I} G_i$, $\tau_i = (|G_i| - 1)^{-1}$, $\pi_0 : \mathcal{A}(\tau) \to \mathcal{B}(\mathcal{H}_0)$ *is a representation and let* $\pi : G \to \mathcal{B}(\mathcal{H})$ *be the induced representation of* G.

- (i) *For every* $x \in G$ *we have* $P_0\pi(x)|_{\mathcal{H}_0} = \pi_0(\delta_{t(x)})$.
- (ii) *If* $\mathcal{H}_0$ *can be decomposed into a direct sum* $\mathcal{H}_0' + \mathcal{H}_0''$ *of* $\mathcal{A}(\tau)$*-invariant closed subspaces, then* $\mathcal{H}$ *can be decomposed into a direct sum* $\mathcal{H}' + \mathcal{H}''$ *of* G*-invariant closed subspaces, and the corresponding representations* π', π'' *of* G *are induced from the corresponding representations* π_0', π_0'' *of* $\mathcal{A}(\tau)$. *If the former sum is orthogonal, then so is the latter.*
- (iii) *If* π_0 *is the left regular representation of* $\mathcal{A}(\tau)$, *then* π *is the left regular representation of* G.
- (iv) *If* π_0 *is a* $*$*-representation, then* π *is a unitary representation of* G.
- (v) *If there are constants* d_m *such that* $\|\pi_0(\delta_u)\| \le d_{|u|}$ *for every* $u \in S(I)$ *and if* $\sum_{m=0}^{\infty} d_m < \infty$, *then* π *is uniformly bounded.*
- (vi) *The set* $\{\pi(x)(e,\xi) : x \in G,\ \xi \in \mathcal{H}_0\}$ *is linearly dense in* $\mathcal{H}$.

Proof. (i) By formula (5.2) and Theorem 5.2 (ii) we have $P_0\pi(x)|_{\mathcal{H}_0} = B_{i_1}\cdots B_{i_m}$, where $B_i = (1+\tau_i)P_i - \tau_i \operatorname{Id}$, so $P_0\pi(x)|_{\mathcal{H}_0} = \pi_0(\delta_{t(x)})$.

(ii) Let P be the projection of $\mathcal{H}_0$ onto $\mathcal{H}_0'$ with $\ker P = \mathcal{H}_0''$. Replacing $\ker P_i$ by $P\ker P_i$ and $(\operatorname{Id} - P)\ker P_i$ in the definition of $\mathcal{H}_i$ we decompose $\mathcal{H}_i$ into a direct sum $\mathcal{H}_i' + \mathcal{H}_i''$ of G_i-invariant subspaces, which leads to the decomposition of all of $\mathcal{H}$ into a direct sum $\mathcal{H}' + \mathcal{H}''$ of G-invariant subspaces.

To prove (iii) we note that the vector δ_e is cyclic for λ_0 and $[\lambda_0(a)\delta_e, \delta_e] = a(e)$ for $a \in \mathcal{A}(\tau)$. In view of the previous point we have

$$[\pi(x)\delta_e, \delta_e] = [P_0\pi(x)\delta_e, \delta_e] = [\pi_0(\delta_{t(x)})\delta_e, \delta_e] = \begin{cases} 1 & \text{if } x = e, \\ 0 & \text{otherwise,} \end{cases}$$

for $x \in G$, and, since the vectors $\pi(x)\xi$, $x \in G$, $\xi \in \mathcal{H}_0$, are linearly dense in $\mathcal{H}$, the vector δ_e is cyclic for π. It means that π is the left regular representation of G.

If π_0 is a $*$-representation, then all the projections P_i are orthogonal and then every representation π_i of G_i is unitary, which implies unitarity of π.

Assume that $\|\pi_0(\delta_u)\| \le d_{|u|}$ for every $u \in S(I)$. Then the norms $\|P_i\|$ are uniformly bounded, which implies that there is a constant C such that $\|\pi_i(g)\| \le$

C for every $i \in I$ and $g \in G_i$. Then for $y = h_1 \cdots h_n$, with $t(y) = j_1 \cdots j_n$, we have

$$\|\pi_{j_1}(h_1)Q_0 \cdots Q_0\pi_{j_n}(h_n)\| = \|\pi_{j_1}(h_1)B_{j_2}B_{j_3} \cdots B_{j_{n-1}}\pi_{j_n}(h_n)\| \\ \le C^2\|B_{j_2} \cdots B_{j_{n-1}}\| \le C^2 d_{n-2},$$

and in view of Theorem 5.2 (iii) we see that π is uniformly bounded.

For (vi) it suffices, by Theorem 5.2 (iv), to prove that for every $i \in I$ the set $\{\pi_i(g)\xi : g \in G_i,\ \xi \in \mathcal{H}_0\}$ is linearly dense in $\mathcal{H}_0 \oplus \mathcal{H}_i$. If G_i is infinite, then it is clear that the linear span of vectors of the form $\delta_g \otimes \xi = \pi_i(g)\xi$, where $g \in G_i \setminus \{e\}$, $\xi \in \ker P_i$, is dense in $\mathcal{H}_i$.

Now, if G_i is finite, then it is sufficient to check, that

$$\left\{\pi_i'(g)F : g \in G_i,\ F \in T_i^{-1}(\ker P_i)\right\} = \mathcal{H}_i'.$$

Fix $F \in \mathcal{H}_i'$. For $g \in G_i$ we define $F_g \in T_i^{-1}(\ker P_i)$ by putting

$$F_g(h) := \begin{cases} \dfrac{1}{1+\tau_i}F(g) & \text{if } h = e, \\ \dfrac{-\tau_i}{1+\tau_i}F(g) & \text{if } h \neq e. \end{cases}$$

Then one can check that

$$\sum_{g \in G_i} \pi_i'(g)F_g = f,$$

which concludes the proof. ◻

For further investigations we will need two lemmas.

Lemma 5.6. *Given a $*$-representation ρ of a $*$-algebra $\mathcal{A}$ acting on a Hilbert space $\mathcal{H}$ and a family $\{\rho_\alpha\}$, $\alpha \in A$, of subrepresentations of ρ, each ρ_α acting on $\mathcal{H}_\alpha \subseteq \mathcal{H}$, such that the set $\bigcup_{\alpha \in A} \mathcal{H}_\alpha$ is linearly dense in $\mathcal{H}$, then ρ is equivalent to a subrepresentation of the direct sum $\bigoplus_{\alpha \in A} \rho_\alpha$.*

Proof. We may assume that the index set A is an ordinal. Put

$$V_\alpha = \overline{\operatorname{lin}\{\mathcal{H}_\beta : \beta < \alpha\}}.$$

Fix α and let W_α be the orthogonal complement of V_α in $V_{\alpha+1}$. Consider $P : \mathcal{H}_\alpha \mapsto W_\alpha$ being the orthogonal projection. Take the polar decomposition $P = US$, where S is positive definite and U is a partial isometry. By definition of $V_{\alpha+1}$, the image of $\mathcal{H}_\alpha$ is dense in W_α so U^{-1} is an isometric embedding of W_α in $\mathcal{H}_\alpha$. Moreover, by the uniqueness of polar decomposition, U^{-1} intertwines the action of $\mathcal{A}$ on W_α with the action of $\mathcal{A}$ on $\mathcal{H}_\alpha$. Let $\widetilde{\rho}_\alpha$ be the restriction of ρ to W_α. By definition

$$\bigoplus_{\alpha\in A} W_\alpha = \overline{\operatorname{lin}\{\mathcal{H}_\beta : \beta \in A\}} = \mathcal{H}$$

so $\bigoplus_{\alpha\in A} \widetilde{\rho}_\alpha = \rho$. Since each $\widetilde{\rho}_\alpha$ is equivalent to a subrepresentation of ρ_α the claim follows. □

Having constructed the induced representation π, we can in turn define a representation of $\mathcal{A}(\tau)$ acting on $\mathcal{H}$. Namely, we put

$$(5.3) \qquad \widetilde{\pi}_0(\delta_i) = \begin{cases} \tau_i \cdot \sum\limits_{g\in G_i\setminus\{e\}} \pi(g) & \text{if } G_i \text{ is finite,} \\ \lim \dfrac{1}{n} \sum\limits_{k=1}^{n} \pi(g_{k,i}) & \text{otherwise,} \end{cases}$$

where $g_{k,i}$ is an arbitrary sequence of distinct elements of G_i and the limit is in the strong operator topology. Note that $\widetilde{\pi}_0$ restricted to $\mathcal{H}_0$ is just π_0 and that if G_i is infinite, then $\widetilde{\pi}_0(\delta_i) = \pi_0(\delta_i)P_0$, where P_0 is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{H}_0$.

Lemma 5.7. *The restriction of $\widetilde{\pi}_0$ to $\mathcal{H}_0^\perp$ is contained in a multiple of the left regular representation.*

Proof. For $i \in I$, $\xi \in \mathcal{H}_i$ we have $\pi(\mu_i)(e,\xi) = -\tau_i(e,\xi)$ and for a vector (w,ξ) in $(\mathcal{H}_0 \oplus \mathcal{H}_i)^\perp$ we have $\pi(\mu_i)(w,\xi) = \tau_i \sum_{g\in G_i\setminus\{e\}}(gw,\xi)$. Therefore for every $w \in G$ and $(w,\xi) \in \mathcal{H}_0^\perp$, the τ-positive definite function

$$u \mapsto [\pi(\mu_u)(w,\xi),(w,\xi)]$$

on $S(I)$ has a finite support and hence belongs to $\ell^2(\tau)$. □

Here we give one application of the induced representation. Recall that for a $*$-algebra $\mathcal{A}$, the *enveloping C^*-algebra $C^*(\mathcal{A})$* is defined as the completion of $\mathcal{A}$ with respect to the norm

$$\|a\| := \sup\Big\{\|\pi(a)\| : \pi \text{ is a } *\text{-representation of } \mathcal{A}\Big\}.$$

If Γ is a discrete group, then $C^*(\mathcal{F}(\Gamma))$ is called the *full C^*-algebra of* Γ and denoted $C^*(\Gamma)$. Define also the *reduced C^*-algebra of* Γ (resp. *of the algebra* $\mathcal{A}(\tau)$), denoted $C^*_r(\Gamma)$ (or $C^*_r(\mathcal{A}(\tau))$), as the closure of $\mathcal{F}(\Gamma)$ (resp. of $\mathcal{F}(S(I))$) in the operator norm $\|\lambda(f)\|$ (resp. $\|\lambda_0(a)\|$).

Proposition 5.8. *Suppose that all the groups G_i are finite, $\tau_i = (|G_i| - 1)^{-1}$ and $G = *_{i\in I}G_i$. Then the map*

$$\mathcal{A}(\tau) \ni a \mapsto \sum_{u\in S(I)} a(u)\mu_u \in \mathcal{F}(G)$$

extends to an isometric embedding of $C^*(\mathcal{A}(\tau))$ into $C^*(G)$ and to an isometric embedding of $C^*_r(\mathcal{A}(\tau))$ into $C^*_r(G)$.

Proof. Put $j(a) := \sum_{u\in S(I)} a(u)\mu_u$. If π is a unitary representation of G, then $\pi\circ j$ is a $*$-representation of $\mathcal{A}(\tau)$, which implies that $\|j(a)\| \le \|a\|$ for $a\in\mathcal{A}(\tau)$. On the other hand, if π_0 is a $*$-representation of $\mathcal{A}(\tau)$, then taking the induced representation π of G we have $\|\pi_0(a)\| \le \|\pi(j(a))\|$, which yields $\|a\| \le \|j(a)\|$.

The second statement holds because the map $a \mapsto \lambda(j(a))$ contains a copy and is contained in a multiple of the left regular representation of $\mathcal{A}(\tau)$. □

Recall that if $\pi:\mathcal{A}\to\mathcal{B}(\mathcal{H})$ and $\sigma:\mathcal{A}\to\mathcal{B}(\mathcal{K})$ are representations of an algebra $\mathcal{A}$, then $T:\mathcal{H}\to\mathcal{K}$ is said to be an *intertwining operator* if $T\pi(a) = \sigma(a)T$ holds for every $a\in\mathcal{A}$. Representations π and σ are called *equivalent* if there is an intertwining isomorphism $T:\mathcal{H}\to\mathcal{K}$. Representations π and σ are said to be *disjoint* if 0 is the only operator intertwining them.

Lemma 5.9. *Let $\mathcal{A}_0$ be a subalgebra of an algebra $\mathcal{A}$ and let $\rho:\mathcal{A}\to\mathcal{B}(\mathcal{H})$, $\sigma:\mathcal{A}\to\mathcal{B}(\mathcal{K})$ be representations. Denote by ρ_0 and σ_0 their restrictions to $\mathcal{A}_0$. We assume that $\mathcal{H} = \mathcal{H}_0\oplus\mathcal{H}_1$, $\mathcal{K} = \mathcal{K}_0\oplus\mathcal{K}_1$ are decompositions into direct sums of $\mathcal{A}_0$-invariant closed subspaces.*

(1) *If $\rho_0|_{\mathcal{H}_0}$ is disjoint from $\sigma_0|_{\mathcal{K}_1}$, $\rho_0|_{\mathcal{H}_1}$ is disjoint from $\sigma_0|_{\mathcal{K}_0}$ and $T:\mathcal{H}\to\mathcal{K}$ intertwines ρ and σ, then $T\mathcal{H}_0\subseteq\mathcal{K}_0$ and $T\mathcal{H}_1\subseteq\mathcal{K}_1$.*
(2) *If $\rho(\mathcal{A})\mathcal{H}_0$ is dense in $\mathcal{H}$ and $T_1, T_2:\mathcal{H}\to\mathcal{K}$ are intertwining operators for ρ and σ such that $T_1|_{\mathcal{H}_0} = T_2|_{\mathcal{H}_0}$, then $T_1 = T_2$.*

Proof. For $\eta = \eta_0+\eta_1\in\mathcal{K}$, with $\eta_j\in\mathcal{K}_j$, we put $Q_j\eta := \eta_j$. We need to prove that both the operators $B := Q_1T|_{\mathcal{H}_0}$ and $C := Q_0T|_{\mathcal{H}_1}$ are zero. But B (resp C) intertwines $\rho_0|_{\mathcal{H}_0}$ and $\sigma_0|_{\mathcal{H}_1}$ (resp. $\rho_0|_{\mathcal{H}_1}$ and $\rho_0|_{\mathcal{K}_0}$), which implies that $B=0$ and $C=0$ and proves the first assertion.

For the second one we note that for $a\in\mathcal{A}$ and $\xi\in\mathcal{H}_0$ we have

$$T_1\rho(a)\xi = \sigma(a)T_1\xi = \sigma(a)T_2\xi = T_2\rho(a)\xi$$

which implies that $T_1 = T_2$. □

Theorem 5.10. *Suppose that π_0 and σ_0 are two representations of $\mathcal{A}(\tau)$ which are disjoint from the regular representation of $\mathcal{A}(\tau)$ and let π and σ be the induced representations of G.*

(1) *If π_0 and σ_0 are not equivalent, then so are π and σ.*
(2) *If π_0 and σ_0 are disjoint, then so are π and σ.*

Proof. If $T:\mathcal{H}\to\mathcal{K}$ is an intertwining operator between π and σ then, by the previous lemma, $T(\mathcal{H}_0)\subseteq\mathcal{K}_0$, and hence $T|\mathcal{H}_0$ intertwines π_0 and σ_0. Therefore if π_0 and σ_0 are not equivalent, then $T|_{\mathcal{H}_0}$, and hence T cannot be an isomorphism. Moreover, if π_0 and σ_0 are disjoint, then $T|_{\mathcal{H}_0} = 0$ and for $\xi\in\mathcal{H}_0$, $x\in G$ we have $T\pi(x)\xi = \sigma(x)T\xi = 0$, which implies that $T=0$. □

Recall that a representation π of a $*$-algebra $\mathcal{A}$ is said to be *weakly contained* in a representation σ of $\mathcal{A}$ if $\|\pi(a)\| \le \|\sigma(a)\|$ for every $a \in \mathcal{A}$. This is equivalent to saying that $\ker(\sigma) \subseteq \ker(\pi)$, where $\ker(\pi)$ and $\ker(\sigma)$ denote the kernel of the extension of π and σ to the enveloping C^*-algebra.

Proposition 5.11. *Suppose that π_0 and σ_0 are $*$-representations of $\mathcal{A}(\tau)$ and let π and σ be the induced representations of $G = *_{i\in I} G_i$.*

(1) *If π_0 is weakly contained in σ_0, then π is weakly contained in σ.*
(2) *If π is weakly contained in σ, then π_0 is weakly contained in $\sigma_0 \oplus \lambda_0$.*

In particular, π is weakly contained in the regular representation of G if and only if π_0 is weakly contained in the regular representation λ_0 of $\mathcal{A}(\tau)$.

Proof. Suppose that π_0 is weakly contained in σ_0 and decompose π_0 into a direct sum $\bigoplus_{\alpha\in A} \pi_0^\alpha$ of cyclic representations. For each $\alpha \in A$ we fix a unit cyclic vector ξ^α for π_0^α and let φ^α be the corresponding state on $\mathcal{A}(\tau)$, i.e., $\varphi^\alpha(a) = \langle \pi_0^\alpha(a)\xi^\alpha, \xi^\alpha\rangle$ for $a \in \mathcal{A}(\tau)$. Then $\pi = \bigoplus_{\alpha\in A} \pi^\alpha$, where π^α is induced from π_0^α, and ξ^α is a cyclic vector also for π^α. We need to show that every π^α is weakly contained in σ. Then the corresponding positive definite function on G is $x \mapsto \varphi^\alpha(\delta_{t(x)})$. Since each π_0^α is weakly contained in σ_0 there is a sequence ζ_n^α of vectors in the space of σ_0 such that $\lim_{n\to\infty}\langle \sigma_0(a)\zeta_n^\alpha, \zeta_n^\alpha\rangle = \varphi^\alpha(a)$ for every $a \in \mathcal{A}(\tau)$. Then for every $x \in G$ we have

$$\lim_{n\to\infty} \langle \sigma(x)\zeta_n^\alpha, \zeta_n^\alpha\rangle = \lim_{n\to\infty} \langle \sigma_0(\delta_{t(x)})\zeta_n^\alpha, \zeta_n^\alpha\rangle = \varphi^\alpha(\delta_{t(x)}),$$

which means that π_α is weakly contained in σ.

If π is weakly contained in σ, then $\ker(\sigma) \subseteq \ker(\pi)$ (the kernels are meant with respect to the full C^*-algebra of G) and hence

$$\ker\left(\sigma|_{C^*(\mathcal{A}(\tau))}\right) \subseteq \ker\left(\pi|_{C^*(\mathcal{A}(\tau))}\right).$$

But the restriction of π (resp. σ) to $C^*(\mathcal{A}(\tau))$ is a direct sum of π_0 (resp. σ_0) and a subrepresentation of a multiple of λ_0 (Lemma 5.7). Therefore we have $\ker(\sigma_0) \cap \ker(\lambda_0) \subseteq \ker(\pi_0) \cap \ker(\lambda_0) \subseteq \ker(\pi_0)$, which proves the second statement. ☐

6. Irreducibility of the Induced Representation

Throughout this section we fix a family $\{G_i\}_{i\in I}$ of groups and a representation π_0 of $\mathcal{A}(\tau)$, acting on a Hilbert space $\mathcal{H}_0$, where $\tau_i := (|G_i| - 1)^{-1}$, and therefore a family $\{P_i\}_{i\in I}$ of projections on $\mathcal{H}_0$, such that $\pi_0(\delta_i) = (1 + \tau_i)P_i - \tau_i \,\mathrm{Id}$. We are going to study irreducibility of the induced representation π of the group $G = *_{i\in I} G_i$.

Proposition 6.1. *Assume that the representation π_0 (and hence the family of projections $\{P_i\}_{i\in I}$) is topologically irreducible (i.e., there is no nontrivial invariant closed subspace) and that there exists $i_0 \in I$ such that G_{i_0} is infinite and $P_{i_0} \neq 0$. Then π is topologically irreducible.*

Proof. Define operators T_i on $\mathcal{H}$ by

$$T_i = \begin{cases} \dfrac{1}{|G_i|} \displaystyle\sum_{g \in G_i} \pi(g) & \text{if } G_i \text{ is finite,} \\ \displaystyle\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \pi(g_{n,i}) & \text{otherwise,} \end{cases}$$

where, as before, $\{g_{n,i}\}$ is a sequence of distinct elements in G_i and the convergence is in the strong sense (cf. Theorem 2.2 in [22]). Then T_i is a projection, $T_i|_{\mathcal{H}_0} = P_i$ and $T_i|_{\mathcal{H}_0^\perp}$ is an orthogonal projection. Moreover, if G_i is infinite, then $T_i|_{\mathcal{H}_0^\perp} = 0$.

Assume that M is an invariant closed subspace of $\mathcal{H}$. Then $T_i M \subseteq M$ for every $i \in I$. If $T_{i_0} M \neq 0$, then the space $M_0 := M \cap \mathcal{H}_0 \neq 0$ is invariant for all T_i, and hence for all P_i. This implies $M_0 = \mathcal{H}_0$ and consequently $M = \mathcal{H}$.

Now assume that $T_{i_0} M = 0$. This gives $T_{i_0} T_i M = 0$, which means that $T_i^* T_{i_0}^* \mathcal{H} = P_i^* P_{i_0}^* \mathcal{H}_0$ is orthogonal to M for every $i \in I$. Since the family $\{P_i\}_{i\in I}$ (and hence $\{P_i^*\}_{i\in I}$) is irreducible in $\mathcal{H}_0$ and $P_{i_0} \neq 0$, we have $\mathcal{H}_0 \perp M$, which implies $M = \{0\}$. ◻

From now on we will be assuming that π_0 is a $*$-representation of $\mathcal{A}(\tau)$, hence all P_i's are selfadjoint and therefore the induced representation π of G is unitary.

Lemma 6.2. *Let $\mathcal{A}_1$ be a $*$-subalgebra of a $*$-algebra $\mathcal{A}_2$ and let $\rho : \mathcal{A}_2 \to \mathcal{B}(\mathcal{H})$ be a $*$-representation with a cyclic vector ξ_0. Assume that $\mathcal{H}_1 \subseteq \mathcal{H}$ is an $\mathcal{A}_1$-invariant, closed subspace of $\mathcal{H}$, with $\xi_0 \in \mathcal{H}_1$, such that the representation $\rho|_{\mathcal{A}_1}$ is irreducible on $\mathcal{H}_1$ and occurs in $\mathcal{H}$ only once. Then ρ is an irreducible representation of $\mathcal{A}_2$.*

Proof. Let M be an $\mathcal{A}_2$-invariant closed subspace of $\mathcal{H}$ and let P be the orthogonal projection onto M. Then $P\rho(a) = \rho(a)P$ for every $a \in \mathcal{A}_2$. Indeed, for $\zeta, \eta \in \mathcal{H}$ we have

$$\begin{aligned} \langle P\rho(a)\zeta, \eta\rangle &= \langle \rho(a)\zeta, P\eta\rangle = \langle \zeta, \rho(a^*)P\eta\rangle \\ &= \langle P\zeta, \rho(a^*)P\eta\rangle = \langle \rho(a)P\zeta, P\eta\rangle = \langle \rho(a)P\zeta, \eta\rangle. \end{aligned}$$

If $P\mathcal{H}_1 = \{0\}$ then, in particular, $P\xi_0 = 0$ and hence $M = \{0\}$. Assume that $M_1 := P\mathcal{H}_1 \neq \{0\}$. Then one can observe that $\rho|_{\mathcal{A}_1}$ acts irreducibly on M_1 (for if M_0 is an $\mathcal{A}_1$-invariant subspace of M_1, then so is $P^{-1}(M_0) \cap \mathcal{H}_1$) and that $P_1 := P|_{\mathcal{H}_1}$ is an isomorphism $\mathcal{H}_1 \to M_1$ (for $\ker P_1$ is an $\mathcal{A}_1$-invariant subspace of $\mathcal{H}_1$). By our assumption this implies $\mathcal{H}_1 = M_1 \subseteq M$ and consequently $M = \mathcal{H}$ as ξ_0 lies in $\mathcal{H}_1$. ◻

Denote

$$I^{\text{fin}} := \{i \in I : G_i \text{ is finite}\}, \quad G^{\text{fin}} := *_{i\in I^{\text{fin}}} G_i$$

and let $(\mathcal{H}^{\mathrm{fin}} = \bigoplus_{w\in G^{\mathrm{fin}}} \mathcal{H}_w^{\mathrm{fin}}, \pi^{\mathrm{fin}})$ be the induced representation of G^{fin} related to the family $\{P_i\}_{i\in I^{\mathrm{fin}}}$. There is a natural embedding of $\mathcal{H}^{\mathrm{fin}}$ into $\mathcal{H}$. Namely, if we put $\mathcal{H}_1 := \bigoplus_{i\in I\setminus I^{\mathrm{fin}}} \mathcal{H}_i$, then $\mathcal{H}_w = \mathcal{H}_w^{\mathrm{fin}} \oplus \mathcal{H}_1$ for every $w \in G^{\mathrm{fin}}$. The orthogonal complement of $\mathcal{H}^{\mathrm{fin}}$ in $\mathcal{H}$ is $\ell^2(G^{\mathrm{fin}}, \mathcal{H}_1) \oplus \bigoplus_{w\in G\setminus G^{\mathrm{fin}}} \mathcal{H}_w$ and $\pi|_{G^{\mathrm{fin}}}$ acts on this subspace as a multiple of the regular representation, while $\pi|_{G^{\mathrm{fin}}}$ on $\mathcal{H}^{\mathrm{fin}}$ is just π^{fin}.

Lemma 6.3. *Assume that $|I^{\mathrm{fin}}| \geq 2$, $P_i = 0$ for every $i \in I \setminus I^{\mathrm{fin}}$ and that the representation π^{fin} of G^{fin} is irreducible. Then π is irreducible too.*

Proof. First of all we note that, by Theorem 5.5 (ii), π_0 is irreducible. If $|I^{\mathrm{fin}}| \geq 2$, then G^{fin} is infinite so no irreducible representation of G^{fin} can be contained in the orthogonal complement of $\mathcal{H}^{\mathrm{fin}}$ in $\mathcal{H}$. Note also that every nonzero vector $\xi_0 \in \mathcal{H}_0$ is cyclic for π. Indeed, by irreducibility of π^{fin}, the closure M of $\mathrm{lin}\{\pi(x)\xi_0 : x \in G\}$ contains $\mathcal{H}^{\mathrm{fin}}$. In particular $\mathcal{H}_0 \subseteq M$, which, by Theorem 5.5 (vi), implies $M = \mathcal{H}$. Applying the previous lemma to $\mathcal{A}_1 := \mathcal{F}(G^{\mathrm{fin}})$ and $\mathcal{A}_2 := \mathcal{F}(G)$, we see that π is irreducible. ◻

For $i \in I \cup \{0\}$ we define a function φ_i on $S(I)$ by

$$\varphi_0(i_1 i_2 \cdots i_m) := (-\tau_{i_1})(-\tau_{i_2}) \cdots (-\tau_{i_m}) \tag{6.1a}$$

and for $i \in I$

$$\varphi_i(i_1 i_2 \cdots i_m) := a_{i_1} a_{i_2} \cdots a_{i_m}, \tag{6.1b}$$

where $a_i := 1$ and $a_j := -\tau_j$ for $j \in I \setminus \{i\}$. We are now ready to present the main theorem of this paper:

Theorem 6.4. *Suppose that we are given a free product group $G := *_{i\in I} G_i$ and an irreducible $*$-representation $\pi_0 : \mathcal{A}(\tau) \to \mathcal{B}(\mathcal{H}_0)$ of the algebra $\mathcal{A}(\tau)$, where $\tau_i := (|G_i| - 1)^{-1}$. Then the induced unitary representation π of G is irreducible unless π_0 is contained in the regular representation λ_0 of $\mathcal{A}(\tau)$, that is, unless either*

(1) *$\mathcal{H}_0 = \mathbb{C}$ and $\pi_0(\delta_i) = -\tau_i$ for every $i \in I$ and*

$$\sum_{i\in I} \frac{1}{|G_i|} < 1,$$

(2) *or $\mathcal{H}_0 = \mathbb{C}$ and there is $i_0 \in I$ such that $\pi_0(\delta_{i_0}) = 1$, $\pi_0(\delta_i) = -\tau_i$ for all $i \in I \setminus \{i_0\}$ and*

$$\sum_{i\in I\setminus\{i_0\}} \frac{1}{|G_i|} < \frac{1}{|G_{i_0}|}.$$

Proof. If π_0 is irreducible and contained in λ_0, then $\dim \pi_0 = 1$ (Theorem 2.11) and either $\pi_0(\delta_i) = -\tau_i$ for every $i \in I$ and $\sum_{i\in I} \tau_i/(1+\tau_i) < 1$ or there is $i_0 \in I$ such that $\pi_0(\delta_{i_0}) = 1$, $\pi_0(\delta_i) = -\tau_i$ for $i \in I \setminus \{e\}$ and $1/(1+\tau_{i_0}) + \sum_{i\in I\setminus\{i_0\}} \tau_i/(1+\tau_i) < 1$. The corresponding character (φ_0 in the first case and φ_{i_0} in the second) of $S(\tau)$ can be written as $\varphi = f_0 \cdot \tau$ and $f_0 \in \ell^2(\tau)$, which means that the composition $\varphi \circ t$ belongs to $\ell^2(G)$ and hence π is not irreducible.

Now assume that π_0 is irreducible and not contained in λ_0. If there is $i \in I \setminus I^{\text{fin}}$ such that $\pi_0(\delta_i) \neq 0$, then π is irreducible in view of Proposition 6.1. If, on the other hand, $\pi_0(\delta_i) = 0$ for every $i \in I \setminus I^{\text{fin}}$, then our assumptions on π_0 imply that $|I^{\text{fin}}| \geq 2$. Now we observe that any nonzero vector $\xi \in \mathcal{H}_0$ is cyclic for π^{fin}. Indeed, by irreducibility of π_0 the closure of $\{\pi_0(a)\xi : a \in \mathcal{A}(\tau)\} = \{\pi_0(a)\xi : a \in \mathcal{A}(\tau|_{I^{\text{fin}}})\}$ is $\mathcal{H}_0$. Hence, by Theorem 5.5 (vi), ξ is cyclic for π^{fin}. Applying Lemma 6.2 to $\mathcal{A}_1 := \mathcal{F}_t(G^{\text{fin}}) \cong \mathcal{A}(\tau|_{I^{\text{fin}}})$ and $\mathcal{A}_2 := \mathcal{F}(G^{\text{fin}})$ we see, in view of Lemma 5.7, that π^{fin} is irreducible. Now we conclude the proof by using Lemma 6.3. ◻

Corollary 6.5. *Let $\mathcal{P}^1(G)$ (resp. $\mathcal{P}^1_t(G)$) denote the convex set of all (type-dependent) positive definite functions φ on $G = *_{i\in I} G_i$ with $\varphi(e) = 1$ and let* $\operatorname{ex}\mathcal{P}^1(G)$ *(resp.* $\operatorname{ex}\mathcal{P}^1_t(G)$*) denote the set of its extreme points.*

(1) *If*

$$\sum_{i\in I} \frac{1}{|G_i|} \geq 1,$$

then $\operatorname{ex}\mathcal{P}^1_t(G) \subseteq \operatorname{ex}\mathcal{P}^1(G)$.

(2) *If*

$$\sum_{i\in I} \frac{1}{|G_i|} < 1 \quad \text{but} \quad \sum_{i\in I\setminus\{j\}} \frac{1}{|G_i|} \geq \frac{1}{|G_j|}$$

for every $j \in I$, then $\operatorname{ex}\mathcal{P}^1_t(G) \setminus \operatorname{ex}\mathcal{P}^1(G) = \{\varphi_0 \circ t\}$.

(3) *Finally, if there is $i_0 \in I$ such that*

$$\sum_{i\in I\setminus\{i_0\}} \frac{1}{|G_i|} < \frac{1}{|G_{i_0}|},$$

then $\operatorname{ex}\mathcal{P}^1_t(G) \setminus \operatorname{ex}\mathcal{P}^1(G) = \{\varphi_0 \circ t, \varphi_{i_0} \circ t\}$.

In other words, if a type-dependent function $\varphi \circ t$ belongs to $\operatorname{ex}\mathcal{P}^1_t(G)$*, then* $\varphi \circ t \in \operatorname{ex}\mathcal{P}^1(G)$ *holds if and only if* $\varphi \circ t \notin \ell^2(G)$.

The exceptional representations from points (1) and (2) of Theorem 6.4 have been studied in [14], and the positive definite function $\varphi_0 \circ t$, in the case when the groups G_i are unipotent, appears in [32, Lemma 1]. The case when all G_i's are infinite was studied in [22].

Nonunitary representations. For nonunitary representations, various definitions of irreducibility are not equivalent.

Definition 6.6. Let ρ be a representation of a complex algebra $\mathcal{A}$ acting on a Hilbert space $\mathcal{K}$. Then ρ is said to be

(1) *algebraically irreducible* if there is no nontrivial invariant subspace of $\mathcal{K}$,
(2) *topologically irreducible* if there is no nontrivial closed invariant subspace of $\mathcal{K}$,
(3) *fully irreducible* if the closure of $\rho(\mathcal{A})$ in the strong operator topology coincides with $\mathcal{B}(\mathcal{K})$.

In view of the Burnside theorem, these notions coincide if $\dim \mathcal{K}$ is finite. In general, full irreducibility implies topological irreducibility.

Lemma 6.7. *Let $\rho_0 : \mathcal{A} \to \mathcal{B}(\mathcal{K}_0)$ be a finite dimensional irreducible representation of a $*$-algebra $\mathcal{A}$ which is not equivalent to a $*$-representation and let $\rho_1 : \mathcal{A} \to \mathcal{B}(\mathcal{K}_1)$ be a $*$-representation of $\mathcal{A}$. Then there is a sequence $a_n \in \mathcal{A}$ satisfying $\|\rho_0(a_n) - \mathrm{Id}_{\mathcal{K}_0}\| \to 0$ and $\|\rho_1(a_n)\| \to 0$.*

Proof. Assume that there is a constant L such that

$$(*) \qquad \|\rho_0(a)\| \leq L\|\rho_1(a)\| \quad \text{for every } a \in \mathcal{A}.$$

Then we take $\mathcal{B}$ to be the norm closure of $\rho_1(\mathcal{A})$. It is a C^*-algebra and by ($*$) we can extend ρ_0 to a representation of $\mathcal{B}$. However, every algebraically irreducible representation of a C^*-algebra is equivalent to a $*$-representation (see Corollary 2.9.6 in [7]). Therefore ($*$) can not be true. Hence there is a sequence $c_n \in \mathcal{A}$ satisfying $\|\rho_0(c_n)\| = 1$ and $\|\rho_1(c_n)\| \to 0$. As $\dim \mathcal{K}_0$ is finite, we may in addition assume that $\|\rho_0(c_n) - C\| \to 0$ for some $C \in \mathcal{B}(\mathcal{K}_0)$ with $\|C\| = 1$. Since ρ_0 is fully irreducible, there exist $b_k, d_k \in \mathcal{A}$, $k \leq M$, such that

$$\sum_{k=1}^{M} \rho_0(b_k) C \rho_0(d_k) = \mathrm{Id}_{\mathcal{K}_0}.$$

Now we can conclude the proof by putting $a_n := \sum_{k=1}^{M} b_k c_n d_k$. □

Now we are ready to prove the following result.

Theorem 6.8. *Assume that we are given an irreducible representation $(\pi_0, \mathcal{H}_0)$ of $\mathcal{A}(\tau)$, with $\dim \mathcal{H}_0 < \infty$, which is not equivalent to a $*$-representation. Then the induced representation π of the free product group $G = *_{i \in I} G_i$ is fully irreducible.*

First we prove a weaker statement.

Lemma 6.9. *Under the assumptions of Theorem 6.8, π is topologically irreducible.*

Proof. Consider the representation $(\tilde{\pi}_0, \mathcal{H})$ of $\mathcal{A}(\tau)$ defined by formula (5.3) and its restrictions ρ_0 and ρ_1 to $\mathcal{H}_0$ and $\mathcal{H}_0^\perp$ respectively, so that $\rho_0 = \pi_0$ and ρ_1 is a multiple of λ_0 (Lemma 5.7). In view of Lemma 6.7 the orthogonal projection P_0 of $\mathcal{H}$ to $\mathcal{H}_0$ belongs to the norm closure of $\tilde{\pi}_0(\mathcal{A}(\tau))$.

Let M be a G-invariant subspace of $\mathcal{H}$ and denote $M_0 := P_0 M \subseteq M$.

If $M_0 \neq \{0\}$, then M_0 it is a nontrivial closed subspace of $\mathcal{H}_0$ which is $\mathcal{A}(\tau)$-invariant. Hence $M_0 = \mathcal{H}_0$, which means that $H_0 \subseteq M$. For fixed $i \in I$ we have $\pi(g)\mathcal{H}_0 \subseteq M$ for each $g \in G_i$, which leads to $\delta_e \otimes \mathcal{H}_i \subseteq M$. Consequently, $\delta_e \otimes \mathcal{H}_e \subseteq M$ and for any $w \in G$ we have $\delta_w \otimes \mathcal{H}_w = \pi(w)(\delta_e \otimes \mathcal{H}_w)$. Therefore $M = \mathcal{H}$.

Now assume that $P_0 M = \{0\}$, i.e., $M \perp \mathcal{H}_0$. For fixed $i \in I$ and for every $g \in G_i$ we have $\pi(g)M \subseteq M \perp \mathcal{H}_0$, so that $M \perp \delta_e \otimes \mathcal{H}_i$, and hence $M \perp \delta_e \otimes \mathcal{H}_e$. Now for $w \in G$ we have $\pi(w^{-1})M \subseteq M \perp \delta_e \otimes \mathcal{H}_e$, which yields $M \perp \delta_w \otimes \mathcal{H}_w$ and we conclude that $M = \{0\}$. ◻

Proof of Theorem 6.8. Let $\mathcal{B}$ denote the closure of $\mathrm{lin}\{\pi(x) : x \in G\}$ in the strong operator topology of $\mathcal{B}(\mathcal{H})$. Then $P_0 \in \mathcal{B}$ by Lemma 6.7 and, consequently, $BP_0 \in \mathcal{B}$ for every $B \in \mathcal{B}(\mathcal{H}_0)$. In particular, $\zeta_0 \otimes \eta_0 \in \mathcal{B}$ for fixed ζ_0, $\eta_0 \in \mathcal{H}_0 \setminus \{0\}$. Now take any $\zeta, \eta \in \mathcal{H}$. Due to the topological irreducibility of the family $\mathcal{B}$ (and hence of $\mathcal{B}^* := \{B^* : B \in \mathcal{B}\}$) there are $S_n, T_n \in \mathcal{B}$ such that $S_n\zeta_0 \to \zeta$ and $T_n^*\eta_0 \to \eta$, which implies

$$S_n(\zeta_0 \otimes \eta_0)T_n = (S_n\zeta_0) \otimes (T_n^*\eta_0) \to \zeta \otimes \eta$$

in the strong topology. Therefore $\mathcal{B}$ contains all operators of finite rank, so $\mathcal{B} = \mathcal{B}(\mathcal{H})$. ◻

7. FREE PRODUCT OF TWO GROUPS

This section is devoted to the case when $|I| = 2$, say $I = \{+, -\}$. Here we will write $S(+,-)$ and $\mathcal{A}(\tau_+, \tau_-)$ instead of $S(I)$ and $\mathcal{A}(\tau)$. A word of the form $u = + - \cdots \pm$ (resp. $u = - + \cdots \pm$), with $|u| = m$, will be denoted $+m$ (resp. $-m$) and here we will denote by 0 the empty word.

Cartwright and Soardi [6] (see also [20, 33]) introduced a family of spherical functions φ_λ (λ is a complex parameter) on the free product of two groups: $\mathbb{Z}_r * \mathbb{Z}_s$, where $r > s \geq 2$. It was shown in [21] that such a function is positive definite if and only if $\lambda \in [-2,\ s-2] \cup [r-2,\ r+s-2]$. Here our aim is to study in detail the corresponding family of representations.

First we are interested in finding all irreducible representations of $\mathcal{A}(\tau_+, \tau_-)$, therefore in finding all (equivalence classes of) irreducible pairs (P_+, P_-) of projections on a Hilbert space $\mathcal{H}_0$. Two such pairs: (P_+, P_-) on $\mathcal{H}_0$ and (Q_+, Q_-) on $\mathcal{K}_0$ are said to be *equivalent* if there is an invertible operator $T : \mathcal{H}_0 \to \mathcal{K}_0$ such that $TP_+ = Q_+T$ and $TP_- = Q_-T$. Putting $Z := 2P_+ - \mathrm{Id}$ and $T := (2P_+ - \mathrm{Id})(2P_- - \mathrm{Id})$ we have $Z^2 = \mathrm{Id}$ and $ZTZ = T^{-1}$ so our question is equivalent to finding all (equivalence classes of) irreducible representations of the semidirect product $\mathbb{Z}_2 \ltimes \mathbb{Z}$. Unitary irreducible representations of $\mathbb{Z}_2 \ltimes \mathbb{Z}$ are known to be at most two-dimensional. We note that the same holds without unitarity if we assume that $\dim \mathcal{H}_0$ is finite, as we now see.

Lemma 7.1. *Let (P_+, P_-) be an irreducible pair of projections in a finitely dimensional Hilbert space $\mathcal{H}_0$. Then $\dim \mathcal{H}_0 \leq 2$. Moreover, two irreducible pairs (P_+, P_-) and (Q_+, Q_-) of projections on a two-dimensional Hilbert space $\mathcal{H}_0$ are equivalent if and only if* $\mathrm{Tr}(P_+P_-) = \mathrm{Tr}(Q_+Q_-)$.

Proof. Put $Z := 2P_+ - \mathrm{Id}$ and $T := (2P_+ - \mathrm{Id})(2P_- - \mathrm{Id})$. Since $\dim \mathcal{H}_0$ is finite, T has an eigenvector $\eta_1 \neq 0$ so that $T\eta_1 = \lambda\eta_1$, with $\lambda \neq 0$. Put $\eta_2 := Z\eta_1$. Then $Z\eta_2 = \eta_1$ and

$$T\eta_2 = TZ\eta_1 = ZT^{-1}\eta_1 = \lambda^{-1}Z\eta_1 = \lambda^{-1}\eta_2,$$

which implies that the vectors η_1, η_2 span an invariant subspace, hence $\dim \mathcal{H}_0 \leq 2$. If $\lambda = \pm 1$, then $T(\eta_1 + \eta_2) = \pm(\eta_1 + \eta_2)$ and $Z(\eta_1 + \eta_2) = \eta_1 + \eta_2$ so $\dim \mathcal{H}_0 = 1$.

Now assume that $\lambda \neq \pm 1$. Then the vectors η_1, η_2 are linearly independent and $\mathrm{Tr}\, T = \lambda + \lambda^{-1}$. If operators W, S on $\mathcal{H}_0$ satisfy $W^2 = \mathrm{Id}$, $WSW = S^{-1}$ and $\mathrm{Tr}\, S = \lambda + \lambda^{-1}$ then, as before, S has eigenvalues γ and γ^{-1} which satisfy $\gamma + \gamma^{-1} = \lambda + \lambda^{-1}$. This implies $\gamma = \lambda$ or $\gamma = \lambda^{-1}$, so there are $\xi_1, \xi_2 \neq 0$ such that $S\xi_1 = \lambda\xi_1$, $S\xi_2 = \lambda^{-1}\xi_2$. Thus the map given by $\xi_1 \mapsto \eta_1$, $\xi_2 \mapsto \eta_2$ defines an equivalence between pairs Z, T and W, S. □

Thus the family of all irreducible pairs P_+, P_- of projections on two-dimensional Hilbert space is parametrised by $w := \mathrm{Tr}(P_+P_-)$, $w \in \mathbb{C} \setminus \{0, 1\}$. Note that for unit vectors $\eta_\pm \in \mathcal{H}_0$ and for $P_\pm := \eta_\pm \otimes \eta_\pm$ we have $\mathrm{Tr}(P_+P_-) = |\langle \eta_+, \eta_- \rangle|^2$ so the points from the interval $(0, 1)$ correspond to pairs of orthogonal projections. We also include the parameters $w = 0$ and $w = 1$ when $\eta_+ \perp \eta_-$ or $\eta_+ = \eta_-$ respectively.

Now fix $\tau_+, \tau_- \geq 0$ and set $B_\pm := (1 + \tau_\pm)P_\pm - \tau_\pm \,\mathrm{Id}$ for a pair (P_+, P_-) with $\mathrm{Tr}(P_+P_-) = w$. We denote by π_w^0 the representation of $\mathcal{A}(\tau_+, \tau_-)$ for which $\pi_w^0(\delta_\pm) = B_\pm$. For $w \in \{0, 1\}$ these representations can be decomposed into one-dimensional ones:

$$\pi_0^0 = \pi_{01}^0 \oplus \pi_{10}^0,\ \pi_1^0 = \pi_{00}^0 \oplus \pi_{11}^0, \quad \text{where } \pi_{\varepsilon_+\varepsilon_-}^0(\delta_\pm) = (1 + \tau_\pm)\varepsilon_\pm - \tau_\pm.$$

Therefore the family $\{\pi_w^0\}$, with $w \in (\mathbb{C} \setminus \{0, 1\}) \cup \{0, 1\}^2$, exhausts all finite-dimensional irreducible representations of $\mathcal{A}(\tau_+, \tau_-)$. The coefficient of $\pi_{\varepsilon_+\varepsilon_-}^0$ is the character:

$$\widetilde{\varphi}_{\varepsilon_+\varepsilon_-}(i_1 i_2 \cdots i_m) = a_{i_1} a_{i_2} \cdots a_{i_m},$$

where $a_i = -\tau_i$ if $\varepsilon_i = 0$ and $a_i = 1$ if $\varepsilon_i = 1$. Applying results from Section 2, concerning the left regular representation λ_0 of $\mathcal{A}(\tau)$, and Theorem 3.1, we have the following result.

Proposition 7.2.

(1) π^0_{00} *is contained in* λ_0 *if and only if* $\tau_+\tau_- < 1$,
(2) π^0_{01} *is contained in* λ_0 *if and only if* $\tau_+ < \tau_-$,
(3) π^0_{10} *is contained in* λ_0 *if and only if* $\tau_+ > \tau_-$,
(4) π^0_{11} *is contained in* λ_0 *if and only if* $\tau_+\tau_- > 1$.

Equivalently,

$$\pi^0_{\varepsilon_+\varepsilon_-} \text{ is contained in } \lambda_0 \iff (\varepsilon_+ + \varepsilon_- - 1)(1 - \tau_+\tau_-) < (\varepsilon_+ - \varepsilon_-)(\tau_+ - \tau_-).$$

Moreover, $\pi^0_{\varepsilon_+\varepsilon_-}$ *is weakly contained in* λ_0 *if and only if the same conditions hold, with "<" replaced by "≤".*

For two-dimensional representations we need to study the eigenvalues of B_+B_-. We will base on the following elementary fact.

Lemma 7.3. *Assume that* $z = u + v\mathrm{i} \in \mathbb{C}$ *and* $a \geq 0$. *Then both the inequalities* $|z \pm \sqrt{z^2 - a}| \leq 1$ *hold if and only if either* $a = 1$, $v = 0$ *and* $-1 \leq u \leq 1$ *or* $0 \leq a < 1$ *and* $4u^2(1+a)^{-2} + 4v^2(1-a)^{-2} \leq 1$. *Strict inequalities* $|z \pm \sqrt{z^2 - a}| < 1$ *hold if and only if* $0 \leq a < 1$ *and* $4u^2(1+a)^{-2} + 4v^2(1-a)^{-2} < 1$.

The equality $|z + \sqrt{z^2 - a}| = |z - \sqrt{z^2 - a}|$ *holds if and only if* z *is real and* $z^2 \leq a$.

Proposition 7.4. *The representation* π^0_w *is uniformly bounded (i.e., the norms* $\|\pi^0_w(\delta_u)\|$, $u \in S(I)$, *are uniformly bounded) if and only if either* $\tau_+\tau_- = 1$ *and* $((1-\tau_+)/(1+\tau_+))^2 \leq w \leq 1$ *or* $\tau_+\tau_- < 1$ *and* $w = x + y\mathrm{i}$ *satisfies*

$$\frac{((1+\tau_+)(1+\tau_-)x - \tau_+ - \tau_-)^2}{(1+\tau_+\tau_-)^2} + \frac{(1+\tau_+)^2(1+\tau_-)^2 y^2}{(1-\tau_+\tau_-)^2} \leq 1. \tag{7.1}$$

For $0 < w < 1$ *the* $*$*-representation* π^0_w *is weakly contained in* λ_0 *if and only if*

$$w \in \left[\frac{(\sqrt{\tau_+} - \sqrt{\tau_-})^2}{(1+\tau_+)(1+\tau_-)}, \frac{(\sqrt{\tau_+} + \sqrt{\tau_-})^2}{(1+\tau_+)(1+\tau_-)}\right].$$

Proof. We have $\det(B_\pm) = -\tau_\pm$ so that $\det(B_+B_-) = \tau_+\tau_-$. We also have

$$\mathrm{Tr}(B_+B_-) = (1+\tau_+)(1+\tau_-)w - (\tau_+ + \tau_-).$$

It remains to apply Lemma 7.3 to the eigenvalues

$$\eta_\pm = \frac{\mathrm{Tr}(B_+B_-)}{2} \pm \sqrt{\left(\frac{\mathrm{Tr}(B_+B_-)}{2}\right)^2 - \det(B_+B_-)}$$

of B_+B_- and to use Theorem 3.1. ◻

Let us denote by $E(\tau_+, \tau_-)$ the closed subset of the complex plane described by (7.1). Its boundary is an ellipse with the following basic points $(x, 0)$:

(7.2a) $$\text{centre: } x = \frac{\tau_+ + \tau_-}{(1+\tau_+)(1+\tau_-)},$$

(7.2b) $$\text{vertices: } x = -\frac{(1-\tau_+)(1-\tau_-)}{(1+\tau_+)(1+\tau_-)} \quad \text{and} \quad x = 1,$$

(7.2c) $$\text{foci: } x = \frac{\left(\sqrt{\tau_+} \pm \sqrt{\tau_-}\right)^2}{(1+\tau_+)(1+\tau_-)}.$$

Let G be the free product of two groups, $G = G_+ * G_-$, and put $\tau_\pm := (|G_\pm| - 1)^{-1}$. For the representation π_w^0 of $\mathcal{A}(\tau_+, \tau_-)$, with $w \in \mathbb{C} \setminus \{0, 1\}$ or $w \in \{0, 1\}^2$, let π_w denote the induced representation of G. Then applying Theorems 5.5, 5.10, 6.4, and 6.8 we get immediately the following result.

Theorem 7.5.

(1) *Assume that $w \in \mathbb{C} \setminus \{0, 1\}$. Then:*
- (i) *π_w is fully irreducible.*
- (ii) *If $w_1 \neq w_2$, then π_{w_1} and π_{w_2} are inequivalent.*
- (iii) *If $w \in \operatorname{Int} E(\tau_+, \tau_-)$, then π_w is uniformly bounded.*
- (iv) *π_w is unitary if and only if $w \in (0, 1)$.*

(2) *For $w \in \{0, 1\}^2$ the representation π_w is unitary and*
- (i) *π_{11} is irreducible.*
- (ii) *π_{01} is irreducible if and only if $|G_+| \leq |G_-|$.*
- (iii) *π_{10} is irreducible if and only if $|G_+| \geq |G_-|$.*
- (iv) *π_{00} is irreducible if and only if $|G_+| = |G_-| = 2$.*

From now on we fix $\tau_+, \tau_- \geq 0$ and $\beta_+, \beta_- > 0$. Assume that $\beta_-(1+\tau_-) \leq \beta_+(1+\tau_+)$. Set $\mu := \beta_+\delta_+ + \beta_-\delta_-$ and denote by $\mathcal{A}(\tau, \mu)$ the commutative unital $*$-subalgebra of $\mathcal{A}(\tau)$ generated by μ. Let $C^*(\tau, \mu)$ and $C_r^*(\tau, \mu)$ be the closure of $\mathcal{A}(\tau, \mu)$ in $C^*(\mathcal{A}(\tau))$ and $C_r^*(\mathcal{A}(\tau))$ (see remarks preceding Proposition 5.8) respectively.

Proposition 7.6. *Denote by $\operatorname{sp}(\mu)$ and $\operatorname{sp}_r(\mu)$ the spectrum of μ in $C^*(\tau, \mu)$ and $C_r^*(\tau, \mu)$ or, equivalently, in $C^*(\mathcal{A}(\tau))$ and $C_r^*(\mathcal{A}(\tau))$, respectively. Then*

$$\operatorname{sp}(\mu) = [-\beta_+\tau_+ - \beta_-\tau_-,\ -\beta_+\tau_+ + \beta_-] \cup [-\beta_-\tau_- + \beta_+,\ \beta_+ + \beta_-],$$
$$\operatorname{sp}_r(\mu) = [x_0 - x_+,\ x_0 - x_-] \cup [x_0 + x_-,\ x_0 + x_+] \cup \Upsilon,$$

where

$$x_0 := \frac{(\beta_+(1-\tau_+) + \beta_-(1-\tau_-))}{2},$$

$$x_\pm := \sqrt{\left(\frac{\beta_+(1+\tau_+) - \beta_-(1+\tau_-)}{2}\right)^2 + \beta_+\beta_-\left(\sqrt{\tau_+} \pm \sqrt{\tau_-}\right)^2},$$

and Υ *consists of those points* (*at most two*)

$$\beta_+(1+\tau_+)\varepsilon_+ + \beta_-(1+\tau_-)\varepsilon_- - (\beta_+\tau_+ + \beta_-\tau_-),$$

with $\varepsilon_\pm \in \{0,1\}$, *for which* $(\varepsilon_+ + \varepsilon_- - 1)(1-\tau_+\tau_-) \le (\varepsilon_+ - \varepsilon_-)(\tau_+ - \tau_-)$.

Proof. Recall that the spectrum of an element a_0 in a commutative Banach algebra $\mathcal{A}$ is the set of all values $\varphi(a_0)$, where φ runs over all multiplicative functionals of $\mathcal{A}$. If φ is a multiplicative functional on $C^*(\tau,\mu)$ then, regarded as a 1-dimensional representation, it can be extended to an irreducible $*$-representation of $C^*(\mathcal{A}(\tau))$ (see 2.10.2 in [7]) and hence of $\mathcal{A}(\tau)$. Therefore we can say that there is $w \in [0,1]$ and a unit eigenvector ξ for $\pi_w^0(\mu)$ such that $\varphi(a) = \langle \pi_w(a)\xi, \xi\rangle$ for $a \in \mathcal{A}(\tau,\mu)$. Put $\alpha_\pm := (1+\tau_\pm)\beta_\pm$ and $B := \alpha_+ P_+ + \alpha_- P_-$. We have $P_\pm = \eta_\pm \otimes \eta_\pm$, $w = |\langle \eta_+, \eta_-\rangle|^2$ and

$$\begin{aligned} B\eta_+ &= \alpha_+\eta_+ + \alpha_-\langle \eta_+,\eta_-\rangle\eta_-,\\ B\eta_- &= \alpha_+\langle \eta_-,\eta_+\rangle\eta_+ + \alpha_-\eta_-, \end{aligned}$$

so that the eigenvalues of B are

$$t_\pm(w) = \frac{\alpha_+ + \alpha_- \pm \sqrt{(\alpha_+ - \alpha_-)^2 + 4w\alpha_+\alpha_-}}{2}.$$

If w runs over $[0,1]$, then $t_\pm(w)$ runs over $[0,\alpha_-] \cup [\alpha_+, \alpha_+ + \alpha_-]$. Knowing that

$$\pi_w^0(\mu) = B - (\tau_+\beta_+ + \tau_-\beta_-)\,\mathrm{Id},$$

we obtain the first statement.

For $\mathrm{sp}_r(\mu)$ we have to take into account those π_w^0 which are contained or weakly contained in the regular representation of $\mathcal{A}(\tau)$. ◻

Now we will study some coefficients of the representations π_w^0 and π_w. For a complex function φ on $S(I)$ and for $f \in \mathcal{F}(S(I))$ we define their *dual right* τ*-convolution* $\varphi \diamond_\tau f$ putting $(\varphi \diamond_\tau f)(u) = \langle \varphi, \delta_u *_\tau f^\vee\rangle$, where $f^\vee(u) := f(u^*)$.

Definition 7.7. Let $\tau_+, \tau_- \ge 0$, $\beta_+, \beta_- > 0$, $\lambda \in \mathbb{C}$. A complex function φ on $S(+,-)$ is said to be $(\tau_+,\tau_-,\beta_+,\beta_-;\lambda)$-*spherical* if

(i) $\varphi(e) = 1$;
(ii) $\varphi(u^*) = \varphi(u)$ for $u \in S(+,-)$;
(iii) $\varphi \diamond_\tau \mu = \lambda\varphi$, where $\tau = (\tau_+,\tau_-)$, $\mu = \beta_+\delta_+ + \beta_-\delta_-$.

It was shown in [21, Proposition 4.2], that such a function φ does exist and is unique for $\lambda \ne x_0 := (\beta_+(1-\tau_+) + \beta_-(1-\tau_-))/2$. If $\beta_+(1+\tau_+) = \beta_-(1+\tau_-)$ and $\lambda = x_0$, then such a function exists but is not unique.

Fix $\tau_+, \tau_- \geq 0$, $\beta_+, \beta_- > 0$ and let φ_λ denote the $(\tau_+, \tau_-, \beta_+, \beta_-; \lambda)$-spherical function. If $\beta_+(1+\tau_+) = \beta_-(1+\tau_-)$, then we define φ_{x_0} as the pointwise limit of φ_λ when $\lambda \to x_0$. Then, as it was explained in the proof of [21, Theorem 4.5], the function $\varphi = T_{\sigma\tau}\varphi_\lambda$ (the map $T_{\sigma\tau}$ was defined in Section 1), for $\sigma_\pm = 0$, is the $(0, 0, \alpha_+, \alpha_-; \gamma)$-spherical function ψ_γ, where

$$\alpha_+ = (1+\tau_+)\beta_+, \quad \alpha_- = (1+\tau_-)\beta_-, \quad \gamma = \lambda + \tau_+\beta_+ + \tau_-\beta_-. \tag{7.3}$$

Now we are going to obtain spherical functions as coefficients of representations of $\mathcal{A}(\tau_+, \tau_-)$. Take $\mathcal{H}_0 := \mathbb{C}^2$, with an orthonormal basis ζ_0, ζ_1. For complex numbers θ, ω define vectors

$$\begin{aligned}
\zeta_+(\theta) &:= \cos\theta \cdot \zeta_0 + \sin\theta \cdot \zeta_1,\\
\zeta_-(\theta) &:= \cos\theta \cdot \zeta_0 - \sin\theta \cdot \zeta_1,\\
\xi(\omega) &:= \cos\omega \cdot \zeta_0 + \sin\omega \cdot \zeta_1.
\end{aligned}$$

If θ, ω are real, then the angle between $\zeta_+(\theta)$ and $\zeta_-(\theta)$ is 2θ and between $\zeta_\pm(\theta)$ and $\xi(\omega)$ is $\theta \pm \omega$.

Now we define one-dimensional projections

$$P_+ := \zeta_+(\theta) \otimes \zeta_+(\overline{\theta}), \quad P_- := \zeta_-(\theta) \otimes \zeta_-(\overline{\theta}).$$

Then $[\zeta_+(\theta), \zeta_-(\overline{\theta})] = [\zeta_-(\theta), \zeta_+(\overline{\theta})] = \cos(2\theta)$ and for any $u = i_1 i_2 \cdots i_n \in S(+,-) \setminus \{0\}$

$$[P_{i_1} P_{i_2} \cdots P_{i_n} \xi(\omega), \xi(\overline{\omega})] = \cos z_{i_1} \cos^{n-1}(2\theta) \cos z_{i_n}$$

where $z_\pm = \theta \pm \omega$.

Now we choose the numbers θ, ω in a special way. Assume that

$$\begin{aligned}
\cos^2(\theta+\omega) &= \frac{\gamma(\alpha_- - \gamma)}{\alpha_+(\alpha_+ + \alpha_- - 2\gamma)},\\
\cos^2(\theta-\omega) &= \frac{\gamma(\alpha_+ - \gamma)}{\alpha_-(\alpha_+ + \alpha_- - 2\gamma)}.
\end{aligned} \tag{7.4}$$

Then we have

$$\alpha_+ \cos^2(\theta+\omega) + \alpha_- \cos^2(\theta-\omega) = \gamma, \tag{7.5}$$

$$\begin{aligned}
\sin^2(\theta+\omega) &= \frac{(\alpha_+ - \gamma)(\alpha_+ + \alpha_- - \gamma)}{\alpha_+(\alpha_+ + \alpha_- - 2\gamma)},\\
\sin^2(\theta-\omega) &= \frac{(\alpha_- - \gamma)(\alpha_+ + \alpha_- - \gamma)}{\alpha_-(\alpha_+ + \alpha_- - 2\gamma)},
\end{aligned} \tag{7.6}$$

and

$$\alpha_+^2 \sin^2(\theta+\omega)\cos^2(\theta+\omega) = \alpha_-^2 \sin^2(\theta-\omega)\cos^2(\theta-\omega).$$

We assume in addition that

$$\alpha_+ \sin(\theta+\omega)\cos(\theta+\omega) = \alpha_- \sin(\theta-\omega)\cos(\theta-\omega). \tag{7.7}$$

Then

$$\cos^2(2\theta) = \frac{(\alpha_+ - \gamma)(\alpha_- - \gamma)}{\alpha_+\alpha_-}, \tag{7.8}$$

and

$$\cos^2(2\omega) = \frac{(\alpha_+ + \alpha_-)^2(\alpha_+ - \gamma)(\alpha_- - \gamma)}{\alpha_+\alpha_-(\alpha_+ + \alpha_- - 2\gamma)^2}. \tag{7.9}$$

Assuming that $\alpha_- \le \alpha_+$, we note that θ is real if and only if $\gamma \in [0, \alpha_-] \cup [\alpha_+, \alpha_+ + \alpha_-]$ and ω is real if and only if either $\alpha_+ = \alpha_-$ (and then $\cos^2(2\omega) = 1$) or $\gamma \in [0, \alpha_-] \cup [\alpha_+, \alpha_+ + \alpha_-]$. One can also check that for $\alpha_+ \neq \alpha_-$ the cases $\gamma = 0, \alpha_-, \alpha_+, \alpha_+ + \alpha_-$ correspond to $\pi_{00}^0, \pi_{01}^0, \pi_{10}^0$, and π_{11}^0 respectively.

The case $\alpha_+ = \alpha_- := \alpha$ is slightly different. Here we have: $\cos^2(\theta \pm \omega) = \gamma/(2\alpha)$, $\cos^2(2\theta) = (\alpha-\gamma)^2/\alpha^2$ and $\cos^2(2\omega) = 1$. If $\gamma = \alpha$, then $\zeta_+(\theta)$ and $\zeta_-(\theta)$ are mutually orthogonal and we may assume that the angle between $\zeta_\pm(\theta)$ and $\xi(\omega)$ is $\pi/4$.

Lemma 7.8. *Under the above choice of parameters, define a function ψ on $S(+,-)$ by putting*

$$\psi(u) := [P_{i_1}P_{i_2}\cdots P_{i_m}\xi(\omega), \xi(\overline{\omega})],$$

for $u = i_1 \cdots i_m \in S(+,-)$. Then ψ is the $(0,0,\alpha_+,\alpha_-;\gamma)$-spherical function ψ_γ on $S(+,-)$. Consequently, the function

$$\varphi(u) := [B_{i_1}B_{i_2}\cdots B_{i_m}\xi(\omega), \xi(\overline{\omega})],$$

$B_\pm = (1+\tau_\pm)P_\pm - \tau_\pm \operatorname{Id}$, *is the $(\tau_+,\tau_-,\beta_+,\beta_-;\lambda)$-spherical function φ_λ on $S(+,-)$.*

Proof. Assume that the last letter of $u = \varepsilon n$ is "$-$". Then, putting $\nu = \alpha_+\delta_+ + \alpha_-\delta_-$, we obtain

$$\begin{aligned}(\psi \diamond_0 \nu)(u) &= \langle \psi, \delta_u *_0 \nu\rangle \\ &= \langle \psi, \alpha_+\delta_{\varepsilon(n+1)} + \alpha_-\delta_{\varepsilon n}\rangle = \alpha_+\psi(\varepsilon(n+1)) + \alpha_-\psi(\varepsilon n) \\ &= \cos z_\varepsilon \cos^{n-1}(2\theta)[\alpha_+\cos(2\theta)\cos(\theta+\omega) + \alpha_-\cos(\theta-\omega)].\end{aligned}$$

Now we apply (7.7) and (7.5):

$$\begin{aligned}
&\alpha_+ \cos(2\theta)\cos(\theta+\omega) + \alpha_- \cos(\theta-\omega)\\
&\quad= \alpha_+ \cos^2(\theta+\omega)\cos(\theta-\omega)\\
&\qquad - \alpha_+ \sin(\theta+\omega)\cos(\theta+\omega)\sin(\theta-\omega) + \alpha_- \cos(\theta-\omega)\\
&\quad= \alpha_+ \cos^2(\theta+\omega)\cos(\theta-\omega)\\
&\qquad - \alpha_- \sin^2(\theta-\omega)\cos(\theta-\omega) + \alpha_- \cos(\theta-\omega)\\
&\quad= [\alpha_+ \cos^2(\theta+\omega) + \alpha_- \cos^2(\theta-\omega)]\cos(\theta-\omega)\\
&\quad= \gamma\cos(\theta-\omega),
\end{aligned}$$

so $(\psi \diamond_0 \nu)(u) = \gamma\psi(u)$.

Analogous proof works for the other nonzero elements of $S(+,-)$ and for $u = 0$ we use (7.5):

$$\begin{aligned}
(\psi \diamond_0 \nu)(0) &= \alpha_+\psi(+) + \alpha_-\psi(-)\\
&= \alpha_+\cos^2(\theta+\omega) + \alpha_-\cos^2(\theta-\omega) = \gamma = \gamma\psi(0). \qquad \square
\end{aligned}$$

If $\beta_-(1+\tau_-) \neq \beta_+(1+\tau_+)$, then for $\lambda = \beta_+\tau_+ - \beta_-\tau_-, -\beta_+\tau_+ + \beta_-, -\beta_-\tau_- + \beta_+$ or $\beta_+ + \beta_-$ the function φ_λ equals $\tilde{\varphi}_{00}$, $\tilde{\varphi}_{01}$, $\tilde{\varphi}_{10}$ or $\tilde{\varphi}_{11}$ respectively. If $\beta_-(1+\tau_-) = \beta_+(1+\tau_+)$ and $\lambda = -\beta_+\tau_+ + \beta_- = -\beta_-\tau_- + \beta_+$, then we have $\varphi_\lambda = \frac{1}{2}(\tilde{\varphi}_{01} + \tilde{\varphi}_{10})$.

Denote

$$x_0 := \frac{\beta_+(1-\tau_+) + \beta_-(1-\tau_-)}{2},$$

$$c_\pm := \left(\frac{\beta_+(1+\tau_+) \pm \beta_-(1+\tau_-)}{2}\right)^2,$$

and

$$b := c_- + \beta_+\beta_-(\tau_+ + \tau_-).$$

Proposition 7.9. *Let φ_λ be the $(\tau_+,\tau_-,\beta_+,\beta_-;\lambda)$-spherical function and assume that $\beta_-(1+\tau_-) \le \beta_+(1+\tau_+)$. Then*

(i) *φ_λ is (τ_+,τ_-)-positive definite if and only if λ is real and $c_- \le (\lambda - x_0)^2 \le c_+$ or, equivalently,*

$$\lambda \in [-\beta_+\tau_+ - \beta_-\tau_-,\ -\beta_+\tau_+ + \beta_-] \cup [-\beta_-\tau_- + \beta_+,\ \beta_+ + \beta_-]. \tag{7.10}$$

(ii) *φ_λ is bounded if and only if either $\tau_+\tau_- = 1$ and*

$$c_- - \beta_+\beta_-(1-\tau_+)(1-\tau_-) \le (\lambda - x_0)^2 \le c_+ \tag{7.11a}$$

or $\tau_+\tau_- < 1$ *and* $(\lambda - x_0)^2 = x + y\mathrm{i}$ *satisfies*

$$\text{(7.11b)} \qquad \left(\frac{x-b}{\beta_+\beta_-(1+\tau_+\tau_-)}\right)^2 + \left(\frac{y}{\beta_+\beta_-(1-\tau_+\tau_-)}\right)^2 \le 1.$$

Proof. The condition (7.10) is equivalent to $\gamma \in [0,\alpha_-] \cup [\alpha_+, \alpha_+ + \alpha_-]$ (see (7.3)). Then θ and ω are real and consequently φ_λ is positive definite. On the other hand if a function φ is (τ_+,τ_-)-positive definite, with $\varphi(0) = 1$, then $-\tau_+ \le \varphi(+) \le 1$ and $-\tau_- \le \varphi(-) \le 1$, which implies (7.10) (see [21]).

We know that π_w^0 is uniformly bounded if and only if $w \in E(\tau_+,\tau_-)$. Now we note that

$$\text{(7.12)} \qquad \begin{aligned} w &= \frac{(\alpha_+ - \gamma)(\alpha_- - \gamma)}{\alpha_+\alpha_-} \\ &= \frac{1}{\alpha_+\alpha_-}\left[\left(\gamma - \frac{\alpha_+ + \alpha_-}{2}\right)^2 - \left(\frac{\alpha_+ - \alpha_-}{2}\right)^2\right] \\ &= \frac{1}{\beta_+\beta_-(1+\tau_+)(1+\tau_-)}\left[(\lambda - x_0)^2 - \left(\frac{\beta_+(1+\tau_+) - \beta_-(1+\tau_-)}{2}\right)^2\right], \end{aligned}$$

which concludes the proof. ∎

Note that if $\lambda_1, \lambda_2 \in \mathbb{C}$ correspond to w_1, w_2 respectively, then $w_1 = w_2$ if and only if either $\lambda_1 = \lambda_2$ or $\lambda_1 + \lambda_2 = 2x_0$.

We denote by $E(\tau_+,\tau_-,\beta_+,\beta_-)$ the closed subset of the complex plane described by (7.11) in the last proposition. Its boundary is an ellipse with the following basic points $(x, 0)$:

$$\text{(7.13a)} \qquad \text{centre: } x = \left(\frac{\beta_+(1+\tau_+) - \beta_-(1+\tau_-)}{2}\right)^2 + \beta_+\beta_-(\tau_+ + \tau_-),$$

$$\text{(7.13b)} \qquad \text{foci: } x = \left(\frac{\beta_+(1+\tau_+) - \beta_-(1+\tau_-)}{2}\right)^2 + \beta_+\beta_-(\sqrt{\tau_+} \pm \sqrt{\tau_-})^2,$$

$$\text{(7.13c)} \qquad \text{left vertex: } x = \left(\frac{\beta_+(1+\tau_+) - \beta_-(1+\tau_-)}{2}\right)^2 - \beta_+\beta_-(1-\tau_+)(1-\tau_-),$$

$$\text{(7.13d)} \qquad \text{right vertex: } x = \left(\frac{\beta_+(1+\tau_+) + \beta_-(1+\tau_-)}{2}\right)^2.$$

Let $F(\tau_+,\tau_-,\beta_+,\beta_-)$ denote the set of such $\lambda \in \mathbb{C}$ that satisfy $(\lambda - x_0)^2 \in E(\tau_+,\tau_-,\beta_+,\beta_-)$.

We can see that $F(\tau_+,\tau_-,\beta_+,\beta_-)$ is connected if $c_- \le \beta_+\beta_-(1-\tau_+)(1-\tau_-)$, and otherwise $F(\tau_+,\tau_-,\beta_+,\beta_-)$ has two components. Note also that if $\tau_+, \tau_- <$

1 and $\beta_+(1+\tau_+) \neq \beta_-(1+\tau_-)$, then there are bounded real spherical functions φ_λ which are not (τ_+, τ_-)-positive definite.

Let us take again the free product group $G = G_+ * G_-$, and let us put $\tau_\pm := (|G_\pm| - 1)^{-1}$. If G_+ and G_- are finite, then the condition

$$\varphi \diamond_\tau (\beta_+\delta_+ + \beta_-\delta_-) = \lambda \cdot \varphi$$

is equivalent to

$$(\varphi \circ t) * (\beta_+\mu_+ + \beta_-\mu_-) = \lambda \cdot (\varphi \circ t)$$

(see Proposition 3.1 in [21]), where

$$\mu_\pm(x) := \begin{cases} \tau_\pm & \text{if } x \in G_\pm \setminus \{e\}, \\ 0 & \text{otherwise.} \end{cases}$$

Let us now specify our results to the particular case which was investigated by Cartwright and Soardi [6]. Here we have $|G_+| = r$, $|G_-| = s$, $\infty > r > s \geq 2$, $\tau_+ = 1/(r-1)$, $\tau_- = 1/(s-1)$, $\beta_+ = r-1$, $\beta_- = s-1$, so that $\varphi_\lambda \circ t$ is an eigenfunction for the convolution with χ_1, the characteristic function of elements of length 1 in G. Lemma 5.6 and Theorem 7.5 yield (cf. Theorem 1 in [6] and Theorem II in [33]) the following result:

Proposition 7.10. *Denote by* $\mathrm{sp}(\chi_1)$ *and* $\mathrm{sp}_r(\chi_1)$ *the spectrum of* χ_1 *in* $C^*(G)$ *and* $C^*_r(G)$ *respectively. Then*

$$\mathrm{sp}(\chi_1) = [-2,\ s-2] \cup [r-2,\ r+s-2]$$

and

$$\mathrm{sp}_r(\chi) = [x_0-x_+,\ x_0-x_-] \cup [x_0+x_-,\ x_0+x_+] \cup \{-2,\ s-2\},$$

where

$$x_0 := \frac{r+s-4}{2}$$

and

$$x_\pm := \sqrt{\left(\frac{r-s}{2}\right)^2 + \left(\sqrt{r-1} \pm \sqrt{s-1}\right)^2}.$$

Our final proposition is a consequence of Theorem 7.5 (cf. Proposition 8 in [6]).

Proposition 7.11. *Let* $\lambda \neq x_0 := (r+s-4)/2$ *and assume that*

$$\lambda \notin \{-2,\ s-2,\ r-2,\ r+s-2\}.$$

Then the spherical function $\varphi_\lambda \circ t$ *on* $G = G_+ * G_-$ *is a coefficient of the representation* π_w*, where* $w = (\lambda+2-r)(\lambda+2-s)/(rs)$*. Moreover,*

(i) *π_w is fully irreducible.*
(ii) *If* $(\lambda - x_0)^2 = x + y\mathrm{i}$ *and*

$$\left(\frac{x-b}{a+1}\right)^2 + \left(\frac{y}{a-1}\right)^2 < 1,$$

where $a := (r-1)(s-1)$ *and* $b := ((r-s)/2)^2 + r + s - 2$, *then* π_w *is uniformly bounded.*
(iii) *π_w is unitary if and only if*

$$\lambda \in (-2, s-2) \cup (r-2, r+s-2)$$

and then $\varphi_\lambda \circ t$ *is positive definite.*

The positive definite spherical functions $\varphi_{-2} \circ t$, $\varphi_{s-2} \circ t$, $\varphi_{r-2} \circ t$, *and* $\varphi_{r+s-2} \circ t$ *are coefficients of the unitary representations* π_{00}, π_{01}, π_{10}, *and* π_{11}, *respectively. The representations* π_{10} *and* π_{11} *are irreducible, while* π_{00} *and* π_{01} *are contained in the regular representation of* G, *and thus they are not irreducible.*

Comparing formulas (7.10) and (7.13c) one can see that if $s > 2$, then there are bounded real spherical functions $\varphi_\lambda \circ t$ which are not positive definite.

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