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Positive Definite Radial Functions on Free Product of Groups.

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Sunto. – *Si presentano alcune classi di funzioni radiali definite positive su prodotto libero di gruppi. Si dà la caratterizzazione di tutte le funzioni sferiche definite positive sul gruppo libero prodotto* $\mathbf{Z}_k \circ \mathbf{Z}_k \circ \ldots \circ \mathbf{Z}_k$ *(N volte) introdotto da A. Iozzi e M. Picardello.*

0. – Introduction.

For fixed integer $N \geqslant 2$ let $G_1, G_2, \ldots, G_N$ be arbitrary discrete groups. We shall consider their *free product* $G = G_1 \circ G_2 \circ \ldots \circ G_N$. Every element $x \in G \setminus \{e\}$ has the unique representation as a *reduced word*

$$x = g_1 g_2 \ldots g_n ,$$

where $g_k \in G_{i_k} \setminus \{e\}$ and $i_k \neq i_{k+1}$. We define on G the block lenght function $x \mapsto \|x\|$, putting $\|x\| = n$ for x as above and $\|e\| = 0$. In [2] M. Bozejko has proved, that the function $P_r(x) = r^{\|x\|}$ is positive definite for $0 < r \leqslant 1$. In this paper we shall show that the function P_r is in the Fourier-Stieltjes algebra of the group G for $(-1)/(N-1) < r \leqslant 1$. Next observe that for $0 < r < 1$ P_r is not irreducible positive definite function.

In case when $G_1 = G_2 = \ldots = G_N = \mathbf{Z}_k$ we obtain complete description of the positive definite spherical functions on the free product group $G_{k,N} = \mathbf{Z}_k \circ \mathbf{Z}_k \circ \ldots \circ \mathbf{Z}_k$ (N-times). These results improve some theorems of A. Iozzi and M. Picardello [7].

1. – Preliminaries.

Let G be arbitrary discrete group. We say that a function φ_1 on G is *positive definite* if

$$\langle \varphi_1, f^* * f \rangle \geqslant 0$$

for every finitely supported function f, where $f^*(x) = \overline{f(x^{-1})}$, $\langle u, v\rangle = \sum_{x\in G} u(x)v(x)$, $(u * v)(x) = \sum_{y\in G} u(xy^{-1})v(y)$. We say that a function φ_2 is *negative definite* if

$$\langle \varphi_2, f^* * f\rangle \leqslant 0$$

for every f with finite support and such that $\sum_{x\in G} f(x) = 0$ (see [1]). By $B(G)$ we denote the space of linear combinations of positive definite functions on G. $B(G)$ is an algebra under pointvise multiplication, called the *Fourier-Stieltjes algebra* of the group G. For the basic definitions and theorems concerning $B(G)$ we refer to the paper of Eymard [4].

From now on let G be a free product group $G_1 \circ G_2 \circ \ldots \circ G_N$. A function f on G is called *radial* when $f(x)$ depends only on $\|x\|$ for every $x \in G$. For a natural number $k \leqslant N$ we define the set

$$S_k = \{x \in G\setminus\{e\}\text{: the first letter of the reduced word } x \text{ belong to } G_k\}.$$

For a function $f\colon G \to \boldsymbol{C}$ let $f_k = f\cdot\chi_{S_k}$, where χ_A denotes the characteristic function of a set $A \subseteq G$. Then we have

$$f = f_1 + f_2 + \ldots + f_N + f(e)\delta_e .$$

2. – The theorem.

For $r \neq 0$ we define the function ψ_r on $G = G_1 \circ G_2 \circ \ldots \circ G_N$ as follows:

$$\psi_r(x) = \begin{cases} 1 & \text{for } x = e \\ \alpha r^{\|x\|} & \text{for } x \neq e \end{cases}$$

where

$$\alpha = \alpha(r) = \frac{(N-1)r + 1}{Nr} .$$

For $r = 0$ we put

$$\psi_0 = \frac{1}{N}\sum_{k=1}^{N} \chi_{G_k} .$$

Let us note, that for $r \neq 0$

$$\psi_r(x) = \alpha r^{\|x\|} + (1-\alpha)\delta_e(x) .$$

We shall use following simple facts:

For $(-1)/(N-1) < r \leqslant 1$, $\quad r \neq 0$

(1) $$0 \leqslant \alpha(r) r \leqslant 1 ,$$

(2) $$1 + (N-2)r - (N-1)r^2 \geqslant 0 ,$$

(3) $$\frac{\alpha(r) - 1}{\alpha(r)} \geqslant 0 ,$$

and for $r \neq 0$, $(-1)/(N-1)$

(4) $$1 + (N-1)\frac{\alpha(r) - 1}{\alpha(r)} = \frac{1}{r\alpha(r)} .$$

LEMMA 1. – *Let $G = G_1 \circ G_2 \circ \ldots \circ G_N$ and let a function $f\colon G \to \boldsymbol{C}$ has a finite support contained in some S_{k_0} $(k_0 \leqslant N)$. Then*

$$(*) \qquad \langle \psi_r, f^* * f \rangle \geqslant \frac{1}{r\alpha(r)} |\langle \psi_r, f \rangle|^2$$

for $r \in \big((-1)/(N-1), 0\big) \cup (0, 1]$.

PROOF. – We will write α and ψ instead of $\alpha(r)$ and ψ_r respectively. We shall prove the inequality $(*)$ by induction with respect to maximal lenght of word in the support of function f.

First let us assume, that $\operatorname{supp} f \subseteq \tilde{G}_{k_0} = G_{k_0} \setminus \{e\}$. Using the fact (1) we get

$$\langle \psi, f^* * f \rangle - \frac{1}{\alpha r} |\langle \psi, f \rangle|^2 = (1 - \alpha r) \sum_x |f|(x)|^2 \geqslant 0 .$$

Now assume that $\|x\| \leqslant n+1$ for every $x \in \operatorname{supp}(f)$. We can represent f in the following form:

$$f = \sum_{a \in \tilde{G}_{k_0}} \delta_a * g_{(a)} ,$$

where $\operatorname{supp}(g_{(a)}) \subseteq \Big(\bigcup_{l \neq k_0} S_l\Big) \cup \{e\}$ and $\|x\| \leqslant n$ for each $x \in \operatorname{supp}(g_{(a)})$. Then

$$\begin{aligned}\langle \psi, f^* * f \rangle - \frac{1}{\alpha r} |\langle \psi, f \rangle|^2 = \sum_{a,b \in \tilde{G}_{k_0}} \Big(&(\langle \psi, g^*_{(b)} * \delta_{b^{-1}} * \delta_a * g_{(a)} \rangle - \\ &- \frac{1}{\alpha r} \langle \psi, \delta_a * g_{(a)} \rangle \overline{\langle \psi, \delta_b * g_{(b)} \rangle} \Big) = \\ &= \sum_{a \in \tilde{G}_{k_0}} \Big(\langle \psi, g^*_{(a)} * g_{(a)} \rangle - \frac{1}{\alpha r} |\langle \psi, \delta_a * g_{(a)} \rangle|^2 \Big) .\end{aligned}$$

Fix $a \in \tilde{G}_{k_0}$ and write g instead of $g_{(a)}$. We have the following decomposition of g:

$$g = \sum_{k=1}^{N} g_k + c\delta_e ,$$

where $\operatorname{supp}(g_k) \subseteq S_k$ and $c \in \boldsymbol{C}$. Note that $g_{k_0} \equiv 0$. Then the following equation holds:

$$S = \langle \psi, g^* * g \rangle - \frac{1}{\alpha r} |\langle \psi, \delta_a * g \rangle|^2 =$$
$$= \sum_{k,l} \left(\langle \psi, g_k^* * g_l \rangle - \frac{r}{\alpha} \overline{\langle \psi, g_k \rangle} \langle \psi, g_l \rangle \right) +$$
$$+ (1-r) \sum_k \left(\bar{c} \langle \psi, g_k \rangle + c \overline{\langle \psi, g_k \rangle} \right) + (1-\alpha r)|c|^2 .$$

Let us observe that for $k \neq l$

$$\langle \psi, f_k^* * f_l \rangle = \frac{1}{\alpha} \overline{\langle \psi, f_k \rangle} \langle \psi, f_l \rangle .$$

If $0 < r \leqslant 1$ then $\alpha > 0$, $(1-r)/\alpha \geqslant 0$. It follows readily that in this case

$$S = \frac{1-r}{\alpha} \left| \sum_{k \neq k_0} \langle \psi, g_k \rangle + \frac{\alpha r}{1+r} c \right|^2 +$$
$$+ \frac{1-r}{\alpha r} \sum_{k \neq k_0} \left| \langle \psi, g_k \rangle + \frac{\alpha r}{1+r} c \right|^2 +$$
$$+ \sum_{k \neq k_0} \left[\langle \psi, g_k^* * g_k \rangle - \frac{1}{\alpha r} |\langle \psi, g_k \rangle|^2 \right] + \frac{1-r}{1+r} \cdot \frac{N-2}{N} |c|^2 .$$

If $r \in \big((-1)/(N-1), 0\big)$ then $\alpha < 0$ and $(1-r)/\alpha < 0$. Therefore we obtain

$$S = -\frac{1-r}{\alpha} \sum_{\substack{k<l \\ k,l \neq k_0}} |\langle \psi, g_k \rangle - \langle \psi, g_l \rangle|^2 + \frac{1+(N-2)r-(N-1)r^2}{\alpha r} \cdot$$
$$\cdot \sum_{k \neq k_0} \left| \langle \psi, g_k \rangle + \frac{\alpha r(1-r)}{1+(N-2)r-(N-1)r^2} c \right|^2 +$$
$$+ \sum_{k \neq k_0} \left[\langle \psi, g_k^* * g_k \rangle - \frac{1}{\alpha r} |\langle \psi, g_k \rangle|^2 \right] + \frac{(1-r)(N-2)}{N} |c|^2 .$$

By the two last chains, (2) and the induction assumption $S \geqslant 0$. The proof is finished.

THEOREM 1. – *Let* $G = G_1 \circ G_2 \circ \ldots \circ G_N$: *If* $r \in [(-1)/(N-1), 1]$ *then the function* ψ_r *is positive definite on* G.

PROOF. – We shall prove that for every function f on G with a finite support the following inequality holds:

$$\langle \psi_r, f^* * f \rangle \geqslant |\langle \psi_r, f \rangle|^2 .$$

First we observe that $\langle \psi_r, f^* * f \rangle - |\langle \psi_r, f \rangle|^2 = \langle \psi_r, f_0^* * f_0 \rangle - |\langle \psi_r, f_0 \rangle|^2$, where $f_0(x) = f(x)$ for $x \neq e$ and $f_0(e) = 0$. Therefore we can assume, that $f(e) = 0$. We have

$$f = f_1 + f_2 + \ldots + f_N ,$$

where $\operatorname{supp}(f_k) \subseteq S_k$. As before we shall write ψ and α instead of ψ_r and $\alpha(r)$. Using Lemma 1 and facts (3) and (4) one readily obtains for each $r \in ((-1)/(N-1), 0) \cup (0, 1]$

$$\begin{aligned}
\langle \psi, f^* * f \rangle &= \sum_{k,l} \langle \psi, f_k^* * f_l \rangle = \sum_k \langle \psi, f_k^* * f_k \rangle + \\
&+ \sum_{k \neq l} \frac{1}{\alpha} \overline{\langle \psi, f_k \rangle} \langle \psi, f_l \rangle \geqslant \frac{1}{\alpha r} \sum_k |\langle \psi, f_k \rangle|^2 + \frac{1}{\alpha} \sum_{k \neq l} \overline{\langle \psi, f_k \rangle} \langle \psi, f_l \rangle = \\
&= \Big| \sum_k \langle \psi, f_k \rangle \Big|^2 + \frac{\alpha - 1}{\alpha} \sum_{k<l} |\langle \psi, f_k \rangle - \langle \psi, f_l \rangle|^2 = \\
&= |\langle \psi, f \rangle|^2 + \frac{\alpha - 1}{\alpha} \sum_{k<l} |\langle \psi, f_k - f_l \rangle|^2 \geqslant |\langle \psi, f \rangle|^2 .
\end{aligned}$$

Since the cases $r = (-1)/(N-1)$ and $r = 0$ are trivial the proof of Theorem 1 is complete.

COROLLARY 1. – *Let* $G = G_1 \circ G_2 \circ \ldots \circ G_N$: *The positive definite function* $G \in x \mapsto P_r(x) = r^{\|x\|}$ *is not irreducible for* $0 < r < 1$ *i.e.* P_r *is a convex combination of different positive definite functions. Namely*

$$P_r = \frac{1}{\alpha(r)} \psi_r + \left(1 - \frac{1}{\alpha(r)}\right) \delta_e .$$

COROLLARY 2. – *For* $(-1)/(N-1) < r \leqslant 1$ *the function* P_r *belongs to the Fourier-Stieltjes algebra of* G.

COROLLARY 3. – (*a*) *The function* $\varrho(x) = \|x\| - 1/N$ *for* $x \neq e$, $\varrho(e) = 0$ *is negative definite on the group* $G = G_1 \circ G_2 \circ \ldots \circ G_N$.

(*b*) *The function* $\tilde{d}(x, y) = d(x, y) - 1/N$ *for* $x \neq y$ *and* $\tilde{d}(x,x) = 0$ *is negative definite on a homogeneous tree* T *of degree* N, *where* d *denotes the usual distance on* T.

For basic facts about trees see book of J.-P. Serre [9], and for main properties of negative definite functions see C. Berg, G. Forst [1].

PROOFS. – Corollaries 1 and 2 follow directly from Theorem 1. To get Corollary 3 (*a*) it is sufficient to calculate, that

$$\varrho(x) = \lim_{r \to 1^-} \frac{1 - \varphi_r(x)}{1 - r}.$$

For the proof of (*b*) one should consider the natural homogeneous tree of degree N connected with the product group $G = \mathbf{Z}_2 \circ \mathbf{Z}_2 \circ \ldots \circ \mathbf{Z}_2$ (N-copies).

REMARK. – Corollary 3 (*b*) is a generalisation of the result of P. Julg and A. Valette ([8], Lemma 2.3) which says that kernel $(x, y) \mapsto d(x, y)$ is a negative definite function on a tree.

3. – The case $G_1 = G_2 = \ldots = G_N = \mathbf{Z}_k$.

This part is connected with the paper [7] of A. Iozzi and Ml Picardello. They have introduced and studied so called spherica. functions on the group $G_{k,N} = \mathbf{Z}_k \circ \mathbf{Z}_k \circ \ldots \circ \mathbf{Z}_k$ (N-times), where $\mathbf{Z}_k$ denotes the cyclic group of order k. In Proposition 4 (page 359 in [7]) the autors have written, that a spherical function φ is positive definite if and only if

$$\varphi(x) \in \left[\frac{2(k-2)}{(k-1)N} - 1, 1\right] \qquad \text{for } \|x\| = 1 .$$

However this is not correct in full generality. We shall give the complete characterisation of positive definite spherical functions on the group $G_{k,N}$.

DEFINITION. – A radial function φ on $G_{k,N}$ is called *spherical* if the functional $Lf = \langle f, \varphi \rangle$ is multiplicative on the convolution algebra of radial, finite supported functions on $G_{k,N}$.

A spherical function φ, being radial, depends only on $\|x\|$. Therefore it is convenient, for simplicity, to use the notation $\varphi(n)$ to denote the value of a spherical function φ at a word of lenght n. Let us recall, that φ is a spherical function if and only if there exists a complex number z such, that $\varphi(n) = P_n(z)$, where P_n are polynomials defined by the formulas:

$$P_0(z) = 1\,, \qquad P_1(z) = z$$

and

$$zP_n(z) = \frac{1}{(k-1)N} P_{n-1}(z) + \frac{k-2}{(k-1)N} P_n(z) + \frac{N-1}{N} P_{n+1}(z)$$

for $n \geqslant 1$ (see [7]).

We have the natural projection E from functions on $G_{k,N}$ onto radial functions on $G_{k,N}$ defined as follows

$$(Ef)(x) = \frac{1}{\# W_n} \sum_{y \in W_n} f(y)$$

where $n = \|x\|$ and $W_n = \{y \in G_{k,N}\colon \|y\| = n\}$ (see [7] page 351). Since the proof of the Lemma 2 in [7] is not clear for us we present our version of the proof using ideas of the U. Haagerup paper [6].

THEOREM 2. – *E maps positive definite functions into positive definite functions.*

To prove Theorem 2 it is convenient to consider $G_{k,N}$ as the set of vertices of a graph $\tilde{G}_{k,N}$ where $\{x, y\}$ is an edge if and only if $\|y^{-1}x\| = 1$. Let us observe, that for any $x, y \in G_{k,N}$, $\|y^{-1}x\|$ is the minimal lenght of paths from y to x (see [9]). It is sufficient to see it for $y = e$ because the group $G_{k,N}$ acts on $\tilde{G}_{k,N}$ by left multiplication as a group of isomorfisms. Therefore each isomorfism of $\tilde{G}_{k,N}$ must be an isometry in the metric $d(x, y) = \|y^{-1}x\|$. Let I_0 be the group of all isomorfisms σ of the graph $\tilde{G}_{k,N}$ such that $\sigma(e) = e$. I_0 is a subgroup of the infinite product

$$H = \prod_{n=0}^{\infty} S(W_n)$$

of the permutation groups $S(W_n)$ of W_n. Since H is compact in the product topology and since the topology coincides with the topology of pointvise convergence in $G_{k,N} = \bigcup_{n=0}^{\infty} W_n$, I_0 is a closed subgroup of H. Therefore I_0 is compact.

LEMMA 2. – *Consider the action of I_0 on $G_{k,N} \times G_{k,N}$ given by*

$$\sigma(x, y) = (\sigma(x), \sigma(y)), \quad x, y \in G_{k,N}.$$

Then I_0 acts transitively on each of the sets

$$E_{k,m,l} = \{(x, y) \in G_{k,N} \times G_{k,N}\colon \|x\| = n, \|y\| = m, \|y^{-1}x\| = l\}.$$

PROOF. – It is clear, that each $E_{n,m,l}$ is a finite set invariant under the action of I_0.

Let us note, that we can identify the graph $\tilde{G}_{k,N}$ with a set X defined bellow:

$$X = \{\omega\colon \omega = ((i_1, k_1), \ldots, (i_\eta, k_\eta)), \eta \geqslant 0, 1 \leqslant i_1 \leqslant N, 1 \leqslant i_p \leqslant N-1 \text{ for } p > 1 \text{ and } 1 \leqslant k_p \leqslant k-1 \text{ for } p \geqslant 1\},$$

in which are two following kinds of connections:

$$((i_1, k_1), \ldots, (i_\eta, k_\eta)) \quad \text{with}$$
$$((i_1, k_1), \ldots, (i_\eta, k_\eta), (i_{\eta+1}, k_{\eta+1}))$$

for $\eta \geqslant 0$ and

$$((i_1, k_1), \ldots, (i_\eta, k_\eta)) \quad \text{with}$$
$$((i_1, k_1), \ldots, (i_{\eta-1}, k_{\eta-1}), (i_\eta, k'_\eta))$$

for $\eta > 0$.

Let now (x, y), (s, t) be two elements of $E_{n,m,l}$: We shall look for $\tau \in I_0$ such that $\tau(x, y) = (s, t)$. Let

$$x = ((i'_1, k'_1), \ldots, (i'_n, k'_n)),$$
$$y = ((i''_1, k''_1), \ldots, (i''_m, k''_m)),$$
$$s = ((j'_1, l'_1), \ldots, (j'_n, l'_n)),$$
$$t = ((j''_1, l''_1), \ldots, (j''_m, l''_m)).$$

Observe, that for any $\omega_0 = ((i_1, k_1), \ldots, (i_\eta, k_\eta)) \in X$, any i ($i \leqslant N$ if $\omega_0 = e$ and $i \leqslant N-1$ if $\omega_0 \neq e$), and any permutation $\tilde{\sigma}$ of the set $\{1, 2, \ldots, k-1\}$ the map

$$\sigma(\omega) = \begin{cases} ((i_1, k_1), \ldots, (i_\eta, k_\eta), (i_{\eta+1}, \tilde{\sigma}(k_{\eta+1})), (i_{\eta+2}, k_{\eta+2}), \ldots, (i_\mu, k_\mu)) \\ \text{when} \quad \omega = ((i_1, k_1), \ldots, (i_\mu, k_\mu)), \mu > \eta \text{ and } i_{\eta+1} = i, \\ \omega \text{ othervise} \end{cases}$$

is in I_0. Therefore we can assume, that

$$k_1' = l_1', \ldots, k_n' = l_n' ,$$
$$k_1'' = l_1'', \ldots, k_m'' = l_m'' .$$

Now assume, that $n+m-l$ is even and let $p = (n+m-l)/2+1$. Then we have

$$i_1' = i_1'', \ldots, i_{p-1}' = i_{p-1}'', i_p' \neq i_p'' ,$$
$$j_1' = j_1'', \ldots, j_{p-1}' = j_{p-1}'', j_p' \neq j_p'' ,$$
$$k_1' = k_1'', \ldots, k_{p-1}' = k_{p-1}'' .$$

Using a map

$$((i_1, k_1), \ldots, (i_\eta, k_\eta)) \mapsto$$
$$\mapsto \big((\sigma_1(i_1), k_1), \ldots, (\sigma_p(i_p), k_p), (i_{p+1}, k_{p+1}), \ldots, (i_\eta, k_\eta)\big)$$

where $\sigma_2, \ldots, \sigma_p$ are permutations of the set $\{1, 2, \ldots, N-1\}$, σ_1 is permutation of the set $\{1, 2, \ldots, N\}$, we can assume, that

$$i_1' = j_1', i_2' = j_2', \ldots, i_p' = j_p' \quad \text{and} \quad i_p'' = j_p'' .$$

Now let $\sigma_{p+1}, \ldots, \sigma_n, \tilde{\sigma}_{p+1}, \ldots, \tilde{\sigma}_m$ are permutations such, that

$$\sigma_{p+1}(i_{p+1}') = j_{p+1}', \ldots, \sigma_n(i_n') = j_n' ,$$
$$\tilde{\sigma}_{p+1}(i_{p+1}'') = j_{p+1}'', \ldots, \tilde{\sigma}_m(i_m'') = j_m'' .$$

We define $\tau \in I_0$ putting for $\omega = ((i_1, k_1), \ldots, (i_\eta, k_\eta))$,

$$\tau(\omega) = ((j_1, k_1), \ldots, (j_\eta, k_\eta))$$

where

a) if $i_p = i_p'$ then $j_q = \sigma_q(i_q)$ for $p+1 \leqslant q \leqslant n$ and $j_q = i_q$ othervise;

b) if $i_p = i_p''$ then $j_q = \tilde{\sigma}_q(i)$ for $p+1 \leqslant q \leqslant m$ and $j_q = i_q$ othervise;

c) if $i_p \neq i_p', i_p''$ then $j_q = i_q$ for each $q \leqslant \eta$.

In the case when $n+m-l$ is odd let $p = (n+m-l+1)/2$.

Then we have

$$i_1' = i_1'', \ldots, i_{p-1}' = i_{p-1}'', \qquad i_p' = i_p'',$$
$$j_1' = j_1'', \ldots, j_{p-1}' = j_{p-1}'', \qquad j_p' = j_p'',$$
$$k_1' = k_1'', \ldots, k_{p-1}' = k_{p-1}'', \qquad k_p' \neq k_p''.$$

Simillary as before we can assume, that

$$i_1' = j_1', \ldots, j_p' = j_p'.$$

Now let $\sigma_{p+1}, \ldots, \sigma_n, \tilde{\sigma}_{p+1}, \ldots, \tilde{\sigma}_m$ are permutations as before. Then we put

$$\tau(\omega) = ((j_1, k_1), \ldots, (j_\eta, k_\eta))$$

where

a) if $k_p = k_p'$ then $j_q = \sigma_q(i_q)$ for $p+1 \leqslant q \leqslant n$ and $j_q = i_q$ othervise;

b) if $k_p = k_p''$ then $j_q = \tilde{\sigma}_q(i_q)$ for $p+1 \leqslant q \leqslant m$ and $j_q = i_q$ othervise;

c) if $i_p \neq i_p', i_p''$ then $j_q = i_q$ for each $q \leqslant \eta$.

In the both cases we have $\tau(x, y) = (s, t)$ and Lemma 2 is proved.

PROOF OF THEOREM 2. – For any complex function f on $G_{k,N}$ we shall prove the following formula

$$(**) \qquad (Ef)(y^{-1}x) = \int_{\mathfrak{J}_0} f(\sigma(y)^{-1}\sigma(x))\, d\mu(\sigma)$$

for all $x, y \in G_{k,N}$, where μ denotes the normalised Haar measure on I_0.

Fix $x, y \in G_{k,N}$ and let $\|x\| = n$, $\|y\| = m$, $\|y^{-1}x\| = l$. By Lemma 2 all subsets of I_0 given by

$$\mathcal{F}_{s,t} = \{\sigma \in I\colon \sigma(x_0, y) = (s, t)\}, \qquad (s, t) \in E_{n,m,l}$$

have the same measure in I_0. Therefore

$$\mu(\mathcal{F}_{s,t}) = \frac{1}{\#(E_{n,m,l})}, \qquad (s, t) \in E_{n,m,l}.$$

Hence

$$\int_{\mathfrak{J}_0} f(\sigma(y)^{-1}\sigma(x))\, d\mu(\sigma) = \frac{1}{\#(E_{n,m,l})} \sum_{(s,t)\in E_{n,m,l}} f(t^{-1}s).$$

For each z of lenght l the number of pairs (s, t) in $E_{n,m,l}$ for which $t^{-1}s = z$ is equal to $(\chi_m * \chi_n)(z)$, where χ_n, χ_m are the characteristic functions of W_n, W_m respectively. Recall, that convolution of two radial functions is again radial (see [7]). Therefore the value $(\chi_m * \chi_n)(z)$ depends only on $\|z\|$. Hence we have

$$\#(W_l)\cdot(\chi_m * \chi_n)(z) = \#(E_{n,m,l})$$

and

$$\frac{1}{\#(E_{n,m,l})}\sum_{(s,t)\in E_{n,m,l}} f(t^{-1}s) = \frac{1}{\#(E_{n,m,l})}\sum_{z\in W_l}(\chi_m * \chi_n)(z)f(z) =$$

$$= \frac{1}{\#(W_l)}\sum_{z\in W_l} f(z) = (Ef)(z) = (Ef)(y^{-1}x)\,.$$

This way we have proved (**) and Theorem 2 follows directly from (**).

LEMMA 3. – *If φ is real valued spherical function on $G_{k,N}$ and if φ belongs to Fourier-Stieltjes algebra $B(G_{k,N})$ then φ is positive definite.*

PROOF. – Let f be any finite supported function. Because φ is radial then we have

$$\langle\varphi, f^* * f\rangle = \langle\varphi, E(f^* * f)\rangle\,.$$

By Theorem 2 the function $E(f^* * f)$ is positive definite and $E(f^* * f)$ belongs to $C^*_{\mathrm{rad}}(G_{k,N})$ the closure of algebra of finite supported radial functions in full C^*-algebra $C^*(G_{k,N})$. Hence we have $E(f^* * f) = g^* * g$ for some $g \in C^*_{\mathrm{rad}}(G_{k,N})$ (see [3].

Let us take a sequence $\{h_n\}$ of finite supported radial functions tending to g in $C^*_{\mathrm{rad}}(G_{k,N})$. Since by the assumption φ is real valued and φ is a multiplicative functional on radial, finite supported functions, we obtain

$$\langle\varphi, g^* * g\rangle = \lim_{n\to\infty}\langle\varphi, h_n^* * h_n\rangle = \lim_{n\to\infty}(\overline{\langle\varphi, h_n\rangle}\langle\varphi, h_n\rangle) \geqslant 0\,.$$

Here we have used the fact, that $B(G)$ is the dual of the full C^*-algebra $C^*(G)$. This completes the proof.

THEOREM 3. – *Let φ be a spherical function on $G_{k,N}$. Then φ is positive definite if and only if $\varphi(1) \in [(-1)/(k-1), 1]$.*

PROOF. – At the beginning let us observe, that the function

$$w_r(x) = \begin{cases} 1 & \text{for } x = e \\ r & \text{for } x \neq e \end{cases}$$

is positive definite on the cyclic group $\boldsymbol{Z}_k$ if and only if $r \in \in [(-1)/(k-1), 1]$. Therefore we get « only if ».

In [7], Theorem 2, page 357 the authors have showed, that if $z \neq \frac{1}{2} + m\pi i/\ln q$ then

$$\varphi_z(x) = c_z q^{-z\|x\|} + c_{1-z} q^{(z-1)\|x\|},$$

where $q = (N-1)(k-1)$, c_z, c_{1-z} are some constances and φ_z is the spherical function for which

$$(***) \qquad \varphi_z(1) = \frac{\sqrt{q}}{N(k-1)}\left(q^{-\frac{1}{2}+z} + q^{\frac{1}{2}-z} + \frac{k-2}{\sqrt{q}}\right).$$

By Lemma 2 it is sufficient to show, that the functions $q^{-z\|x\|}$, $q^{(z-1)\|x\|}$ are both in the Fourier-Stieltjes algebra $B(G_{k,N})$.

First assume, that $k > N$. By Corollary 2 the function $P_r(x) = = r^{\|x\|}$ is in $B(G_{k,N})$ for $(-1)/(N-1) < r \leqslant 1$. Hence φ_z is positive definite when

$$q^{-z+\frac{1}{2}},\ q^{z-\frac{1}{2}} \in \left(\frac{-\sqrt{q}}{N-1}, \sqrt{q}\right).$$

It occurs if and only if

$$q^{-z+\frac{1}{2}} \in \left(-\frac{\sqrt{q}}{N-1}, -\frac{N-1}{\sqrt{q}}\right) \cup \left[\frac{1}{\sqrt{q}}, \sqrt{q}\right]$$

and both of the intervals are nonempty. One can check, that the image of

$$\left(-\frac{\sqrt{q}}{N-1}, -\frac{N-1}{\sqrt{q}}\right) \cup \left[\frac{1}{\sqrt{q}}, \sqrt{q}\right]$$

by the function $s \mapsto s + 1/s$ is exactly the set

$$\left(-\frac{N+k-2}{\sqrt{q}}, -2\right) \cup \left[2, \frac{q+1}{\sqrt{q}}\right].$$

Using (***) we infer, that if

$$\varphi(1) \in \left(\frac{-1}{k-1}, \frac{k-2-2\sqrt{q}}{(k-1)N}\right] \cup \left[\frac{k-2+2\sqrt{q}}{(k-1)N}, 1\right],$$

then φ is positive definite.

Now let $k \leqslant N$. If $(-1)/(k-1) \leqslant r \leqslant 1$ then the function w_r defined before is positive definite on the cyclic group $\boldsymbol{Z}_k$. Using the fact, that the function P_r on $G_{k,N}$ is a free product of w_r (see [2]) and using Bozejko Theorem in [2] stating, that free product of positive definite functions is again positive definite, we obtain, that P_r is positive definite for $(-1)/(k-1) \leqslant r \leqslant 1$. Therefore φ_z is positive definite when

$$q^{-z+\frac{1}{2}}, q^{z-\frac{1}{2}} \in \left[\frac{-\sqrt{q}}{k-1}, 1\right].$$

We infer as before, that φ is positive definite when

$$\varphi(1) \in \left[\frac{-1}{k-1}, \frac{k-2-2\sqrt{q}}{(k-1)N}\right] \cup \left[\frac{k-2+2\sqrt{q}}{(k-1)N}, 1\right].$$

Recall, tha the interval

$$\left[\frac{k-2-2\sqrt{q}}{(k-1)N}, \frac{k-2+2\sqrt{q}}{(k-1)N}\right]$$

corresponds to the principal series of unitary representations (see [7]). Since pointvise limit of positive definite functions is again positive definite we have Theorem 2.

REMARK. – This result was know in the case $k = 2$ (see for example [5], Lemma 3.2.).

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