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Positive Definite Functions on Free Product of Groups.

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Sunto. – *Si presenta una costruzione di alcune funzioni definite positive su prodotti liberi di gruppi. In alcuni casi si ottengono descrizioni complete di tutte le funzioni radiali definite positive.*

0. – Introduction.

Let $G = \bigcirc_{i \in I} G_i$ be the free product of discrete groups. In this paper we present a construction of positive definite functions on G. Namely, let $A = (a_{ij})_{i,j \in I}$ be a complex matrix such, that $(a_{ij} - 1)_{i,j \in I}$ is positive definite. We define the function ψ_A on G in the following way: $\psi_A(e) = 1$ and if

$$x = g_1 g_2 \dots g_n\,, \qquad \text{where } g_k \in G_{i_k} \setminus \{e\} \text{ and } i_k \neq i_{k+1}$$

then we put

$$\psi_A(x) = a_{i_1 i_1}^{-1} a_{i_1 i_2} a_{i_2 i_2}^{-1} a_{i_2 i_3} \dots a_{i_n i_n}^{-1}\,.$$

We shall show that ψ_A is a positive definite function on G. Such a function is *type-dependent* and we prove, that the set of type-dependent positive definite functions on G depends in fact on the cardinalities of the factor groups G_i and doesn't depend on the group structure of the groups G_i.

Next we turn our attention to the case $G_{k,N} = \boldsymbol{Z}_k \circ \dots \circ \boldsymbol{Z}_k$ (N times) which was considered in [7, 8] and we characterize the set of all positive definite radial functions (a complex function f on the free product $G = \bigcirc_{i \in I} G_i$ is called *radial* if $f(x)$ depends only on $\|x\|$—*the length* of x, where we define $\|e\| = 0$ and for x as above $\|x\| = n$). This characterization is analogous to the Bochner theorem in the case of abelian groups. Finally we consider the cases $k = \infty$ and $N = \infty$. For example, let $\boldsymbol{F}_N = \boldsymbol{Z} \circ \dots \circ \boldsymbol{Z}$ (N times) be the free group with N generators. Then a radial function φ is a positive definite on F_N if and only if there exists a bounded nonnegative Radon measure on the interval $[0, 1]$ such, that for each $x \in F_N$

$$\varphi(x) = \int_0^1 \psi_r(x)\, d\mu(r) ,$$

where

$$\psi_r(x) = \begin{cases} 1 & \text{if } x = e , \\ r\left(\dfrac{Nr-1}{N-1}\right)^{n-1} & \text{if } \|x\| = n \geqslant 1 . \end{cases}$$

Similar theorems, with other definition of length on F_N have been given in [1, 3, 6].

1. – Notation.

Let I be a fixed set. A complex matrix $(a_{ij})_{i,j\in I}$ is called *positive definite* if for any function $\alpha\colon I \to C$ with finite support

$$\sum_{i,j\in I} a_{ij}\alpha(i)\overline{\alpha(j)} \geqslant 0 .$$

Recall, that *the Schur theorem* tells, that if $A = (a_{ij})_{i,j\in I}$, $B = (b_{ij})_{i,j\in I}$ are positive definite then $A \circ B = (a_{ij}b_{ij})_{i,j\in I}$ is also positive definite.

If G is a group then a function $\psi\colon G \to C$ is called *positive definite* if for any function $f\colon G \to C$ with finite support

$$\sum_{x,y\in G} \psi(y^{-1}x) f(x)\overline{f(y)} \geqslant 0 .$$

Let $\{G_i\}_{i\in I}$ be a family of discrete groups. We can consider their *free product* $\bigcirc_{i\in I} G_i$, in which each element x can be uniquely expressed as *a reduced word*

$$x = g_1 g_2 \ldots g_n ,$$

where $n \geqslant 0$, $g_k \in G_{i_k}\setminus\{e\}$, $i_k \neq i_{k+1}$ for $k = 1, 2, \ldots, n-1$. For such a word x we define its *type* $t(x)$ as follows

$$t(x) = (i_1, i_2, \ldots, i_n)$$

and its *block length* $\|x\| = n$. See also book of J.-P. Serre [9]. A function $f\colon \bigcirc_{i\in I} G_i \to \boldsymbol{C}$ is called *radial* if the value $f(x)$ depends only on $\|x\|$.

2. – The construction.

Let us suppose, that $0 \notin I$ and denote $I_0 = I \cup \{0\}$. We start with the following lemma:

LEMMA 1. – *Let $(u_{ij})_{i,j\in I_0}$ be a hermitian matrix (i.e. $u_{ij} = \overline{u}_{ji}$). Suppose, that $u_{kk} > 0$ for some $k \in I_0$. Then the matrix $(u_{ij})_{i,j\in I_0}$ is positive definite if and only if the matrix*

$$\left(u_{ij} - \frac{u_{ik}u_{kj}}{u_{kk}}\right)_{i,j\in I_0\setminus\{k\}}$$

is positive definite.

PROOF. – Suppose, that $(u_{ij})_{i,j\in I_0}$ is a positive definite matrix. For any two functions $f, g\colon I_0 \to \boldsymbol{C}$ with finite supports by the Schwarz inequality we have:

$$|[f, g]_u|^2 \leqslant [f, f]_u [g, g]_u ,$$

where

$$[f, g]_u = \sum_{i,j\in I_0} u_{ij} f(i) \overline{g(j)} .$$

Choose g in following way

$$g(i) = \begin{cases} 1 & \text{if } i = k , \\ 0 & \text{if } i \neq k . \end{cases}$$

Then we have

$$\Big|\sum_{i\in I_0} u_{ik} f(i)\Big|^2 \leqslant u_{kk} \sum_{i,j\in I_0} u_{ij} f(i) \overline{f(j)} .$$

Using the assumption, that $(u_{ij})_{i,j\in I_0}$ is a hermitian matrix, we obtain

$$0 \leqslant \sum_{i,j\in I_0} u_{ij} f(i)\overline{f(j)} - \frac{1}{u_{kk}} \sum_{i,j\in I_0} u_{ik} f(i) \overline{u}_{jk} \overline{f(j)} =$$

$$= \sum_{i,j\in I_0} \left(u_{ij} - \frac{u_{ik}u_{kj}}{u_{kk}}\right) f(i)\overline{f(j)} = \sum_{i,j\in I_0\setminus\{k\}} \left(u_{ij} - \frac{u_{ik}u_{kj}}{u_{kk}}\right) f(i)\overline{f(j)} .$$

Conversely, if $(u_{ij} - (u_{ik}u_{kj})/u_{kk})_{i,j\in I_0\setminus\{k\}}$ is positive definite and if $f\colon I_0 \to C$ is a finite supported function then we have

$$0 \leqslant \sum_{i,j\in I_0\setminus\{k\}} \left(u_{ij} - \frac{u_{ik}u_{kj}}{u_{kk}}\right) f(i)\overline{f(j)} = \sum_{i,j\in I_0} \left(u_{ij} - \frac{u_{ik}u_{kj}}{u_{kk}}\right) f(i)\overline{f(j)} .$$

Therefore

$$\sum_{i,j\in I_0} u_{ij} f(i)\overline{f(j)} \geqslant \frac{1}{u_{kk}} \left| \sum_{i\in I_0} u_{ik} f(i) \right|^2$$

and the proof is finished.

Let $\{G_i\}_{i\in I}$ be any family of discrete groups and let $A = (a_{ij})_{i,j\in I}$ be any complex matrix such, that for $i\in I$ $a_{ii}\neq 0$. We define a function ψ_A on $G = \bigcirc_{i\in I} G_i$ in the following way: $\psi_A(e) = 1$ and if $t(x) = (i_1, i_2, \dots, i_n)$ we put

$$\psi_A(x) = a_{i_1i_1}^{-1} a_{i_1i_2} a_{i_2i_2}^{-1} a_{i_2i_3} \dots a_{i_ni_n}^{-1} .$$

Let us define *the extended matrix* $A_0 = (a_{ij})_{i,j\in I_0}$ putting $a_{i0} = a_{0j} = a_{00} = 1$. Let us note, that Lemma 1 follows, that A_0 is positive definite if and only if the matrix $(a_{ij} - 1)_{i,j\in I}$ is positive definite.

THEOREM 1. – *Let* $A = (a_{ij})_{i,j\in I}$ *be a complex matrix such that the extended matrix* A_0 *is positive definite. Then* ψ_A *is a positive definite function on the free product group* $G = \bigcirc_{i\in I} G_i$.

PROOF. – Let g be any complex function on G. For any $w\in G$ define the function g_w by

$$g_w(x) = \begin{cases} g(wx) & \text{if } \|wx\| = \|w\| + \|x\| , \\ 0 & \text{otherwise} . \end{cases}$$

For any $k\in I_0$ define the function $g^{(k)}$ in the following way:

$$g^{(0)}(x) = \begin{cases} g(e) & \text{if } x = e , \\ 0 & \text{otherwise} , \end{cases}$$

and for $k \neq 0$

$$g^{(k)}(x) = \begin{cases} g(x) & \text{if } x\neq e ,\ t(x) = (i_1, \dots, i_n) ,\ i_1 = k , \\ 0 & \text{otherwise} . \end{cases}$$

If $x\neq e$ and $t(x) = (i_1, \dots, i_n)$ we put $k(x) = i_n$.

Let f be any finitely supported function. We shall prove, that

$$(1)\quad \sum_{x,y\in G}\psi_A(y^{-1}x)f(x)\overline{f(y)} = \sum_{i,j\in I_0} a_{ji}\langle f^{(i)},\psi_A\rangle\overline{\langle f^{(j)},\psi_A\rangle} + \\ + \sum_{k\in I}\sum_{\substack{w\neq e\\ k(w)=k}}\sum_{i,j\in I_0\setminus\{k\}}\left(a_{.i}-\frac{a_{jk}a_{ki}}{a_{kk}}\right)\langle (f_w)^{(i)},\psi_A\rangle\overline{\langle (f_w)^{(j)},\psi_A\rangle}$$

where we design $\langle f, g\rangle = \sum_{x\in G} f(x)g(x)$. Once the formula (1) is established, it follows by Lemma 1 that ψ_A is positive definite.

The right hand side of (1) can be written as follows

$$\sum_{x,y\in G} v(x,y)f(x)\overline{f(y)}$$

and we only need to show, that $v(x,y)=\psi_A(y^{-1}x)$. Fix $x, y\in G$ and let $t(x)=(i_1,\ldots,i_n)$, $t(y)=(j_1,\ldots,j_m)$. We consider two cases:

(α) $\quad x=ux'\,,\ y=uy'\,,\ \|x\|=\|u\|+\|x'\|\,,\ \|y\|=\|u\|+\|y'\|\,,$
$t(u)=(i_1,\ldots,i_s)\,,\ i_1=j_1,\ldots,i_s=j_s\,,\ i_{s+1}\neq j_{s+1}\,,$

(β) $\quad x=ug_1x'\,,\ y=ug_2y'\,,\ \|x\|=\|u\|+1+\|x'\|\,,$
$\|y\|=\|u\|+1+\|y'\|\,,\ t(u)=(i_1,\ldots,i_s)\,,$
$i_1=j_1,\ldots,i_s=j_s\,,\ i_{s+1}=j_{s+1}\,,\ g_1,g_2\in G_{i_{s+1}}\setminus\{e\}\,,\ g_1\neq g_2\,.$

Let us take u of the form $u=u_1u_2\ldots u_s$, $u_t\in G_{i_t}\setminus\{e\}$. In both cases the term $f(x)\overline{f(y)}$ appears in the second sum of the right hand side of (1) if and only if $w=u_1u_2\ldots u_t$, $1\leqslant t\leqslant s$. Therefore the coefficient $v(x,y)$ has the following form

$$v(x,y)=a_{j_1i_1}(a^{-1}_{i_1i_1}a_{i_1i_2}\ldots a^{-1}_{i_ni_n})(\overline{a^{-1}_{j_1j_1}a_{j_1j_2}\ldots a^{-1}_{j_mj_m}}) + \\ +\sum_{t=1}^{s}\left(a_{j_{t+1}i_{t+1}}-\frac{a_{j_{t+1}j_t}a_{i_ti_{t+1}}}{a_{j_ti_t}}\right)(a^{-1}_{i_{t+1}i_{t+1}}a_{i_{t+1}j_{t+2}}\ldots a^{-1}_{i_ni_n})\cdot \\ \cdot(\overline{a^{-1}_{j_{t+1}j_{t+1}}a_{j_{t+1}j_{t+2}}\ldots a^{-1}_{j_mj_m}}) = (a^{-1}_{j_mj_m}a_{j_mj_{m-1}}\ldots a^{-1}_{j_1j_1})a_{j_1i_1}(a^{-1}_{i_1i_1}a_{i_1i_2}\ldots a^{-1}_{i_ni_n}) + \\ +\sum_{t=1}^{s}(a^{-1}_{j_mj_m}\ldots a^{-1}_{j_{t+1}j_{t+1}})\left(a_{j_{t+1}i_{t+1}}-\frac{a_{j_{t+1}j_t}a_{i_ti_{t+1}}}{a_{j_ti_t}}\right)(a^{-1}_{i_{t+1}i_{t+1}}\ldots a^{-1}_{i_ni_n})\,.$$

The sum is telescopic and in both cases we obtain $v(x,y)=$ $=\psi_A(y^{-1}x)$. The cases $n=s$ or $m=s$ can be considered in similar way. Therefore the formula (1) and so the theorem is proved.

COROLLARY 1. - *Let* $G=\bigcirc_{i\in I} G_i$ *and suppose, that* card $(I)=N$,

$1 < N < \infty$. *For any number r define the function ψ_r on G as*

$$\psi_r(x) = \begin{cases} 1 & \textit{if } x = e\,, \\ r\left(\dfrac{Nr-1}{N-1}\right)^{n-1} & \textit{if } \|x\| = n \geqslant 1\,. \end{cases}$$

Then ψ_r is positive definite on G for any $r \in [0, 1]$.

PROOF. – We can suppose, that $I = \{1, 2, \ldots, N\}$. It is easy to see, that for $r \neq 0$ $\psi_r = \psi_A$, where $A = (a_{ij})_{i,j=1}^{N}$,

$$a_{ii} = \frac{1}{r} \text{ and for } i \neq j\,, \quad a_{ij} = \frac{Nr-1}{(N-1)r}\,.$$

Hence we have $a_{ij} - 1 = ((1-r)/r)\, b_{ij}$, where

$$b_{ii} = 1 \text{ and for } i \neq j\,, \quad b_{ij} = \frac{-1}{N-1}\,.$$

Straightforward calculations show, that $(b_{ij})_{i,j=1}^{N}$ is a positive definite matrix. Hence, by Lemma 1 the matrix A satisfies the assumption of Theorem 1. The case $r = 0$ is trivial.

REMARK. – The functions ψ_r with slight modification have been considered in [8] where one can find the different proof of Corollary 1.

The following proposition shows, that in some cases the condition on the matrix in Theorem 1 is necessary.

PROPOSITION 1. – *Let $\{G_i\}_{i\in I}$ be a family of infinite discrete groups and let $A = (a_{ij})_{i,j\in I}$ be a complex matrix. If the function ψ_A defined as before is positive definite on $G = \bigcirc_{i\in I} G_i$ then the extended matrix A_0 is positive definite.*

PROOF. Let α be any finitely supported complex function on $I_0 = I \cup \{0\}$. Fix $n \in I$ and let for any $i \in I$ B_i be a fixed subset of $G_i \setminus \{e\}$ consisting of n elements. Let us define the following function on G:

$$f(x) = \begin{cases} \alpha(0) & \text{if } x = e\,, \\ \dfrac{a_{ii}\alpha(i)}{n} & \text{if } x \in B_i\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

One can see, that we have

$$0 \leqslant \sum_{x,y\in G} \psi_A(y^{-1}x) f(x) \overline{f(y)} =$$

$$= |\alpha(0)|^2 + \sum_{\substack{i,j\in I_0 \\ i\neq j}} a_{ji}\alpha(i)\overline{\alpha(j)} + \sum_{i\in I}\left(\frac{n-1}{n} + \frac{a_{ii}}{n}\right) a_{ii}|\alpha(i)|^2 .$$

Passing with n to infinity we obtain $\sum_{i,j\in I_0} a_{ji}\,\alpha(i)\overline{\alpha(j)} \geqslant 0$ and this finishes the proof.

Let us recall, that a complex function φ on group G is called *negative definite* if $\varphi(e)\geqslant 0$, $\varphi(x^{-1}) = \overline{\varphi(x)}$ and for any finitely supported f on G such, that $\sum_{x\in G} f(x) = 0$ we have

$$\sum_{x,y\in G} \varphi(y^{-1}x) f(x) f(y) \leqslant 0 .$$

It is easy to see, that any nonnegative constant function is negative definite.

THEOREM 2. – *Let $C = (c_{ij})_{i,j\in I}$ be a positive definite matrix. We define the function φ_c on $G = \bigcirc_{i\in I} G_i$ in the following way: $\varphi_c(e) = 0$ and if $t(x) = (i_1, \ldots, i_n)$ we put*

$$\varphi_c(x) = c_{i_1 i_1} - c_{i_1 i_2} + c_{i_2 i_2} - c_{i_2 i_3} + \ldots + c_{i_n i_n} .$$

Then the function φ_c is negative definite on G.

PROOF. – Fix a number $t > 0$ and let

$$A_t = (\exp[tc_{ij}])_{i,j\in I} = \left(1 + \sum_{k=1}^{\infty} \frac{(tc_{ij})^k}{k!}\right)_{i,j\in I} .$$

By Lemma 1 and by the Shur theorem A_t satisfies the condition of Theorem 1. Therefore ψ_{A_t} is positive definite and hence $(1-\psi_{A_t})/t$ is negative definite. Since

$$\varphi_c(x) = \lim_{t\to 0^+} \frac{1-\psi_{A_t}(x)}{t}$$

then φ_c is the negative definite function on G.

The following proposition shows, that the set of type-dependent positive definite functions on the free product $\bigcirc_{i\in I} G_i$ depends only on the cardinalities of G_i.

PROPOSITION 2. – *Let* $\{G_i^{(1)}\}_{i\in I}$, $\{G_i^{(2)}\}_{i\in I}$ *be two families of discrete groups and let for every* $i \in I$ $\operatorname{card}(G_i^{(1)}) \leqslant \operatorname{card}(G_i^{(2)})$. *Let* f *be a fixed complex function on the set* $\{(i_1, i_2, \ldots, i_n)\colon n \geqslant 0,\ i_k \in I,\ i_k \neq i_{k+1}\}$. *Let us define the following functions* $f_s\colon G^{(s)} \to \boldsymbol{C}$ $(G^{(s)} = \bigcirc_{i\in I} G_i^{(s)},\ s = 1, 2)$ *as follows:*

$$f_1(x) = f(t_1(x))\,, \qquad f_2(u) = f(t_2(u))\,,$$

where t_s *is the type in* $G^{(s)}$. *Then*

a) *if* f_2 *is positive definite on* $G^{(2)}$ *then* f_1 *is positive definite on* $G^{(1)}$,

b) *if* f_2 *is negative definite on* $G^{(2)}$ *then* f_1 *is negative definite on* $G^{(1)}$.

PROOF. – Let for each $i \in I$, $h_i\colon G_i^{(1)} \setminus \{e\} \to G_i^{(2)} \setminus \{e\}$ be arbitrary one-to-one function. We define a function $h\colon G^{(1)} \to G^{(2)}$ in following way: $h(e) = e$ and if $x = g_1 g_2 \ldots g_n$, $g_k \in G_{i_k} \setminus \{e\}$, $i_k \neq i_{k+1}$ we put $h(x) = h_{i_1}(g_1) h_{i_2}(g_2) \ldots h_{i_n}(g_n)$. Then h is also one-to-one. Observe, that for each $x, y \in G^{(1)}$

$$t_1(y^{-1}x) = t_2(h(y)^{-1}h(x))\,.$$

Let $\alpha\colon G^{(1)} \to \boldsymbol{C}$ be any finitely supported function. If we consider $\tilde{\alpha}\colon G^{(2)} \to \boldsymbol{C}$

$$\tilde{\alpha}(u) = \begin{cases} \alpha(x) & \text{if } u = h(x)\,, \\ 0 & \text{if } u \notin h(G^{(1)})\,, \end{cases}$$

we get

$$\sum_{x,y\in G^{(1)}} f_1(y^{-1}x)\alpha(x)\overline{\alpha(y)} = \sum_{x,y\in G^{(1)}} f(t_1(y^{-1}x))\,\alpha(x)\,\overline{(\alpha)y} =$$

$$= \sum_{x,y\in G^{(1)}} f\Big(t_2(h(y)^{-1}h(x))\Big)\,\tilde{\alpha}(h(x))\,\overline{\tilde{\alpha}(h(y))} =$$

$$= \sum_{u,v\in G^{(2)}} f_2(v^{-1}u)\,\tilde{\alpha}(u)\,\overline{\tilde{\alpha}(v)}$$

and this finishes the proof.

3. – The case $\boldsymbol{Z}_k \circ \ldots \circ \boldsymbol{Z}_k$ (N times).

In this part of the paper we will consider the group $G_{k,N} = \boldsymbol{Z}_k \circ \ldots \circ \boldsymbol{Z}_k$ (N times), where $\boldsymbol{Z}_k$ denotes the cyclic group of order k.

Let $W_n = \{x \in G_{k,N} \colon \|x\| = n\}$ and let μ_n be the probability measure equidistributed over W_n. It was proved in [7] that for $n \geqslant 1$

$$\mu_1 * \mu_n = \frac{1}{(k-1)N}\mu_{n-1} + \frac{k-2}{(k-1)N}\mu_n + \frac{N-1}{N}\mu_{n-1}\,. \tag{2}$$

Let $\mathfrak{R}$ be the set of finitely supported radial functions on $G_{k,N}$. By (2) $\mathfrak{R}$ forms an abelian convolution algebra generated by μ_1. Since for radial function f the value $f(x)$ depends only on $\|x\|$, we will often use the notation $f(n)$ to denote the value of f at a word of length n. A radial function φ on $G_{k,N}$ is called *spherical* if the functional $Lf = \langle f, \varphi\rangle$ is multiplicative on the convolution algebra $\mathfrak{R}$. Since $\varphi(n) = \langle \mu_n, \varphi\rangle$ for every radial φ then by (2) φ is spherical iff there exists a complex number z such, that $\varphi(n) = P_n(z; k, N)$, where $P_n(z; k, N)$ are polynomials defined by the formulas

$$\begin{cases} P_0(z;k,N) = 1\,, \qquad P_1(z;k,N) = z\,, \\ zP_n(z;k,N) = \dfrac{1}{(k-1)N}P_{n-1}(z;k,N) + \\ \qquad + \dfrac{k-2}{(k-1)N}P_n(z;k,N) + \dfrac{N-1}{N}P_{n+1}(z;k,N)\,. \end{cases} \tag{3}$$

Such a spherical function will be denoted by φ_z.

There is a natural map E from the space of all functions on $G_{k,N}$ onto the space of all radial functions given by

$$(Ef)(n) = \langle f, \mu_n\rangle\,.$$

One can check, that if f is radial, then

$$\langle f, g\rangle = \langle f, Eg\rangle \tag{4}$$

for every function g on $G_{k,N}$.

By [8, Theorem 2] E maps positive definite functions onto positive definite radial functions.

Let $C^*(G_{k,N})$ denote the full C^*-algebra of the group $G_{k,N}$ (see [4]) and $C^*_\#(G_{k,N})$ be the closure of $\mathfrak{R}$ in $C^*(G_{k,N})$. Then $C^*_\#(G_{k,N})$ is an abelian C^*-algebra.

PROPOSITION 3. – *The Gelfand space of $C^*_\#(G_{k,N})$ coincides with the interval $[-1/(k-1), 1]$ and the Gelfand transform is given by*

$$\hat{f}(t) = \langle f, \varphi_t\rangle\,.$$

PROOF. – Since $\mathfrak{R}$ is dense in $C^*_\#(G_{k,N})$ then L is a multiplicative functional iff L has the form $Lf = \langle f, \varphi_z \rangle$ on $\mathfrak{R}$ for some $z \in \boldsymbol{C}$ and this functional is bounded in $C^*_\#(G_{k,N})$. We shall use the following fact given in [8, Theorem 3]: spherical function φ_z is positive definite on $G_{k,A}$ iff $z \in [-1/(k-1), 1]$.

If $t \in [-1/(k-1), 1]$ then φ_t is positive definite and hence the functional $f \mapsto \langle f, \varphi_t \rangle$ is continuous on $C^*(G_{k,N})$ (see [4, 5]).

Conversely, suppose, that L is continuous on $C^*_\#(G_{k,N})$. Let g be any finitely supported function on $G_{k,N}$. Since $g^* * g$ is positive definite (where $g^*(x) = \overline{g(x^{-1})}$), $E(g^* * g)$ is a radial positive definite function. Therefore we have $E(g^* * g) = h^* * h$ for some $h \in C^*_\#(G_{k,N})$ (see [4]). Hence, by (4)

$$\langle g^* * g, \varphi_t \rangle = \langle E(g^* * g), \varphi_t \rangle = \langle h^* * h, \varphi_t \rangle = |\langle h, \varphi_t \rangle|^2 \geqslant 0 .$$

It means, that φ_t is positive definite and so $t \in [-1/(k-1), 1]$.

Now we are in a position to describe the set of all positive definite radial functions on $G_{k,N}$.

THEOREM 3. – *Let $G = G_1 \circ G_2 \circ \ldots \circ G_N$ be a free product of finite groups such, that* card $(G_i) = k$, $1 < k < \infty$, $1 < N < \infty$. *A radial function φ on G is positive definite if and only if there exists a nonnegative bounded Radon measure μ on $[-1/(k-1), 1]$ such, that*

$$\varphi(n) = \int_{-1/(k-1)}^{1} P_n(t; k, N)\, d\mu(t) .$$

Such a measure is unique.

PROOF. – By Proposition 2 it is sufficient to consider the group $G_{k,N}$. Suppose, that φ is positive definite. Then the functional $f \mapsto \langle f, \varphi \rangle$ can be extended from $\mathfrak{R}$ to a positive functional on $C^*_\#(G_{k,N})$. By Proposition 3, the Gelfand-Naimark theorem and the Riesz theorem there exists a nonnegative bounded Radon measure on $[-1/(k-1), 1]$ such, that for every $f \in \mathfrak{R}$

$$\langle f, \varphi \rangle = \int_{-1/(k-1)}^{1} \hat{f}(t)\, d\mu(t) = \int_{-1/(k-1)}^{1} \langle f, \varphi_t \rangle\, d\mu(t) .$$

Therefore

$$\varphi(n) = \langle \mu_n, \varphi \rangle = \int_{-1/(k-1)}^{1} \langle \mu_n, \varphi_t \rangle\, d\mu(t) = \int_{-1/(k-1)}^{1} \varphi_t(n)\, d\mu(t) = \int_{-1/(k-1)}^{1} P_n(t; k, N)\, d\mu(t) .$$

The part «if» is obvious and uniqueness follows from the fact. that degree of the polynomial $P_n(z; k, N)$ is equal to n.

By the formula (3) it is natural to design $P_0(z; \infty, N) = 1$ and for $n \geqslant 1$ $P_n(z; \infty, N) = z((Nz-1)/(N-1))^{n-1}$. Since degree of $P_n(z; k, N)$ is equal to n and by (3) coefficients of $P_n(z; k, N)$ tend to corresponding coefficients of $P_n(z; \infty, N)$ then $P_n(z; k, N)$ tends to $P_n(z; \infty, N)$ with $k \to \infty$ uniformly on each bounded subset of $\boldsymbol{C}$. Thus we can extend the previous result to the case $k = \infty$.

COROLLARY 2. – *Let $G = G_1 \circ G_2 \circ \ldots \circ G_N$ be a free product of infinite, discrete groups. A radial function φ is positive definite on G iff there exists a nonnegative bounded Radon measure on* $[0, 1]$ *such, that*

$$\varphi(n) = \int_0^1 P_n(t; \infty, N)\, d\mu(t)\,.$$

Such a measure is unique.

PROOF. – One part of the proof follows from Corollary 1. On the other hand suppose, that φ is positive definite on G. By Proposition 2 for each $k \in \boldsymbol{N}$ the function $u \mapsto \varphi(\|u\|)$ is positive definite on the group $G_{k,N}$. Thus there exists measure $\tilde{\mu}_k$ on $[-1/(k-1), 1]$ such, that

$$\varphi(n) = \int_{-1/(k-1)}^1 P_n(t; k, N)\, d\tilde{\mu}_k(t)\,.$$

Since $\varphi(0) = \tilde{\mu}_k([-1/(k-1), 1])$ $\{\mu_k\}$ has a weak-convergent subsequence. Let μ be its limit. Then μ is supported on $[0, 1]$ and we have

$$\left|\varphi(n) - \int_0^1 P_n(t; \infty, N)\, d\mu(t)\right| =$$

$$= \left| \int_{-1/(k-1)} P_n(t; k, N)\, d\tilde{\mu}_k(t) - \int_0^1 P_n(t; \infty, N)\, d\mu(t)\right| \leqslant$$

$$\leqslant \int_{-1/(k-1)}^1 |P_n(t; k, N) - P_n(t; \infty, N)|\, d\tilde{\mu}_k(t) +$$

$$+ \left| \int_{-1/(k-1)} P_n(t; \infty, N)\, d\tilde{\mu}_k(t) - \int_0^1 P_n(t; \infty, N)\, d\mu(t)\right|.$$

Since $P_n(t; k, N)$ tends to $P_n(t; \infty, N)$ uniformly on $[-1, 1]$ then the last sum can be arbitrary small. The uniqueness of μ is obvious.

Now we can take $N = \infty$:

COROLLARY 3. – *Let $\{G_i\}_{i\in I}$ be an infinite family of groups such, that for each $i \in I$* card $(G_i) = k$, $1 < k < \infty$. *The radial function φ on $G = \bigcirc_{i\in I} G_i$ is positive definite iff there exists a nonnegative bounded Radon measure μ on $[-1/(k-1), 1]$ such, that*

$$\varphi(n) = \int_{-1/(k-1)}^{1} t^n \, d\mu(t) \, .$$

Such a measure is unique.

PROOF. – We can again suppose, that $G_i = \boldsymbol{Z}_k$. One can check, that for $t \in [-1/(k-1), 1]$ the function

$$w_t(t) = \begin{cases} 1 & \text{if } x = e \, , \\ t & \text{if } x \neq e \, , \end{cases}$$

is positive definite on $\boldsymbol{Z}_k$. Bożejko's theorem states, that a free product of positive definite functions is again positive definite (see [2]). Therefore the function $x \mapsto t^{\|x\|}$ is positive definite on $G = \bigcirc_{i\in I} G_i$ for $t \in [-1/(k-1), 1]$. Thus we obtain one side of the theorem. On the other hand if φ is positive definite on G then φ is positive definite on the group $\bigcirc_{i\in K} G_i$ for each finite set $K \subset I$ and the rest of the proof is the same as before.

Finally, we can look at the case $N = \infty$, $k = \infty$.

COROLLARY 4. – *Let $\{G_i\}_{i\in I}$ be an infinite family of infinite discrete groups. A radial function φ is positive definite on $G = \bigcirc_{i\in I} G_i$ iff there exists a nonnegative bounded Radon measure on* $[0, 1]$ *such, that*

$$\varphi(n) = \int_0^1 t^n \, d\mu(t) \, .$$

PROOF. – The case « if » follows from Theorem 1 or from Bożejko's theorem [2] and the rest of the proof we can obtain from Corollary 3 in similar way as the proof of Corollary 2.

REMARK. – Let us consider the group $G_{2,N} = \boldsymbol{Z}_2 \circ \boldsymbol{Z}_2 \circ \ldots$ (N times, $N = \infty, 1, 2, \ldots$). The associated Cayley graph is the homogeneous tree of degree N and for any $x, y \in G_{2,N}$ we have $\|y^{-1}x\| = d(x, y)$ where d denotes the usual geodesic distance in tree. Hence, if we put $k = 2$, Theorem 3 and Corollary 3 describe such functions $\varphi: \{0, 1, 2, ...\} \to \boldsymbol{C}$, that the kernel $(x, y) \to \varphi(d(x, y))$ is positive definite on the homogeneous tree of degree N. Take the free group $\boldsymbol{F}_M = \boldsymbol{Z} \circ \boldsymbol{Z} \circ ...$ (M times, $M = \infty, 1, 2, ...$) endowed with the usual length i.e. if $x = g_1 g_2 ... g_n$ is the reduced word in $\boldsymbol{F}_M$ we put $|x| = |g_1| + |g_2| + ... + |g_n|$. Since the associated Cayley graph is the homogeneous tree of degree $2M$ and for any $x, y \in \boldsymbol{F}_M$ $|y^{-1}x| = d(x, y)$ we obtain a characterization of positive definite radial (in sense of $|\cdot|$) functions on $\boldsymbol{F}_M$ putting in Theorem 3 and Corollary 3 $k = 2$ and $N = 2M$.

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REFERENCES

[1] J. P. Arnaud, *Fonctions spheriques et fonctions defines positives sur l'arbre homogène*, C. R. Acad. Sci. Paris I Math., **290**, 2 (1980), A99-A101.

[2] M. Bożejko, *Positive definite functions on the free group and the non-commutative Riesz product*, Boll. Un. Mat. Ital., (6) **5**-A (1986), 13-21.

[3] J. M. Cohen - L. de Michele, *Radial Fourier-Stieltjes algebra on free* groups, Contemp. Math., **40** (Operator algebras and K-theory), (1982).

[4] J. Dixmier, *Les C*-algèbres et leurs represéntations*, Gauthier-Villars, Paris, 1969.

[5] P. Eymard, *L'algebre de Fourier d'un groupe localement compact*, Bull. Soc. Mat. France, **92** (1964), 181-236.

[6] A. Figà-Talamanca - M. A. Picardello, *Harmonic analysis on free groups*, Marcel Dekker, New York-Basel, 1983.

[7] A. Iozzi - M. A. Picardello, *Spherical functions on symmetric graphs*, in *Harmonic Analysis, Proceedings, Cortona, Italy*, 1982, Lectures Notes in Math., **992**.

[8] W. Młotkowski, *Positive definite radial functions on free product of groups*, Boll. Un. Mat. Ital., (7) **2**-B (1988), 53-66.

[9] J. P. Serre, *Arbres, amalgames, SL_2*, Astérisque, **46** (1977).

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