

On freely quasi-infinitely divisible distributions

Ikkei Hotta, Wojciech Młotkowski, Noriyoshi Sakuma and Yuki Ueda

Department of Applied Science, Yamaguchi University 2-16-1 Tokiwadai, Ube 755-8611, Japan.
E-mail address: ihotta@yamaguchi-u.ac.jp

Instytut Matematyczny, Uniwersytet Wrocławski, Plac Grunwaldzki 2, 50-384 Wrocław, Poland.
E-mail address: mlotkow@math.uni.wroc.pl

Graduate School of Sciences, Nagoya City University, 1 Yamanohata, Mizuho-cho, Mizuho-ku, Nagoya, Aichi 467-8501, Japan.
E-mail address: sakuma@nsc.nagoya-cu.ac.jp

Department of Mathematics, Hokkaido University of Education, 9 Hokumon-cho, Asahikawa 070-8621, Japan.
E-mail address: ueda.yuki@a.hokkyodai.ac.jp

Abstract. Inspired by the notion of quasi-infinite divisibility (QID), we introduce and study the class of freely quasi-infinitely divisible (FQID) distributions on \mathbb{R} , i.e. distributions which admit the free Lévy-Khintchine-type representation with signed Lévy measure. We prove several properties of the FQID class, some of them in contrast to those of the QID class. For example, a FQID distribution may have negative Gaussian component, and the total mass of its signed Lévy measure may be negative. Finally, we extend the Bercovici-Pata bijection, providing a characteristic triplet, with the Lévy measure having nonzero negative part, which is at the same time classical and free characteristic triplet.

1. Introduction

In classical and free probability theories, infinitely divisible distributions are defined by the classical convolution operation and connected with Lévy processes, which are stochastic processes with independent increments and time-homogenous stationary distributions. More precisely, if μ is an infinitely divisible distribution on \mathbb{R} , then one can construct a Lévy process $\{X_t\}_{t \geq 0}$ such that a marginal law of X_1 coincides with μ . Conversely, if a stochastic process $\{X_t\}_{t \geq 0}$ is a Lévy process, then its marginal distribution is infinitely divisible. Because of this relation, infinite divisibility plays a crucial role in research on the marginal laws of Lévy processes. Readers may consult [Sato \(1999\)](#) for the classical case, and [Barndorff-Nielsen and Thorbjørnsen \(2002, 2006\)](#); [Biane \(1998\)](#) for the free case.

Received by the editors March 31st, 2022; accepted February 24th, 2023.

2010 Mathematics Subject Classification. 46L54, 60E07.

Key words and phrases. Freely quasi-infinitely divisible distributions, quasi-Lévy measure, Voiculescu transform, Bercovici-Pata bijection.

Infinitely divisible distributions are characterized in terms of analytic tools: the characteristic function in the classical case Sato (1999) and the R -transform or the Voiculescu transform in the free case Barndorff-Nielsen and Thorbjørnsen (2006); Bercovici and Voiculescu (1993). These representations are called Lévy-Khintchine representations. One of the most beautiful discoveries in this area is that the Lévy-Khintchine representation in both classical and free probability can be parametrized by $a \geq 0$, $\gamma \in \mathbb{R}$, and a Lévy measure ν . The triplet (a, ν, γ) is called characteristic triplet and free characteristic triplets, respectively. They give a bijection between the classes of classical and free infinitely divisible distributions (see Bercovici and Pata (1999, Theorem 1.2)). It is called the Bercovici-Pata bijection now. The importance of this bijection comes from the study of the limit theorem in free probability. Indeed, research on free infinitely divisible distributions has made rapid progress in the last two decades that the rich structure of the class of free infinitely divisible distributions has been revealed.

Going back to the classical probability case, many researchers have faced similar interesting examples in which the distributions are not infinitely divisible, but have a Lévy-Khintchine-type representation. Namely, in the Lévy-Khintchine representation, the Lévy measure is a signed measure. A precise definition is given in Section 2. Such a distribution is called a quasi-infinitely divisible distribution. Although the class of quasi-infinitely divisible distributions has been considered for a long time since being investigated in Linnik and Ostrovskiĭ (1977), few systematic approaches for this class have been reported. However, there has been some progress in the past decade. In particular, Lindner, Pan, and Sato in Lindner et al. (2018) gathered many examples and studied the distributional properties of quasi-infinitely divisible distributions using the Fourier transform. They investigate quasi-infinitely divisible distributions from the Lévy-Khintchine representation and the characteristic triplet. They also proved in particular, that if (a, ν, γ) is the characteristic triplet of a quasi-infinitely divisible distribution, then the Gaussian component a must be nonnegative.

The purpose of this paper is to investigate the freely quasi-infinitely divisible (FQID) distributions. The class of FQID distributions is a natural extension of the class of freely infinitely divisible distributions to explore correspondence between free and classical probability. Section 2 presents some preliminary results that are used in the remainder of the paper. In Section 3, we define the FQID distributions and identify several properties (convolution, atoms, convergence, and support) of FQID distributions. In Section 4, we present some examples of FQID distributions that are not freely infinitely divisible. Moreover, we discover different points about classical and free quasi-infinitely divisible distributions through several examples. In Section 5, we focus on a question raised by Bożejko of whether the Bercovici-Pata bijection can be extended to a larger class. Finally, we succeeded in making it from a subclass of QID, which includes ID, to a subclass of FQID, which includes FID. In order to answer his question, we find examples that come from the Lévy measure with the Cauchy distribution and symmetric distribution.

2. Preliminaries

2.1. *Infinitely divisible distributions.* Let \mathcal{B} be the set of all Borel sets in \mathbb{R} and $\mathcal{P}(\mathbb{R})$ the set of all (Borel) probability measures on \mathbb{R} . For $\mu \in \mathcal{P}(\mathbb{R})$, its characteristic function is defined by

$$\widehat{\mu}(t) := \int_{\mathbb{R}} e^{itx} \mu(dx), \quad t \in \mathbb{R}.$$

A probability measure $\mu \in \mathcal{P}(\mathbb{R})$ is said to be *infinitely divisible* (for short, ID) if, for any $n \in \mathbb{N}$, there exists some $\mu_n \in \mathcal{P}(\mathbb{R})$ such that $\mu = \underbrace{\mu_n * \cdots * \mu_n}_{n \text{ times}} = \mu_n^{*n}$, where $*$ means the (classical)

convolution, that is, for $\mu, \nu \in \mathcal{P}(\mathbb{R})$, we define $\mu * \nu$ as the probability measure on \mathbb{R} , characterized by $\widehat{\mu * \nu}(t) = \widehat{\mu}(t)\widehat{\nu}(t)$ for all $t \in \mathbb{R}$. The class of ID distributions, denoted by $\text{ID}(*)$, comes from

limit theorems and plays a crucial role in constructing stochastic processes in connection to the study of Lévy processes and its applications in mathematical statistics, finance, and physics.

If μ is ID, then its characteristic function $\widehat{\mu}(t)$ admits the following representation (namely, the *Lévy-Khintchine representation*):

$$\widehat{\mu}(t) = \exp\left(i\gamma t - \frac{a}{2}t^2 + \int_{\mathbb{R}}(e^{itx} - 1 - itx\mathbf{1}_{[-1,1]}(x))\nu(dx)\right), \quad t \in \mathbb{R}, \tag{2.1}$$

where $\gamma \in \mathbb{R}$, $a \geq 0$, and ν is a Lévy measure¹ on \mathbb{R} . Conversely, if a characteristic function of $\mu \in \mathcal{P}(\mathbb{R})$ is written by the Lévy-Khintchine representation, then μ is ID. Each $\mu \in \text{ID}(\ast)$ determines a unique triplet (a, ν, γ) , called a *characteristic triplet for μ* (see Sato (1999) for further information).

The characteristic function of an ID distribution has another representation. Let μ be an ID distribution with a characteristic triplet (a, ν, γ) . One can see that, if we define

$$b = \gamma, \tag{2.2}$$

$$\zeta(B) = a\delta_0(B) + \int_B(1 \wedge x^2)\nu(dx), \quad B \in \mathcal{B},$$

then ζ is a finite measure on \mathbb{R} and

$$\widehat{\mu}(t) = \exp\left(ibt + \int_{\mathbb{R}}g_c(x, t)\zeta(dx)\right), \quad t \in \mathbb{R}, \tag{2.3}$$

where the function g_c is defined by

$$g_c(x, t) := \begin{cases} (e^{itx} - 1 - itx\mathbf{1}_{[-1,1]}(x))/(1 \wedge x^2), & x \neq 0, \\ -t^2/2, & x = 0, \end{cases} \quad t \in \mathbb{R}.$$

Observe that the kernel function $\mathbb{R} \rightarrow \mathbb{R} : x \mapsto g_c(x, t)$ is bounded for each $t \in \mathbb{R}$ and continuous at 0. It is easy to verify that $\mu \in \text{ID}(\ast)$ determines a unique pair (b, ζ) , called the *characteristic pair for μ* . Conversely, if $\mu \in \text{ID}(\ast)$ has a characteristic pair (b, ζ) , then $\widehat{\mu}(t)$ is written by the Lévy-Khintchine representation with a characteristic triplet (a, ν, γ) fulfilled the relation (2.2).

2.2. *Quasi-infinitely divisible distributions.* In this section, we summarize the work of Lindner, Pan, and Sato (Lindner et al., 2018) for quasi-infinitely divisible distributions.

A function $\rho : \mathcal{B} \rightarrow [-\infty, \infty]$ is called a *signed measure* on \mathbb{R} if $\rho(\emptyset) = 0$ and $\rho(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \rho(A_n)$ for any pairwise disjoint sets $\{A_n\}_n$ in \mathcal{B} . We define the *total variation* of a signed measure ρ by

$$|\rho|(A) := \sup_{\{A_j\}:\text{partition of } A} \sum_j |\rho(A_j)|. \tag{2.4}$$

It is known that $|\rho|$ is a measure on \mathbb{R} . A signed measure ρ is said to be *finite* if $|\rho|$ is a finite measure. By the Hahn-Jordan decomposition theorem (see e.g., Rudin (1987, p.127)), for a finite signed measure ρ , there exist disjoint Borel sets C^+, C^- in \mathbb{R} and finite measures ρ^+, ρ^- on \mathcal{B} satisfying $\rho = \rho^+ - \rho^-$ so that $\rho^+(\mathbb{R} \setminus C^+) = \rho^-(\mathbb{R} \setminus C^-) = 0$. We call ρ^+ and ρ^- the *positive part* and the *negative part* of the signed measure ρ , respectively. The measures ρ^+ and ρ^- are uniquely determined by ρ . It holds that

$$\rho^+ = \frac{1}{2}(|\rho| + \rho), \quad \rho^- = \frac{1}{2}(|\rho| - \rho) \quad \text{and} \quad |\rho| = \rho^+ + \rho^-. \tag{2.5}$$

We now define quasi-infinitely divisible distributions on \mathbb{R} .

Definition 2.1. A probability measure $\mu \in \mathcal{P}(\mathbb{R})$ is said to be *quasi-infinitely divisible* (for short, QID) if its characteristic function $\widehat{\mu}(t)$ admits the representation (2.3) for some $b \in \mathbb{R}$ and some finite signed measure ζ on \mathbb{R} .

¹A positive Borel measure ν on \mathbb{R} is called a Lévy measure if $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}}(1 \wedge x^2)\nu(dx) < \infty$.

The pair (b, ζ) is uniquely determined by a QID distribution μ and is called a *characteristic pair* for a QID distribution μ . The measure μ is QID if and only if there exist $\mu_1, \mu_2 \in \text{ID}(\ast)$ such that $\mu \ast \mu_1 = \mu_2$ by applying the Hahn-Jordan decomposition to a finite signed measure ζ appeared from (2.3).

We can rephrase this in terms of the characteristic triplets. We seek a characteristic triplet for the QID distribution μ in terms of the Lévy-Khintchine representation in the sense of (2.1). We need to use a little caution here. If there exist $\mu_1, \mu_2 \in \text{ID}(\ast)$ such that $\mu_1 \ast \mu = \mu_2$ and that ν_1 and ν_2 are Lévy measures of μ_1 and μ_2 satisfying $\nu_1(\mathbb{R}) = \nu_2(\mathbb{R}) = \infty$, respectively, then the difference $\nu_1 - \nu_2$ may not be a signed measure.

To justify the meaning of difference between two Lévy measures, we introduce two subclasses of \mathcal{B} and the class of specific functions as follows.

Definition 2.2 (see Lindner et al. (2018) for details).

- (1) $\mathcal{B}_r := \{B \in \mathcal{B}; B \cap (-r, r) = \emptyset\}$ for $r > 0$ and $\mathcal{B}_0 := \bigcup_{r>0} \mathcal{B}_r$.
- (2) Denote by Π the set of functions $\nu : \mathcal{B}_0 \rightarrow [-\infty, \infty]$ such that the following condition holds:
 - (C) $\nu|_{\mathcal{B}_r}$ is a finite signed measure on a measurable space $(\mathbb{R} \setminus (-r, r), \mathcal{B}_r)$ for each $r > 0$.
 The condition (C) ensures that the total variation $|\nu|$, the positive part ν^+ , and the negative part ν^- are unique and well-defined measures on $(\mathbb{R}, \mathcal{B})$ satisfying

$$|\nu|(\{0\}) = \nu^+(\{0\}) = \nu^-(\{0\}) = 0,$$

and

$$|\nu|(A) = \left| \nu|_{\mathcal{B}_r} \right|(A), \quad \nu^+(A) = \nu^+|_{\mathcal{B}_r}(A), \quad \nu^-(A) = \nu^-|_{\mathcal{B}_r}(A) \quad \text{for } A \in \mathcal{B}_r.$$

Note that \mathcal{B}_0 is not a σ -algebra, and so $\nu \in \Pi$ is not a signed measure. Whenever we can extend the definition of ν to \mathcal{B} such that ν will be a signed measure, we identify the symbol ν with its extension to \mathcal{B} and speak of ν as a signed measure. In this case, we can verify that $\nu(\{0\}) = 0$ and its total variation $|\nu|$, positive part ν^+ and negative part ν^- as defined in Definition 2.2 coincide with the corresponding notions from (2.4) and (2.5) for the signed measure ν .

Definition 2.3. A function $\nu : \mathcal{B}_0 \rightarrow [-\infty, \infty]$ is called a *quasi-Lévy-type measure* if $\nu \in \Pi$ and it satisfies that

$$\int_{\mathbb{R}} (1 \wedge x^2) |\nu|(dx) < \infty.$$

For a QID distribution μ with a characteristic pair (b, ζ) , the characteristic function of the QID distribution μ admits the following representation:

$$\hat{\mu}(t) = \exp \left(-\frac{a}{2}t^2 + i\gamma t + \int_{\mathbb{R}} (e^{itx} - 1 - itx\mathbf{1}_{[-1,1]}(x))\nu(dx) \right), \quad t \in \mathbb{R},$$

for some $a, \gamma \in \mathbb{R}$ and some function $\nu : \mathcal{B}_0 \rightarrow [-\infty, \infty]$ defined by the relation (2.2). Note that, the function ν is a quasi-Lévy-type measure. Conversely, if the characteristic function of $\mu \in \mathcal{P}(\mathbb{R})$ admits the above representation for some $a, \gamma \in \mathbb{R}$ and a quasi-Lévy-type measure ν on \mathbb{R} , then μ is QID with the characteristic pair (b, τ) fulfilled the relation (2.2). Note that, $b \in \mathbb{R}$ and τ is a finite signed measure on \mathbb{R} .

The triplet (a, ν, γ) is uniquely determined by the QID distribution μ and is called a *characteristic triplet* of μ .

Definition 2.4. A quasi-Lévy-type measure ν is called a *quasi-Lévy measure* if there exist a QID distribution μ and $a, \gamma \in \mathbb{R}$ such that (a, ν, γ) is a characteristic triplet of μ .

It is known that $a \geq 0$ for any QID distribution with a characteristic triplet (a, ν, γ) , see Lindner et al. (2018, Lemma 2.7). Moreover, not every quasi-Lévy-type measure is a quasi-Lévy measure, see Lindner et al. (2018, Example 2.9).

We say that a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is *integrable with respect to a quasi-Lévy-type measure* ν if it is integrable with respect to $|\nu|$ (and therefore with respect to ν^+ and ν^- as well). We then define

$$\int_B f(x)\nu(dx) := \int_B f(x)\nu^+(dx) - \int_B f(x)\nu^-(dx), \quad B \in \mathcal{B}.$$

Note that, for each $t \in \mathbb{R}$, the function $x \mapsto e^{itx} - 1 - it\mathbf{1}_{[-1,1]}(x)$ is integrable with respect to ν .

2.3. Free probability theory. In non-commutative probability theory, self-adjoint operators are interpreted as (real-valued) random variables. A remarkable feature is that various notions of independence exist for those random variables. In particular, free independence has found applications in operator algebras and random matrices and has been intensively studied, see [Nica and Speicher \(2006\)](#); [Mingo and Speicher \(2017\)](#) and references therein.

2.3.1. Free additive convolution. Let X and Y be freely independent random variables affiliated with a von Neumann algebra \mathcal{M} equipped with a normal faithful tracial state τ . Note that, by the definition of the affiliated operator, $E_X(B)$ and $E_Y(B)$ belong to \mathcal{M} for any Borel set B in \mathbb{R} , where E_X is the spectral projection of X . Denote a probability distribution of X with respect to τ as μ_X , i.e., $\mu_X((-\infty, \cdot]) = \tau(E_X((-\infty, \cdot]))$. A probability distribution of $X + Y$ with respect to τ is denoted by $\mu_X \boxplus \mu_Y$ with the operation \boxplus , called *free additive convolution*. This was first introduced in [Voiculescu \(1986\)](#), where μ and ν have compact supports. The concept was later extended to probability measures with finite moments; see [Maassen \(1992\)](#). Finally, it was generalized by [Bercovici and Voiculescu \(1993\)](#) to the case of probability measures μ and ν on \mathbb{R} . Denote the n -fold free additive convolution

$$\underbrace{\mu \boxplus \cdots \boxplus \mu}_{n \text{ times}}$$

of $\mu \in \mathcal{P}(\mathbb{R})$ as $\mu^{\boxplus n}$.

Free additive convolution is characterized by the Voiculescu transform (see [Bercovici and Voiculescu \(1993\)](#)). A *Cauchy-Stieltjes transform* G_μ of $\mu \in \mathcal{P}(\mathbb{R})$ is defined by

$$G_\mu(z) = \int_{\mathbb{R}} \frac{1}{z - x} \mu(dx), \quad z \in \mathbb{C}^+,$$

where \mathbb{C}^+ (resp. \mathbb{C}^-) denotes the set of complex numbers with strictly positive (resp. strictly negative) imaginary parts. Note that $\text{Im}(G_\mu(z)) < 0$ for any $z \in \mathbb{C}^+$. A reciprocal Cauchy transform $F_\mu := 1/G_\mu$ is called an *F-transform*. It is easy to see that F_μ is a Pick function, namely, an analytic function from \mathbb{C}^+ to \mathbb{C}^+ , and $F_\mu(iy)/iy \rightarrow 1$ as $y \rightarrow \infty$. Conversely, a Pick function with a certain asymptotic behavior at ∞ can be expressed by an *F-transform* of some probability measure on \mathbb{R} .

Proposition 2.5. (see [Bercovici and Voiculescu \(1993, Proposition 5.2\)](#)) *If an analytic function $F : \mathbb{C}^+ \rightarrow \mathbb{C}^+$ satisfies*

$$\lim_{y \rightarrow \infty} \frac{F(iy)}{iy} = 1,$$

then there exists some $\mu \in \mathcal{P}(\mathbb{R})$ such that $F = F_\mu$.

In classical probability theory, additive convolution $*$ is characterized by the cumulant transform, that is, the logarithm of the characteristic function: for $\mu, \nu \in \mathcal{P}(\mathbb{R})$ we have that $\log \widehat{\mu * \nu}(t) = \log \widehat{\mu}(t) + \log \widehat{\nu}(t)$ for all $t \in \mathbb{R}$ (see e.g., [Sato \(1999\)](#)). In free probability theory, there is a similar characterization of free additive convolution using the free analog of the cumulant transform. For any $\mu \in \mathcal{P}(\mathbb{R})$ and $\lambda > 0$, there exist positive numbers α, β , and M such that F_μ is univalent on the

set $\Gamma_{\alpha,\beta} := \{z \in \mathbb{C}^+ \mid \text{Im}(z) > \beta, |\text{Re}(z)| < \alpha \text{Im}(z)\}$ and $F_\mu(\Gamma_{\alpha,\beta}) \supset \Gamma_{\lambda,M}$. Thus, the compositional right inverse F_μ^{-1} is defined on $\Gamma_{\lambda,M}$. The Voiculescu transform φ_μ is defined by

$$\varphi_\mu(z) := F_\mu^{-1}(z) - z, \quad z \in \Gamma_{\lambda,M}.$$

It has been shown [Bercovici and Voiculescu \(1993\)](#) that $\varphi_{\mu \boxplus \nu}(z) = \varphi_\mu(z) + \varphi_\nu(z)$ on the intersection of the domains where these three Voiculescu transforms $\varphi_{\mu \boxplus \nu}, \varphi_\mu$ and φ_ν are defined.

2.3.2. Freely infinitely divisible distributions. Since the work of [Bercovici and Pata \(1999\)](#), who found a bijection between the classical and freely infinitely divisible distributions, the class of infinitely divisible distributions in free probability has been intensively studied. A probability measure $\mu \in \mathcal{P}(\mathbb{R})$ is said to be *freely infinitely divisible* (for short, FID) if, for any $n \in \mathbb{N}$, there exists some $\mu_n \in \mathcal{P}(\mathbb{R})$ such that $\mu = \mu_n^{\boxplus n}$. The class of FID distributions is written as $\text{ID}(\boxplus)$, introduced in [Bercovici and Voiculescu \(1993\)](#). Recently, several FID distributions have been discovered: the normal distributions [Belinschi et al. \(2011\)](#), some of the classical stable laws [Hasebe et al. \(2020\)](#), some of the gamma distributions, including the chi-square distributions [Hasebe \(2014\)](#), some of the classical Meixner distributions [Bożejko and Hasebe \(2013\)](#), and of the Fuss-Catalan distributions [Młotkowski et al. \(2020\)](#); see e.g. [Arizmendi and Hasebe \(2016\)](#); [Hasebe \(2016\)](#); [Morishita and Ueda \(2020\)](#) for further examples.

FID distributions can be characterized by those Voiculescu transforms via complex analysis. [Bercovici and Voiculescu \(1993\)](#) gave the following important result on FID distributions:

Proposition 2.6. *For $\mu \in \mathcal{P}(\mathbb{R})$, the following conditions are equivalent.*

- (i) μ is in $\text{ID}(\boxplus)$.
- (ii) The Voiculescu transform φ_μ has an analytic extension (denoted by the same symbol φ_μ) defined on \mathbb{C}^+ with values in $\mathbb{C}^- \cup \mathbb{R}$.
- (iii) There exist $b \in \mathbb{R}$ and a finite positive measure τ on \mathbb{R} such that

$$\varphi_\mu(z) = b + \int_{\mathbb{R}} \frac{1+xz}{z-x} \tau(dx), \quad z \in \mathbb{C}^+. \tag{2.6}$$

The pair (b, τ) is uniquely determined by μ . Conversely, given $b \in \mathbb{R}$ and a finite positive measure τ on \mathbb{R} , there exists $\mu \in \text{ID}(\boxplus)$ whose Voiculescu transform admits the form (2.6).

The pair (b, τ) is called the *free characteristic pair* (or *free generating pair*, see [Barndorff-Nielsen and Thorbjørnsen \(2002, Definition 2.9\)](#)) for the FID distribution μ .

The FID distribution is also characterized by admitting the Lévy-Khintchine representation in terms of the R -transform. The R -transform (or *free cumulant transform*) R_μ of $\mu \in \mathcal{P}(\mathbb{R})$ is defined by

$$R_\mu(z) = z\varphi_\mu\left(\frac{1}{z}\right), \quad \text{for all } z \in \mathbb{C}^- \text{ such that } 1/z \in \Gamma_{\lambda,M}.$$

Note that $\omega \mapsto 1/\omega$ maps $\Gamma_{\lambda,M}$ to $\Delta_{\lambda,1/M}$, where

$$\Delta_{\alpha,\beta} := \{z \in \mathbb{C}^- : |\text{Re}(z)| < -\alpha \text{Im}(z), |z| < \beta\}, \quad \alpha, \beta > 0. \tag{2.7}$$

Thus, the R -transform R_μ is defined on some subset in $\Delta_{\lambda,1/M}$.

Remark 2.7. This $R_\mu(z)$ is what [Nica and Speicher \(2006\)](#) refer to as the (combinatorial) R -transform. It is identical to the free cumulant transform $\mathcal{C}_\mu(z)$ in [Barndorff-Nielsen and Thorbjørnsen \(2002, 2006\)](#). The formula for the original Voiculescu's R -transform \mathcal{R}_μ is $\mathcal{R}_\mu(z) = \varphi_\mu\left(\frac{1}{z}\right)$. We will employ this combinatorial R -transform to think about Lévy-Khintchine type representation for both classical and free QID.

The free version of the Lévy-Khintchine representation amounts to the statement that $\mu \in \text{ID}(\boxplus)$ if and only if there exist $a \geq 0$, $\gamma \in \mathbb{R}$ and a Lévy measure ν such that

$$R_\mu(z) = az^2 + \gamma z + \int_{\mathbb{R}} \left(\frac{1}{1-zx} - 1 - zx\mathbf{1}_{[-1,1]}(x) \right) \nu(dx), \quad z \in \mathbb{C}^-. \tag{2.8}$$

The triplet (a, ν, γ) , referred to as the *free characteristic triplet* for μ , is uniquely determined by $\mu \in \text{ID}(\boxplus)$. The positive number a is called the *semicircular component* of μ and the measure ν is called the *free Lévy measure* for μ . Furthermore, the representation in (2.8) is called the *free Lévy-Khintchine representation* of an FID distribution μ . The relation between the free characteristic triplet (a, ν, γ) and free characteristic pair (b, τ) for the FID distribution μ is as follows (see [Barndorff-Nielsen and Thorbjørnsen \(2006\)](#)):

$$\begin{aligned} a &= \tau(\{0\}), \\ \nu(dx) &= \frac{1+x^2}{x^2} \cdot \mathbf{1}_{\mathbb{R} \setminus \{0\}}(x) \tau(dx), \\ \gamma &= b + \int_{\mathbb{R}} x \left(\mathbf{1}_{[-1,1]}(x) - \frac{1}{1+x^2} \right) \nu(dx). \end{aligned} \tag{2.9}$$

Let us consider $\mu \in \mathcal{P}(\mathbb{R})$. Then, a discrete semigroup $\{\mu^{\boxplus n}\}_{n \in \mathbb{N}}$ can be embedded in a continuous family $\{\mu_t\}_{t \geq 1}$ of probability measures on \mathbb{R} , so that $\mu_1 = \mu$ and $\mu_t \boxplus \mu_s = \mu_{t+s}$ for $t, s \geq 1$. Initially, [Bercovici and Voiculescu \(1995\)](#) showed the existence of $\{\mu_t\}$ for sufficiently large t and for $\mu \in \mathcal{P}(\mathbb{R})$ having a compact support. Later, [Nica and Speicher \(1996\)](#) improved their result to $t \geq 1$, but μ still required a compact support. Finally, [Belinschi and Bercovici \(2004\)](#) extended this further to a general probability measure.

This property can be expressed in the following way: for any $\mu \in \mathcal{P}(\mathbb{R})$ and $t \geq 1$, there exists $\mu_t \in \mathcal{P}(\mathbb{R})$ such that $R_{\mu_t}(z) = tR_\mu(z)$ on the intersection of the domains of R_{μ_t} and R_μ . We denote the above probability measure μ_t as $\mu^{\boxplus t}$. Furthermore, [Bercovici and Voiculescu \(1993\)](#) showed that a partial semigroup $\{\mu^{\boxplus t}\}_{t \geq 1}$ can be embedded in the continuous semigroup $\{\mu^{\boxplus t}\}_{t \geq 0}$, so that $\mu^{\boxplus 0} = \delta_0$, $\mu^{\boxplus 1} = \mu$, and $\mu^{\boxplus t} \boxplus \mu^{\boxplus s} = \mu^{\boxplus t+s}$ for $t, s \geq 0$ when $\mu \in \text{ID}(\boxplus)$. This satisfies $R_{\mu^{\boxplus t}}(z) = tR_\mu(z)$ for all $z \in \mathbb{C}^-$.

2.3.3. The Bercovici-Pata bijection. Let $\Lambda : \text{ID}(\ast) \rightarrow \text{ID}(\boxplus)$ be the Bercovici-Pata bijection (see [Bercovici and Pata \(1999, Theorem 1.2\)](#) and [Barndorff-Nielsen and Thorbjørnsen \(2006\)](#) for details). This bijection satisfies

$$\Lambda(\mu \ast \nu) = \Lambda(\mu) \boxplus \Lambda(\nu)$$

for all $\mu, \nu \in \text{ID}(\ast)$, and is continuous with respect to the weak convergence. Moreover, if $\mu \in \text{ID}(\ast)$ has a characteristic triplet (a, ν, γ) , then $\Lambda(\mu)$ coincides with the FID distribution with a free characteristic triplet (a, ν, γ) . For example, we obtain

- (1) $\Lambda(\mathbf{N}(m, \sigma^2)) = \mathbf{S}(m, \sigma^2)$ for $m \in \mathbb{R}$ and $\sigma > 0$;
- (2) $\Lambda(\mathbf{Po}(\lambda)) = \mathbf{MP}(\lambda)$ for $\lambda > 0$;
- (3) $\Lambda(\mathbf{C}_a) = \mathbf{C}_a$ for $a > 0$,

where $\mathbf{N}(m, \sigma^2)$ is the normal distribution with mean $m \in \mathbb{R}$ and variance σ^2 , $\mathbf{S}(m, \sigma^2)$ is the semicircle law with mean $m \in \mathbb{R}$ and variance σ^2 , $\mathbf{Po}(\lambda)$ is the Poisson distribution with a parameter $\lambda > 0$, $\mathbf{MP}(\lambda)$ is the Marchenko-Pastur law with a parameter $\lambda > 0$ and \mathbf{C}_a is the symmetric Cauchy distribution with parameter $a > 0$.

3. Freely quasi-infinitely divisible distributions

3.1. Definition and convolution properties. In this section, we introduce the class of freely quasi-infinitely divisible distributions. Recall that, a measure $\mu \in \mathcal{P}(\mathbb{R})$ is FID if and only if its Voiculescu

transform φ_μ has an analytic extension to \mathbb{C}^+ with values in $\mathbb{C}^- \cup \mathbb{R}$ (see Proposition 2.6). Next, we will define an extended concept of class $\text{ID}(\boxplus)$.

Definition 3.1. A probability measure $\mu \in \mathcal{P}(\mathbb{R})$ is said to be *freely quasi-infinitely divisible* (for short, FQID) if φ_μ extends as an analytic function on \mathbb{C}^+ , and can be expressed as

$$\varphi_\mu(z) = b + \int_{\mathbb{R}} \frac{1+xz}{z-x} \tau(dx), \quad z \in \mathbb{C}^+,$$

for some $b \in \mathbb{R}$ and a finite signed measure τ on \mathbb{R} . We call (b, τ) a *free characteristic pair* of the FQID distribution μ .

One immediate consequence of the definition is the following property.

Proposition 3.2. *If μ is FQID, then the Voiculescu transform φ_μ has an analytic continuation to \mathbb{C}^+ .*

Example 3.3. Consider $\mu = \frac{1}{2}(\delta_{-1} + \delta_1)$. A simple computation gives

$$F_\mu(z) = \frac{z^2 - 1}{z} \quad \text{and} \quad \varphi_\mu(z) = \frac{-z + \sqrt{z^2 + 4}}{2},$$

where $\sqrt{\omega} := |\omega|^{\frac{1}{2}} e^{\frac{i \arg(\omega)}{2}}$ for $0 \leq \arg(\omega) < 2\pi$. The function $z \mapsto \sqrt{z^2 + 4}$ has a zero at $z = 2i \in \mathbb{C}^+$, and therefore φ_μ does not have an analytic continuation to \mathbb{C}^+ . Hence μ is not FQID by Proposition 3.2. In Remark 3.9, we will give a general result on atoms of FQID distributions.

Below, we state and prove some basic properties of FQID distributions.

Proposition 3.4. *For an FQID distribution, the free characteristic pair is uniquely determined.*

Proof: Let μ be an FQID distribution on \mathbb{R} , and let (b_1, τ_1) and (b_2, τ_2) be free characteristic pairs of μ . Let (τ_1^+, τ_1^-) and (τ_2^+, τ_2^-) be pairs of finite nonnegative measures given by the Hahn-Jordan decomposition of τ_1 and τ_2 , respectively. Then, we have that

$$\begin{aligned} b_1 + \int_{\mathbb{R}} \frac{1+xz}{z-x} \tau_1^+(dx) - \int_{\mathbb{R}} \frac{1+xz}{z-x} \tau_1^-(dx) \\ = \varphi_\mu(z) = b_2 + \int_{\mathbb{R}} \frac{1+xz}{z-x} \tau_2^+(dt) - \int_{\mathbb{R}} \frac{1+xz}{z-x} \tau_2^-(dt), \quad z \in \mathbb{C}^+. \end{aligned}$$

Therefore, we have

$$(b_1 - b_2) + \int_{\mathbb{R}} \frac{1+xz}{z-x} (\tau_1^+ + \tau_2^-)(dt) = \int_{\mathbb{R}} \frac{1+xz}{z-x} (\tau_1^- + \tau_2^+)(dt), \quad z \in \mathbb{C}^+.$$

Both sides correspond to the Voiculescu transform of some FID probability measure ν . Hence, the pairs $(b_1 - b_2, \tau_1^+ + \tau_2^-)$ and $(0, \tau_1^- + \tau_2^+)$ are free characteristic pairs of the FID distribution ν . By the uniqueness of free characteristic pairs of FID distributions (see Bercovici and Voiculescu (1993)), we have that $b_1 - b_2 = 0$ and $\tau_1^+ + \tau_2^- = \tau_1^- + \tau_2^+$. Therefore, $b_1 = b_2$ and $\tau_1 = \tau_2$. \square

Let $cB = \{cx \mid x \in B\}$. If ρ is a Borel measure on \mathbb{R} and c is a nonzero real constant, then the dilation of ρ by c is the measure $\mathbf{D}_c(\rho)$ given by

$$\mathbf{D}_c(\rho)(B) = \rho(c^{-1}B),$$

for any Borel set B .

Proposition 3.5. *We have the following properties.*

- (i) μ is FQID if and only if there exist $\mu_1, \mu_2 \in \text{ID}(\boxplus)$ such that $\mu_1 \boxplus \mu = \mu_2$.
- (ii) If μ is FQID and $\mu^{\boxplus t}$ exists for some $t > 0$, then $\mu^{\boxplus t}$ is also FQID.
- (iii) If μ, ν are FQID, $c \neq 0$ and $a \in \mathbb{R}$, then so are $\mu \boxplus \nu$, $\mathbf{D}_c(\mu)$ and $\mu \boxplus \delta_a$.
- (iv) If μ is FID and $\mu \boxplus \nu$ is FQID, then ν is also FQID.

(v) If μ is FQID and ν is not FQID, then $\mu \boxplus \nu$ is not FQID.

Proof: (i) is obvious from the Hahn-Jordan decomposition theorem. (ii) is a consequence of (i).

We prove (iii). By (i), there exist $\mu_1, \mu_2, \nu_1, \nu_2 \in \text{ID}(\boxplus)$ such that $\mu_1 \boxplus \mu = \mu_2$ and $\nu_1 \boxplus \nu = \nu_2$. Therefore,

$$(\mu_1 \boxplus \nu_1) \boxplus (\mu \boxplus \nu) = (\mu_1 \boxplus \mu) \boxplus (\nu_1 \boxplus \nu) = \mu_2 \boxplus \nu_2.$$

Since $\mu_1 \boxplus \nu_1, \mu_2 \boxplus \nu_2 \in \text{ID}(\boxplus)$, the measure $\mu \boxplus \nu$ is FQID. Using (i) again, we have that $\mu_1 \boxplus \mu = \mu_2$. Hence,

$$\mathbf{D}_c(\mu_1) \boxplus \mathbf{D}_c(\mu) = \mathbf{D}_c(\mu_1 \boxplus \mu) = \mathbf{D}_c(\mu_2).$$

Because $\mathbf{D}_c(\mu_1), \mathbf{D}_c(\mu_2) \in \text{ID}(\boxplus)$, the measure $\mathbf{D}_c(\mu)$ is FQID. Furthermore, it follows from (i) that $\mu \boxplus \mu_1 = \mu_2$, and so

$$(\mu \boxplus \delta_a) \boxplus \mu_1 = \delta_a \boxplus (\mu \boxplus \mu_1) = \delta_a \boxplus \mu_2.$$

Since $\delta_a \boxplus \mu_1, \delta_a \boxplus \mu_2 \in \text{ID}(\boxplus)$, the measure $\mu \boxplus \delta_a$ is also FQID.

Next, we show (iv). As $\mu \boxplus \nu$ is FQID, there exist $\mu_1, \mu_2 \in \text{ID}(\boxplus)$ such that

$$(\mu_1 \boxplus \mu) \boxplus \nu = \mu_1 \boxplus (\mu \boxplus \nu) = \mu_2.$$

Because $\mu_1 \boxplus \mu$ is FID, we conclude that ν is FQID.

Finally, we show (v). Assume that $\mu \boxplus \nu$ is FQID. We then obtain $\sigma_1, \sigma_2 \in \text{ID}(\boxplus)$ such that $(\mu \boxplus \nu) \boxplus \sigma_1 = \sigma_2$ by (i). Moreover, there exist $\mu_1, \mu_2 \in \text{ID}(\boxplus)$ such that $\mu \boxplus \mu_1 = \mu_2$, because μ is also FQID. Then,

$$\begin{aligned} \sigma_2 \boxplus \mu_1 &= \sigma_1 \boxplus (\mu \boxplus \nu) \boxplus \mu_1 \\ &= \sigma_1 \boxplus (\mu \boxplus \mu_1) \boxplus \nu = (\sigma_1 \boxplus \mu_2) \boxplus \nu. \end{aligned}$$

Since $\sigma_2 \boxplus \mu_1, \sigma_1 \boxplus \mu_2 \in \text{ID}(\boxplus)$, the measure ν is FQID, which is a contradiction. □

Definition 3.6. For $\mu_1, \mu_2, \mu \in \mathcal{P}(\mathbb{R})$, such that $\mu_1 \boxplus \mu = \mu_2$ we will write $\mu = \mu_2 \boxminus \mu_1$. The operation \boxminus is called *free deconvolution* (see Arizmendi et al. (2020) for more information).

Using the operation \boxminus , we obtain a similar property to that of Proposition 3.5(i), whereby μ is FQID if and only if $\mu = \mu_2 \boxminus \mu_1$ for some FID distributions μ_1, μ_2 .

3.2. *Atoms of FQID distributions.* In this section, we give a general result on atoms of FQID distributions.

Theorem 3.7. *An FQID distribution has at most one atom.*

Theorem 3.7 can be shown by following the same argument as in Bercovici and Voiculescu (1993, Proposition 5.12(iii)), because the Voiculescu transform of an FQID distribution has an analytic continuation to \mathbb{C}^+ . Nevertheless, we include a proof for readers' convenience.

First, we need the following lemma.

Lemma 3.8. *Let μ be an FQID distribution. Then,*

- (i) *We have that $F_\mu(z) + \varphi_\mu(F_\mu(z)) = z$ for all $z \in \mathbb{C}^+$.*
- (ii) *Let Ω_μ be the component of $\{z \in \mathbb{C}^+ : \text{Im}(z + \varphi_\mu(z)) > 0\}$ that contains iy for some large number $y > 0$. Then, $F_\mu(\mathbb{C}^+) = \Omega_\mu$.*

Proof: (i) Note that $\varphi_\mu(z) = F_\mu^{-1}(z) - z$ for all $z \in \Gamma_{\lambda, M}$ for some $\lambda, M > 0$. Therefore,

$$\varphi_\mu(F_\mu(z)) = z - F_\mu(z), \quad z \in F_\mu^{-1}(\Gamma_{\lambda, M}) \subset \mathbb{C}^+.$$

Since φ_μ has an analytic continuation to \mathbb{C}^+ (denoted by the same symbol φ_μ), we obtain

$$F_\mu(z) + \varphi_\mu(F_\mu(z)) = z, \quad z \in \mathbb{C}^+$$

by the identity theorem.

(ii) Take $w \in F_\mu(\mathbb{C}^+)$ and $z \in \mathbb{C}^+$ such that $w = F_\mu(z)$. Then,

$$w + \varphi_\mu(w) = F_\mu(z) + \varphi_\mu(F_\mu(z)) = z \in \mathbb{C}^+,$$

and therefore $\text{Im}(w + \varphi_\mu(w)) > 0$. Thus, $F_\mu(\mathbb{C}^+) \subset \Omega_\mu$.

Recall that $F_\mu(z + \varphi_\mu(z)) = z$ for all $z \in F_\mu(\mathbb{C}^+)$. The identity extends by analytic continuation to Ω_μ . Finally, for all $z \in \Omega_\mu$, we have that $z + \varphi_\mu(z) \in \mathbb{C}^+$, and therefore $z = F_\mu(z + \varphi_\mu(z)) \in F_\mu(\mathbb{C}^+)$. This means that $\Omega_\mu \subset F_\mu(\mathbb{C}^+)$. \square

Proof of Theorem 3.7: Assume that μ has an atom $a \in \mathbb{R}$ with the mass $\beta = \mu(\{a\}) > 0$. We obtain

$$|G_\mu(a + iy)| \geq |\text{Im}G_\mu(a + iy)| = \left| \int_{\mathbb{R}} \frac{-y}{(a - t)^2 + y^2} \mu(dt) \right| \geq \frac{\beta}{y}.$$

We then have that

$$|\text{Re}F_\mu(a + iy)| \leq |F_\mu(a + iy)| \leq \frac{y}{\beta} \leq \frac{\text{Im}F_\mu(a + iy)}{\beta},$$

where the last inequality holds by [Bercovici and Voiculescu \(1993, Corollary 5.3\)](#). Therefore, $F_\mu(a + iy) \in \Gamma_{1/\beta, 0}$ for all $y > 0$. Hence, the curve $C := F_\mu(a + i(0, \infty))$ approaches zero nontangentially. Moreover, if we define $u_\mu(z) := z + \varphi_\mu(z)$, the function u maps C to \mathbb{C}^+ , because $C \subset F_\mu(\mathbb{C}^+) = \Omega_\mu$ holds by [Lemma 3.8\(ii\)](#). Finally, [Lemma 3.8\(i\)](#) implies that

$$\lim_{z \rightarrow 0, z \in C} u_\mu(z) = \lim_{y \rightarrow 0} (F_\mu(a + iy) + \varphi(F_\mu(a + iy))) = \lim_{y \rightarrow 0} (a + iy) = a.$$

A consequence of [Bercovici and Voiculescu \(1993, Lemma 5.11\)](#) is that the uniqueness of a nontangential limit a of u_μ at the origin holds. Hence the desired result is obtained. \square

Remark 3.9. In [Example 3.3](#), it was shown that $\frac{1}{2}(\delta_{-1} + \delta_1)$ is not FQID by using the functional property of its Voiculescu transform. In general, for any $a, b \in \mathbb{R}$ with $a \neq b$ and any $p \in (0, 1)$, [Theorem 3.7](#) implies that the Bernoulli distribution $p\delta_a + (1 - p)\delta_b$ is not FQID. However, $p\delta_a + (1 - p)\delta_b$ is QID for $p \neq 1/2$ (see [Lindner et al. \(2018\)](#)). Therefore, quasi-infinite divisibility does not exhibit complete similarity between the classical and free probability theories.

3.3. Convergence of FQID distributions. We say that a sequence $\{\mu_n\}_n \subset \mathcal{P}(\mathbb{R})$ converges weakly to $\mu \in \mathcal{P}(\mathbb{R})$ if, for all $f \in C_b(\mathbb{R})$,

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} f(x)\mu_n(dx) = \int_{\mathbb{R}} f(x)\mu(dx), \tag{3.1}$$

where $C_b(\mathbb{R})$ is the set of all bounded continuous functions on \mathbb{R} . In this case, we write $\mu_n \xrightarrow{w} \mu$. The weak convergence of probability measures can be characterized by Voiculescu transforms, see [Bercovici and Voiculescu \(1993\)](#); [Bercovici and Pata \(1996\)](#).

Lemma 3.10. *Let $\{\mu_n\}_n \subset \mathcal{P}(\mathbb{R})$. Then, the following conditions are equivalent.*

- (i) *There exists some $\mu \in \mathcal{P}(\mathbb{R})$ such that $\mu_n \xrightarrow{w} \mu$ as $n \rightarrow \infty$.*
- (ii) *There exist positive numbers λ and M and a function φ such that all functions φ_{μ_n} and φ are defined on $\Gamma_{\lambda, M}$, and such that*
 - (a) $\lim_{n \rightarrow \infty} \varphi_{\mu_n}(z) = \varphi(z)$ *uniformly on compact subsets of $\Gamma_{\lambda, M}$;*
 - (b) $\sup_{n \in \mathbb{N}} \left| \frac{\varphi_{\mu_n}(z)}{z} \right| \rightarrow 0$ *as $|z| \rightarrow \infty$ with $z \in \Gamma_{\lambda, M}$.*
- (iii) *There exist positive numbers λ and M such that all functions φ_{μ_n} are defined on $\Gamma_{\lambda, M}$, and such that*
 - (a) $\lim_{n \rightarrow \infty} \varphi_{\mu_n}(iy)$ *exists for all $y \geq M$;*

$$(b) \sup_{n \in \mathbb{N}} \left| \frac{\varphi_{\mu_n}(iy)}{y} \right| \rightarrow 0 \text{ as } y \rightarrow \infty.$$

In this case, we have $\varphi = \varphi_\mu$ on $\Gamma_{\lambda, M}$.

A sequence $\{\mu_n\}_n$ of finite (Borel) measures is said to be *uniformly bounded* if

$$\sup_{n \in \mathbb{N}} \mu_n(\mathbb{R}) < \infty. \tag{3.2}$$

A sequence $\{\mu_n\}_n$ of finite measures is said to be *tight* if, for any $\epsilon > 0$, there exists a positive number $T > 0$ such that

$$\sup_{n \in \mathbb{N}} \mu_n(\mathbb{R} \setminus [-T, T]) \leq \epsilon. \tag{3.3}$$

The above three notions relating to weak convergence, uniform boundedness, and tightness are also defined for a sequence of finite signed measures on \mathbb{R} . If $\{\mu_n\}_n$ is a sequence of finite signed measures on \mathbb{R} , then we say that μ_n *converges weakly to some finite signed measure* μ if the measures μ_n and μ satisfy condition (3.1), and the family $\{\mu_n\}_n$ is said to be *uniformly bounded and tight* if its total variation $|\mu_n|$ satisfies conditions (3.2) and (3.3), respectively.

The following characterization of uniform boundedness and tightness is useful.

Lemma 3.11. *We have the following properties.*

- (i) *Let $\{\mu_n\}_n$ be a sequence of finite (signed) measures. Then, the sequence $\{\mu_n\}_n$ is tight if and only if every subsequence of $\{\mu_n\}_n$ has a weakly convergent subsequence.*
- (ii) *A weakly convergent sequence of finite (signed) measures is uniformly bounded and tight (see Bogachev (2007, Theorem 8.6.2)).*

The class $ID(\boxplus)$ is closed with respect to the weak topology. Furthermore, the (weak) convergence of FID distributions is equivalent to that of free characteristic pairs.

Lemma 3.12. *We have the following properties.*

- (i) *The class $ID(\boxplus)$ is weakly closed, that is, if $\{\mu_n\}_n$ is a sequence of FID distributions, $\mu \in \mathcal{P}(\mathbb{R})$, and $\mu_n \xrightarrow{w} \mu$ as $n \rightarrow \infty$, then μ is also FID.*
- (ii) *Let $\{\mu_n\}_n$ be a sequence of FID distributions μ_n with the free characteristic pair (b_n, τ_n) . The following conditions are equivalent.*
 - (a) *There exists some $\mu \in \mathcal{P}(\mathbb{R})$ such that $\mu_n \xrightarrow{w} \mu$ as $n \rightarrow \infty$;*
 - (b) *There exist $b \in \mathbb{R}$ and a finite measure τ on \mathbb{R} such that $b_n \rightarrow b$ and $\tau_n \xrightarrow{w} \tau$ as $n \rightarrow \infty$. In this case, μ is an FID distribution with the free characteristic pair (b, τ) .*

The following theorem extends the implication (b) \Rightarrow (a) in Lemma 3.12(ii) to FQID distributions. The proof is a simple modification of Barndorff-Nielsen and Thorbjørnsen (2006, Theorem 5.13(ii) \Rightarrow (i)) to $|\tau_n|$. However, we include the proof for readers' convenience.

Theorem 3.13. *Let $\{\mu_n\}_n$ be a sequence of FQID distributions with free characteristic pairs (b_n, τ_n) . If there exist $b \in \mathbb{R}$ and a finite signed measure τ on \mathbb{R} such that $b_n \rightarrow b$ and $\tau_n \xrightarrow{w} \tau$, then there exists some $\mu \in \mathcal{P}(\mathbb{R})$ such that $\mu_n \xrightarrow{w} \mu$ as $n \rightarrow \infty$. Moreover, μ is an FQID distribution with the free characteristic pair (b, τ) .*

Proof: We verify Lemma 3.10(iii) to prove this theorem. As μ_n is FQID, its Voiculescu transform φ_{μ_n} is defined on \mathbb{C}^+ . For any $y > 0$, the map $t \mapsto \frac{1+tiy}{iy-t}$ is continuous and bounded with respect to $t \in \mathbb{R}$. Hence, the convergence of γ_n and the weak convergence of τ_n imply that

$$\varphi_{\mu_n}(iy) = \gamma_n + \int_{\mathbb{R}} \frac{1+tiy}{iy-t} \tau_n(dt) \xrightarrow{n \rightarrow \infty} \gamma + \int_{\mathbb{R}} \frac{1+tiy}{iy-t} \tau(dt), \quad y > 0.$$

Therefore, $\lim_{n \rightarrow \infty} \varphi_{\mu_n}(iy)$ exists for all $y > 0$. Condition (a) in Lemma 3.10(iii) holds.

For any $n \in \mathbb{N}$ and $y > 0$, we have that

$$\frac{\varphi_{\mu_n}(iy)}{y} = \frac{\gamma_n}{y} + \int_{\mathbb{R}} \frac{1 + tiy}{y(iy - t)} \tau_n(dt).$$

Since $\{\gamma_n\}_n$ is a convergent sequence, it is bounded; therefore, it suffices to show that

$$\sup_{n \in \mathbb{N}} \left| \int_{\mathbb{R}} \frac{1 + tiy}{y(iy - t)} \tau_n(dt) \right| \rightarrow 0, \quad y \rightarrow \infty$$

to check condition (b) in Lemma 3.10(iii). For any $t \in \mathbb{R}$ and $y > 0$, we have that

$$\left| \frac{1 + tiy}{y(iy - t)} \right| \leq \frac{1}{y(y^2 + t^2)^{1/2}} + \frac{|t|}{(y^2 + t^2)^{1/2}}.$$

For any $y \geq 1$, we also have

$$\sup_{t \in \mathbb{R}} \left| \frac{1 + tiy}{y(iy - t)} \right| \leq 2.$$

For any $N \in \mathbb{N}$ and $y \geq 1$, we have that

$$\sup_{t \in [-N, N]} \left| \frac{1 + tiy}{y(iy - t)} \right| \leq \frac{N + 1}{y}.$$

Therefore, for any $N \in \mathbb{N}$ and $y \geq 1$, we obtain

$$\begin{aligned} \sup_{n \in \mathbb{N}} \left| \int_{\mathbb{R}} \frac{1 + tiy}{y(iy - t)} \tau_n(dt) \right| &\leq \sup_{n \in \mathbb{N}} \int_{\mathbb{R}} \left| \frac{1 + tiy}{y(iy - t)} \right| |\tau_n|(dt) \\ &\leq \frac{N + 1}{y} \sup_{n \in \mathbb{N}} |\tau_n|([-N, N]) + 2 \sup_{n \in \mathbb{N}} |\tau_n|(\mathbb{R} \setminus [-N, N]) \\ &\leq \frac{N + 1}{y} \sup_{n \in \mathbb{N}} |\tau_n|(\mathbb{R}) + 2 \sup_{n \in \mathbb{N}} |\tau_n|(\mathbb{R} \setminus [-N, N]). \end{aligned}$$

As $\tau_n \xrightarrow{w} \tau$, the sequence $\{|\tau_n|\}_n$ is uniformly bounded and tight by Lemma 3.11(ii). Consider an arbitrary $\epsilon > 0$. Since $\{|\tau_n|\}_n$ is tight, there exists some $N \in \mathbb{N}$ such that $\sup_{n \in \mathbb{N}} |\tau_n|(\mathbb{R} \setminus [-N, N]) < \epsilon/4$. As $\sup_{n \in \mathbb{N}} |\tau_n|(\mathbb{R}) < \infty$, for any $N \in \mathbb{N}$, there is some $y_0 \geq 1$ such that

$$\frac{N + 1}{y} \sup_{n \in \mathbb{N}} |\tau_n|(\mathbb{R}) < \frac{\epsilon}{2}, \quad y \geq y_0.$$

Therefore, if $y \geq y_0$, then

$$\sup_{n \in \mathbb{N}} \left| \int_{\mathbb{R}} \frac{1 + tiy}{y(iy - t)} \tau_n(dt) \right| \leq \frac{N + 1}{y} \sup_{n \in \mathbb{N}} |\tau_n|(\mathbb{R}) + 2 \sup_{n \in \mathbb{N}} |\tau_n|(\mathbb{R} \setminus [-N, N]) < \epsilon.$$

Finally, Lemma 3.10(iii) holds. Therefore, there is a probability measure μ on \mathbb{R} such that $\mu_n \xrightarrow{w} \mu$ as $n \rightarrow \infty$, and

$$\varphi_{\mu}(z) = b + \int_{\mathbb{R}} \frac{1 + tz}{z - t} \tau(dt), \quad z \in \mathbb{C}^+.$$

Hence, μ is an FQID distribution with the free characteristic pair (b, τ) . □

Remark 3.14. The assumptions in Theorem 3.13 cannot be disregarded. To see this, we refer to Theorem 4.1: let $\{a_n\}_n$ and $\{\lambda_n\}_n$ be sequences of positive numbers such that $a_n \geq 2$ and $0 < \lambda_n \leq 1$. According to Theorem 4.1, a FQID μ_n distribution exists when $a = a_n$, $\lambda = \lambda_n$, and $c = 1$. Its free characteristic pair $(b_n, \tau_n) = (-\frac{\lambda_n}{2}, \frac{a_n dt}{\pi(t^2 + 1)} - \frac{\lambda_n}{2} \delta_1)$, and its Voiculescu transform is

$$\varphi_{\mu_n}(z) = -a_n i - \frac{\lambda_n z}{z - 1}, \quad z \in \mathbb{C}^+.$$

Thus, it is not FID. If either $a_n \rightarrow \infty$ holds or $\lim_{n \rightarrow \infty} \lambda_n$ does not exist (e.g. $\lambda_n = |\sin n|$), then either b_n or τ_n (or both of b_n and τ_n) does not fulfill the assumption in Theorem 3.13. In both cases, it is easy to see that $\lim_{z \rightarrow \infty} \varphi_{\mu_n}(iy)$ does not exist for any $y > 0$. Hence, by Lemma 3.10, μ_n does not converge weakly to any probability measure.

Remark 3.15. Whether the converse implication of Theorem 3.13 is true remains an open question. More precisely, we wish to know whether, if $\mu_n \xrightarrow{w} \mu$, then $b_n \rightarrow b$ and $\tau_n \rightarrow \tau$ as $n \rightarrow \infty$. According to the proof in Barndorff-Nielsen and Thorbjørnsen (2006), if μ_n is an FID distribution with the free characteristic pair (b_n, τ_n) , then the positivity of τ_n can be used to show the implication (i) \Rightarrow (ii) in Lemma 3.12(ii). However, if μ_n is FQID, then τ_n is a signed measure, and we cannot apply the techniques used in Barndorff-Nielsen and Thorbjørnsen (2006).

One can obtain the converse of Theorem 3.13 under some additional assumptions regarding free quasi-generating pairs.

Theorem 3.16. *Let $\{\mu_n\}_n$ be a sequence of FQID distributions μ_n with free characteristic pairs (b_n, τ_n) and let μ be an FQID distribution with the free characteristic pair (b, τ) . Suppose that $\tau_n = \tau_n^+ - \tau_n^-$ for each $n \in \mathbb{N}$, where τ_n^\pm are the positive and negative parts of τ . If $\mu_n \xrightarrow{w} \mu$ as $n \rightarrow \infty$, and $\{\tau_n^-\}_n$ is tight and uniformly bounded, then $b_n \rightarrow b$ and $\tau_n \xrightarrow{w} \tau$ as $n \rightarrow \infty$.*

To prove this theorem, we investigate convergence, tightness, and uniform boundedness for free characteristic pairs. Note that the proofs of Lemma 3.17 and Theorem 3.16 are obtained by the direct application of the results in Lindner et al. (2018).

Lemma 3.17. *Let $\{\mu_n\}_n$ be a sequence of FQID distributions with free characteristic pairs (b_n, τ_n) . If the sequence $\{\mu_n\}_n$ is tight and the sequence $\{\tau_n^-\}_n$ is tight and uniformly bounded, then the sequence $\{b_n\}_n$ is bounded and the sequences $\{\tau_n^+\}_n$ and $\{|\tau_n|\}_n$ are tight and uniformly bounded.*

Proof: As the sequence $\{\mu_n\}_n$ is tight and $\{\tau_n^-\}_n$ is tight and uniformly bounded, for any subsequences $\{k_n\}_n$ in \mathbb{N} , the subsequence $\{\mu_{k_n}\}_n$ is also tight and the subsequence $\{\tau_{k_n}^-\}_n$ is also tight and uniformly bounded. By Lemma 3.11(i), there exists a further subsequence $\{l_{k_n}\}_n$ of the sequence $\{k_n\}_n$ such that the subsequences $\{\mu_{l_{k_n}}\}_n$ and $\{\tau_{l_{k_n}}^-\}_n$ converge weakly to some probability measure μ and some finite measure τ^- , respectively.

Let $\rho_{l_{k_n}}$ be an FID distribution with the free characteristic pair $(0, \tau_{l_{k_n}}^-)$ for each n . By Lemma 3.12, there is an FID distribution ρ with the free characteristic pair $(0, \tau^-)$ such that $\rho_{l_{k_n}} \xrightarrow{w} \rho$ as $n \rightarrow \infty$. Thus, we have that $\mu_{l_{k_n}} \boxplus \rho_{l_{k_n}} \xrightarrow{w} \mu \boxplus \rho$ as $n \rightarrow \infty$. Since $\mu_{l_{k_n}} \boxplus \rho_{l_{k_n}}$ is FID with the free characteristic pair $(b_{l_{k_n}}, \tau_{l_{k_n}}^+)$ and Lemma 3.12(i) holds, the measure $\mu \boxplus \rho$ is also FID. Therefore, the sequence $\{\tau_{l_{k_n}}^+\}_n$ converges weakly and the sequence $\{b_{l_{k_n}}\}_n$ converges as $n \rightarrow \infty$ by Lemma 3.12(ii).

To summarize the above discussion, for any subsequence $\{k_n\}_n$ in \mathbb{N} , there is a further subsequence $\{l_{k_n}\}_n$ of the sequence $\{k_n\}_n$ such that the measure $\tau_{l_{k_n}}^+$ converges weakly and the sequence $\{b_{l_{k_n}}\}_n$ converges. Consequently, the sequence $\{b_n\}_n$ is bounded and the sequence $\{\tau_n^+\}_n$ is tight and uniformly bounded; therefore, so is the sequence $\{|\tau_n|\}_n$. \square

Proof of Theorem 3.16: By our assumption, the sequence $\{\mu_n\}_n$ is tight, and therefore the sequence $\{b_n\}_n$ is bounded and the sequence $\{\tau_n^+\}_n$ is tight and uniformly bounded by Lemma 3.17. Thus, $\{\tau_n\}_n$ is also tight and uniformly bounded. If either (or both) of the sequences $\{b_n\}_n$ and $\{\tau_n\}_n$ does not satisfy the statement of this theorem, then the tightness and (uniform) boundedness of these sequences imply that the sequence $\{\mu_n\}_n$ does not converge weakly, which contradicts the weak convergence of the sequence $\{\mu_n\}_n$. Hence, there exist $b^* \in \mathbb{R}$ and a finite signed measure τ^* such that $b_n \rightarrow b^*$ and $\tau_n \xrightarrow{w} \tau^*$ as $n \rightarrow \infty$. By the uniqueness of free characteristic pairs (see Proposition 3.4) and the convergence $\mu_n \xrightarrow{w} \mu$, we have that $b = b^*$ and $\tau = \tau^*$. \square

We mention that the weak convergence preserves the free quasi-infinite divisibility of probability measures under some additional assumption.

Proposition 3.18. *Let $\{\mu_n\}_n$ be a sequence of FQID distributions. For each $n \in \mathbb{N}$, let us set $\mu_{1,n}, \mu_{2,n} \in \text{ID}(\boxplus)$ satisfying $\mu_{1,n} \boxplus \mu_n = \mu_{2,n}$. Assume that there exists $\mu \in \mathcal{P}(\mathbb{R})$ such that $\mu_n \xrightarrow{w} \mu$. If there exists $\mu_2 \in \mathcal{P}(\mathbb{R})$ (actually, $\mu_2 \in \text{ID}(\boxplus)$) such that $\mu_{2,n} \xrightarrow{w} \mu_2$, then there exists $\mu_1 \in \mathcal{P}(\mathbb{R})$ (actually also, $\mu_1 \in \text{ID}(\boxplus)$) such that $\mu_{1,n} \xrightarrow{w} \mu_1$ and μ is FQID satisfying $\mu_1 \boxplus \mu = \mu_2$.*

Proof: Due to Lemma 3.10, there exist $M > 0$ such that the limits $\lim_{n \rightarrow \infty} \varphi_{\mu_n}(iy)$ and $\lim_{n \rightarrow \infty} \varphi_{\mu_{2,n}}(iy)$ exist for all $y \geq M$, and therefore so is $\lim_{n \rightarrow \infty} \varphi_{\mu_{1,n}}(iy)$. By the triangle inequality, we have

$$\sup_{n \in \mathbb{N}} \left| \frac{\varphi_{\mu_{1,n}}(iy)}{y} \right| \leq \sup_{n \in \mathbb{N}} \left| \frac{\varphi_{\mu_n}(iy)}{y} \right| + \sup_{n \in \mathbb{N}} \left| \frac{\varphi_{\mu_{2,n}}(iy)}{y} \right| \rightarrow 0,$$

as $n \rightarrow \infty$. Hence $\mu_{1,n}$ satisfies the assumptions in Lemma 3.10(a), (b). Thus there exists $\mu_1 \in \mathcal{P}(\mathbb{R})$ such that $\mu_{1,n} \xrightarrow{w} \mu_1$ and $\mu_1 \boxplus \mu = \mu_2$. Since the class $\text{ID}(\boxplus)$ is weakly closed, μ_1 is FID, and therefore μ is FQID. □

Problem 3.19. *In the classical case, the class of all QID distributions is not closed under weak convergence; see Lindner et al. (2018, Section 4) for a concrete example using the Bernoulli distributions. In the free setting, this example does not work well because the Bernoulli distributions are not FQID. At the moment, we still do not find an example of a sequence of FQID distribution in which its weak limit is not FQID. Thus, whether the class of all FQIDs is closed under weak convergence remains an open problem.*

3.4. *Free Lévy-Khintchine type representations of FQID distributions.* In this section, we describe how \mathcal{R} -transforms of FQID distributions give the free Lévy-Khintchine representations.

Lemma 3.20. *Let τ be a finite signed measure on \mathbb{R} . Then a function $\nu : \mathcal{B}_0 \rightarrow [-\infty, \infty]$ defined by*

$$\nu(B) = \int_B \frac{1+x^2}{x^2} \mathbf{1}_{\mathbb{R} \setminus \{0\}}(x) \tau(dx), \quad B \in \mathcal{B}_0,$$

is a quasi-Lévy-type measure.

We prove that, for any $r > 0$, $\nu|_{\mathcal{B}_r}$ is finite signed measure on $(\mathbb{R} \setminus (-r, r), \mathcal{B}_r)$. It follows from the kernel $\frac{1+x^2}{x^2}$ is finite on $\mathbb{R} \setminus (-r, r)$ and Hahn-Jordan decomposition.

Proof: We fix $r > 0$. The Hahn-Jordan decomposition implies the existence of a finite measure τ^+ and τ^- such that $\tau = \tau^+ - \tau^-$. Since $\frac{1+x^2}{x^2}$ is finite on $\mathbb{R} \setminus (-r, r)$, we get

$$\nu|_{\mathcal{B}_r}(B) = \int_B \frac{1+x^2}{x^2} \mathbf{1}_{\mathbb{R} \setminus \{0\}}(x) \tau^+(dx) - \int_B \frac{1+x^2}{x^2} \mathbf{1}_{\mathbb{R} \setminus \{0\}}(x) \tau^-(dx), \quad B \in \mathcal{B}_r.$$

And also, the two functions $B \mapsto \int_B \frac{1+x^2}{x^2} \mathbf{1}_{\mathbb{R} \setminus \{0\}}(x) \tau^\pm(dx)$ are finite (positive) measures on \mathcal{B}_r . Hence $\nu|_{\mathcal{B}_r}$ is a finite signed measure on a measure space $(\mathbb{R} \setminus (-r, r), \mathcal{B}_r)$. Moreover, we get

$$\begin{aligned} \int_{\mathbb{R} \setminus \{0\}} (1 \wedge x^2) |\nu|(dx) &= \int_{(-1,1) \setminus \{0\}} (1 \wedge x^2) |\nu|(dx) + \int_{\mathbb{R} \setminus (-1,1)} (1 \wedge x^2) |\nu|(dx) \\ &= \int_{(-1,1) \setminus \{0\}} x^2 \cdot \frac{1+x^2}{x^2} |\tau|(dx) + \int_{\mathbb{R} \setminus (-1,1)} \frac{1+x^2}{x^2} |\tau|(dx) \\ &\leq 2 \{ |\tau|(((-1, 1) \setminus \{0\})) + |\tau|(\mathbb{R} \setminus (-1, 1)) \} \\ &\leq 4|\tau|(\mathbb{R}) < \infty, \end{aligned}$$

as desired. □

Let μ be an FQID distribution with a free characteristic pair (b, τ) . Then the R -transform of μ is of the form (2.8) for some $a, \gamma \in \mathbb{R}$ and some function $\nu : \mathcal{B}_0 \rightarrow [-\infty, \infty]$, fulfilled the relation (2.9). Due to Lemma 3.20, the function ν is a quasi-Lévy-type measure.

Proposition 3.21. *Let μ be an FQID distribution with a free characteristic pair (b, τ) . Then, we have*

$$R_\mu(z) = az^2 + \gamma z + \int_{\mathbb{R}} \left(\frac{1}{1 - xz} - 1 - xz \mathbf{1}_{[-1,1]}(x) \right) \nu(dx), \quad z \in \mathbb{C}^- \tag{3.4}$$

for some $a, \gamma \in \mathbb{R}$ and a quasi-Lévy-type measure ν on \mathbb{R} . Conversely, if the R -transform of an FQID distribution μ is of the form (3.4), then μ has a free characteristic pair (b, τ) fulfilled the relation (2.9).

Moreover, such a triplet (a, ν, γ) is uniquely determined by the FQID distribution μ . The triplet (a, ν, γ) is called a free characteristic triplet of the FQID distribution μ .

Definition 3.22. A quasi-Lévy-type measure ν is called a free quasi-Lévy measure if there exist an FQID distribution μ and some $a, \gamma \in \mathbb{R}$ such that a triplet (a, ν, γ) is a free characteristic triplet of μ .

Note that, the kernel function of the R -transform of an FQID distribution is integrable with respect to the quasi-Lévy-type measure.

Proposition 3.23. *For $z \in \mathbb{C}^-$, the function on \mathbb{R} defined by*

$$x \mapsto \frac{1}{1 - zx} - 1 - zx \mathbf{1}_{[-1,1]}(x) =: k(x, z)$$

is integrable with respect to a quasi-Lévy-type measure ν .

Proof: Let ν^+ and ν^- be the positive and negative parts of ν , respectively. Then,

$$\begin{aligned} \int_{\mathbb{R}} k(x, z) \nu^+(dx) &= z^2 \int_{-1}^1 \frac{x^2}{1 - zx} \nu^+(dx) + z \int_{\mathbb{R} \setminus [-1,1]} \frac{x}{1 - zx} \nu^+(dx) \\ &= z^2 \int_{-1}^1 \frac{1}{1 - zx} (1 \wedge x^2) \nu^+(dx) + z \int_{\mathbb{R} \setminus [-1,1]} \frac{x}{1 - zx} \nu^+|_{\mathcal{B}_1}(dx). \end{aligned}$$

We show that the first term on the right-hand side of the above equality is integrable. For $z = u + iv \in \mathbb{C}^-$ and $x \in [-1, 1]$, we obtain

$$\left| \frac{1}{1 - zx} \right|^2 = \frac{1}{(u^2 + v^2)x^2 - 2ux + 1} \leq \frac{u^2 + v^2}{v^2} = \left(\frac{|z|}{\text{Im}(z)} \right)^2.$$

As $\int_{\mathbb{R}} (1 \wedge x^2) \nu^+(dx) < \infty$, we have that

$$\left| z^2 \int_{-1}^1 \frac{1}{1 - zx} (1 \wedge x^2) \nu^+(dx) \right| \leq \frac{|z|^3}{|\text{Im}(z)|} \int_{-1}^1 (1 \wedge x^2) \nu^+(dx) < \infty, \quad z \in \mathbb{C}^-.$$

Next, we prove that the second term on the right-hand side of the above equality is also integrable. For $z = u + iv \in \mathbb{C}^-$ and $x \in \mathbb{R}$, we have

$$\left| \frac{x}{1 - zx} \right|^2 = \frac{|x|^2}{(1 - ux)^2 + v^2x^2} \leq \frac{1}{v^2} = \frac{1}{\text{Im}(z)^2}.$$

As $\nu^+|_{\mathcal{B}_1}$ is finite, we obtain

$$\left| z \int_{\mathbb{R} \setminus [-1,1]} \frac{x}{1 - zx} \nu^+|_{\mathcal{B}_1}(dx) \right| \leq \left| \frac{z}{\text{Im}(z)} \right| \nu^+|_{\mathcal{B}_1}(\mathbb{R} \setminus [-1, 1]) < \infty, \quad z \in \mathbb{C}^-.$$

Therefore, the function $k(\cdot, z)$ is integrable with respect to ν^+ . Similarly, it is also integrable with respect to ν^- , and hence is so with respect to ν . □

If μ is FQID with a free characteristic triplet (a, ν, η) , then

$$R_\mu(z) = az^2 + \eta z + \int_{\mathbb{R}} k(x, z) \nu^+(dx) - \int_{\mathbb{R}} k(x, z) \nu^-(dx), \quad z \in \mathbb{C}^-,$$

by Proposition 3.23. It follows that there exist FID distributions μ^+ and μ^- with free characteristic triplets (a, ν^+, γ) and $(0, \nu^-, 0)$, respectively, such that $\mu^- \boxplus \mu = \mu^+$.

3.5. FQID distributions with compact support. It is known that, for $\mu \in \text{ID}(\boxplus)$ with a free characteristic pair (b, τ) , the measure μ has a compact support if and only if so does τ , see e.g. [Bercovici and Voiculescu \(1992\)](#). The purpose of the section is to investigate whether can extend the above statement to the class of FQID distributions.

We say that a sequence $\{s_n\}_{n=0}^\infty$ of complex numbers *grows at most exponentially* if there exists a constant $c > 0$ such that

$$|s_n| \leq c^n, \quad \text{for all } n \in \mathbb{N}.$$

A finite measure ρ has a compact support if and only if the n -th moment $m_n(\rho)$ of ρ exists for all $n \in \mathbb{N}$ and the sequence $\{m_n(\rho)\}_{n=0}^\infty$ grows at most exponentially (see [Nica and Speicher \(2006, Lemma 13.13\)](#)).

Theorem 3.24. *Let μ be an FQID distribution with the free characteristic pair (b, τ) . If the finite signed measure τ is compactly supported, then so is the measure μ . Furthermore, we have that $\tau(\mathbb{R}) \geq -m_2(\tau)$.*

Proof: By the Hahn-Jordan decomposition theorem, we can find two finite measures τ^+ and τ^- such that $\tau = \tau^+ - \tau^-$. As τ has a compact support, so do τ^+ and τ^- . Let μ^+ and μ^- be FID distributions with free characteristic pairs (b, τ^+) and $(0, \tau^-)$, respectively. Then $\mu = \mu^+ \boxplus \mu^-$ and μ^+, μ^- are compactly supported. By [Nica and Speicher \(2006, Lemma 13.13 and Proposition 13.15\)](#), there exist $c_+, c_- > 0$, such that $|\kappa_n(\mu^\pm)| \leq c_\pm^n$, respectively, where $\kappa_n(\rho)$ is the n -th free cumulant of $\rho \in \mathcal{P}(\mathbb{R})$. Hence there exists $c > 0$ such that

$$|\kappa_n(\mu)| \leq |\kappa_n(\mu^+)| + |\kappa_n(\mu^-)| \leq c^n.$$

Hence $\{\kappa_n(\mu)\}_{n=1}^\infty$ grows at most exponentially. Also, by [Nica and Speicher \(2006, Lemma 13.13 and Proposition 13.15\)](#), the measure μ is compactly supported. It is elementary to verify that $\kappa_n(\mu) = m_{n-2}(\tau) + m_n(\tau)$ for $n \in \mathbb{N}$, where $m_{-1}(\tau) = b$. Hence, the desired result is obtained by considering the case of $n = 2$. \square

According to [Lindner et al. \(2018, Example 3.11\)](#), there exists a compactly supported QID distribution such that its quasi-Lévy measure is not compactly supported. That is, the converse implication of Theorem 3.24 is not true in classical case. On the other hand, we still do not know whether the converse statement of Theorem 3.24 is true or not in free case.

Problem 3.25. *Suppose that μ is an FQID distribution with compact support and with the free characteristic pair (b, τ) . Is the support of τ necessarily compact?*

Note also a related problem:

Problem 3.26. *Suppose that μ is an FQID distribution with compact support. Is it true that there exist FID distributions μ_1, μ_2 with compact support such that $\mu_1 \boxplus \mu = \mu_2$?*

The affirmative answer for the latter would imply that for the former.

4. Examples

Let $\mathbb{R}_+ := [0, \infty)$. In the following, the principal square root of z is defined by

$$\sqrt{z} := |z|^{\frac{1}{2}} e^{\frac{i \arg(z)}{2}}, \quad 0 \leq \arg(z) < 2\pi, \tag{4.1}$$

namely, the square root function is defined using the nonnegative real axis \mathbb{R}_+ as a branch cut.

In this section, various examples of FQID distributions (or free characteristic triplets) and their distributional properties are provided. For this purpose, let us recall three important families of distributions in free probability (see [Bercovici and Voiculescu \(1993\)](#); [Nica and Speicher \(2006\)](#); [Mingo and Speicher \(2017\)](#) for details).

- Let \mathbf{C}_a be the Cauchy distribution with the parameter $a > 0$, that is,

$$\mathbf{C}_a(dx) := \frac{a}{\pi} \frac{1}{x^2 + a^2} \mathbf{1}_{\mathbb{R}}(x) dx.$$

The special case $a = 1$, the standard Cauchy distribution \mathbf{C}_1 , is denoted by \mathbf{C} . Note that $\mathbf{C}_a = \mathbf{D}_a(\mathbf{C}) = \mathbf{C}^{\boxplus a}$ for all $a > 0$. The Cauchy distribution is FID and its Voiculescu transform is given by

$$\varphi_{\mathbf{C}_a}(z) = \int_{\mathbb{R}} \frac{1 + zt}{z - t} \frac{a}{\pi(1 + t^2)} dt = -ai, \quad z \in \mathbb{C}^+.$$

In this case, the free characteristic triplet is $(0, \frac{a}{\pi a^2} du, 0)$.

- Let $\mathbf{MP}(\lambda)$ be the Marchenko-Pastur law with the parameter (variance) $\lambda > 0$. The Voiculescu transform $\varphi_{\mathbf{MP}(\lambda)}$ is given by

$$\varphi_{\mathbf{MP}(\lambda)}(z) = \frac{\lambda}{2} + \int_{\mathbb{R}} \frac{1 + tz}{z - t} \frac{\lambda}{2} \delta_1(dt) = \frac{\lambda z}{z - 1}, \quad z \in \mathbb{C}^+.$$

The measure $\mathbf{MP}(\lambda)$ is FID and the free characteristic triplet is $(0, \lambda \delta_1, 0)$. Note that, for $c \neq 0$, the free characteristic triplet of $\mathbf{D}_c(\mathbf{MP}(\lambda))$ is $(0, \lambda \delta_c, 0)$ and the Voiculescu transform $\varphi_{\mathbf{D}_c(\mathbf{MP}(\lambda))}$ is given by

$$\varphi_{\mathbf{D}_c(\mathbf{MP}(\lambda))}(z) = \frac{\lambda}{c^2 + 1} + \int_{\mathbb{R}} \frac{1 + tz}{z - t} \frac{\lambda c}{c^2 + 1} \delta_c(dt) = \frac{\lambda cz}{z - c}.$$

- Let $\mathbf{S}(m, \sigma^2)$ be the semicircle law with mean $m \in \mathbb{R}$ and variance $\sigma^2 > 0$, i.e.,

$$\mathbf{S}(m, \sigma^2)(dx) := \frac{1}{2\pi\sigma^2} \sqrt{4\sigma^2 - (x - m)^2} \mathbf{1}_{[m-2\sigma, m+2\sigma]}(x) dx.$$

The special case $\mathbf{S}(0, 1)$ is simply denoted by \mathbf{S} . This is FID and the Voiculescu transform $\varphi_{\mathbf{S}(m, \sigma^2)}$ has the form

$$\varphi_{\mathbf{S}(m, \sigma^2)}(z) = m + \int_{\mathbb{R}} \frac{1 + zt}{z - t} \sigma^2 \delta_0(dt) = m + \frac{\sigma^2}{z}, \quad z \in \mathbb{C}^+.$$

Its free characteristic triplet is $(\sigma^2, 0, m)$. Note that $\mathbf{D}_a(\mathbf{S}(m, \sigma^2)) = \mathbf{S}(am, a^2\sigma^2)$.

4.1. *Free deconvolution with the Cauchy distribution.* In this section, we construct explicit FQID distributions from the Cauchy distribution. The support of the free Lévy measure of the Cauchy distribution is unbounded, whereas the support of the free Lévy measure of compactly supported FID distributions is bounded. Thus, the difference between the free Lévy measure of the Cauchy distribution and that of a compactly supported FID distribution becomes a signed measure. We investigate whether there exist FQID probability measures whose quasi-free Lévy measure is a signed measure.

This idea comes from the results in Arizmendi et al. (2020). If $\mu \in \mathcal{P}(\mathbb{R})$ has a finite variance σ^2 , then by Arizmendi et al. (2020, Theorem 1.2), there exists some $\rho(\mu) \in \mathcal{P}(\mathbb{R})$ such that

$$\mu \boxplus \rho(\mu) = \mathbf{C}_{2\sqrt{2}\sigma}. \tag{4.2}$$

Furthermore, if μ is an FID distribution with a compact support, then $\rho(\mu)$ is FQID. The probability measures $\mathbf{MP}(\lambda)$ and $\mathbf{S}(0, \sigma^2)$ are examples of such μ , as both distributions have finite variances and are FID with compact supports. That is,

- $\mathbf{MP}(\lambda) \boxplus \rho(\mathbf{MP}(\lambda)) = \mathbf{C}_{2\sqrt{2}\lambda}$ and $\rho(\mathbf{MP}(\lambda))$ is FQID,
- $\mathbf{S}(0, \sigma^2) \boxplus \rho(\mathbf{S}(0, \sigma^2)) = \mathbf{C}_{2\sqrt{2}\sigma}$ and $\rho(\mathbf{S}(0, \sigma^2))$ is FQID.

In Sections 4.1.1, 4.1.2, and 4.1.3, we investigate signed measures generated from the Lévy measure of the Cauchy distribution and the Marchenko-Pastur, semicircle, and free Meixner distributions, respectively.

4.1.1. *Case of the Marchenko-Pastur law.* From now on we will denote $\mathbf{MP}(c, \lambda) := \mathbf{D}_c(\mathbf{MP}(\lambda))$. Let $a, \lambda > 0$ and $c \neq 0$. Consider the analytic function

$$\varphi(z) = \varphi_{\mathbf{C}_a}(z) - \varphi_{\mathbf{MP}(c, \lambda)}(z) = -ai - \frac{\lambda cz}{z - c}, \quad z \in \mathbb{C}^+. \tag{4.3}$$

We will prove that φ is the Voiculescu transform of some probability measure, or equivalently, that the function $K : \mathbb{C}^+ \rightarrow \mathbb{C}^+$ defined by

$$K(z) := \varphi(z) + z = -ai - \frac{\lambda cz}{z - c} + z, \quad z \in \mathbb{C}^+$$

is the compositional right inverse of an F -transform of some probability measure.

A straightforward calculation gives

$$F(z) := K^{-1}(z) = \frac{z + c(\lambda + 1) + ai + \sqrt{q(z)}}{2}, \quad z \in \mathbb{C}^+, \tag{4.4}$$

with an appropriate branch of the square root, where

$$q(z) := (z + c(\lambda + 1) + ai)^2 - 4c(z + ai).$$

A strategy to prove that F is an F -transform of some probability measure, is use of Proposition 2.5. To carry out the proposition, we will show that F is a Pick function which satisfies $\lim_{y \rightarrow \infty} F(iy)/iy = 1$. It is easy to verify the asymptotic condition for F . Hence, it suffices to show that $q(\mathbb{C}^+) \subset \mathbb{C} \setminus \mathbb{R}_+$ (recall that the branch cut of the square root is \mathbb{R}_+ in our setting). Then, $\sqrt{q(z)} \in \mathbb{C}^+$ is well-defined on \mathbb{C}^+ , which yields $\text{Im}F(z) > 0$ for all $z \in \mathbb{C}^+$. Let $z = x + iy \in \mathbb{C}^+$ (namely $x \in \mathbb{R}$ and $y > 0$). We have that

$$q(x + iy) = (x + c(\lambda + 1))^2 - 4cx - (y + a)^2 + 2i(y + a)(x + c(\lambda - 1)).$$

Then, $q(x + iy) \in \mathbb{R}$ if and only if $x = c(1 - \lambda)$. From the equation

$$q(c(1 - \lambda) + iy) = 4c^2\lambda - (y + a)^2,$$

it follows that, if $0 < \lambda \leq (\frac{a}{2c})^2$, then $q(x + iy) \notin \mathbb{R}_+$, which implies that $F(\mathbb{C}^+) \subset \mathbb{C}^+$. We conclude that, in this case, the function F in (4.4) is a Pick function.

By Proposition 2.5, there exists a probability measure $\rho_{a,c,\lambda}$ on \mathbb{R} such that $F = F_{\rho_{a,c,\lambda}}$. Finally, we obtain

$$\mathbf{MP}(c, \lambda) \boxplus \rho_{a,c,\lambda} = \mathbf{C}_a,$$

and so $\rho_{a,c,\lambda}$ is FQID for all $a, \lambda > 0$ and $c \neq 0$ with $0 < \lambda \leq (\frac{a}{2c})^2$.

We further observe that, if $0 < y < \frac{c^2\lambda}{a}$,

$$\operatorname{Im}\varphi_{\rho_{a,c,\lambda}}(c + iy) = -a + \frac{c^2\lambda}{y} > 0.$$

Hence, the measure $\rho_{a,c,\lambda}$ is not FID. Some calculations indicate that the nontangential limit of $G_{\rho_{a,c,\lambda}}$ at x exists for any $x \in \mathbb{R}$. Thus, $\rho_{a,c,\lambda}$ has no singular parts. We summarize the results.

Theorem 4.1. *Let $a, \lambda > 0$, $c \neq 0$ with $0 < \lambda \leq (\frac{a}{2c})^2$. Then, there exists some $\rho_{a,c,\lambda} \in \mathcal{P}(\mathbb{R})$ such that its Voiculescu transform is*

$$\varphi_{\rho_{a,c,\lambda}}(z) = -ai - \frac{\lambda cz}{z - c}, \quad z \in \mathbb{C}^+.$$

It has the following properties:

- (1) $\mathbf{MP}(c, \lambda) \boxplus \rho_{a,c,\lambda} = \mathbf{C}_a$. In other words, $\rho_{a,c,\lambda} = \mathbf{C}_a \boxminus \mathbf{MP}(c, \lambda)$ is FQID, rather than FID. Moreover, it is absolutely continuous with respect to the Lebesgue measure.
- (2) Its free characteristic triplet is given by $(0, \frac{adu}{\pi u^2} - \lambda\delta_c, 0)$.
- (3) Its free characteristic pair is given by $(-\frac{\lambda}{c^2+1}, \frac{adu}{\pi(u^2+1)} - \frac{\lambda c}{c^2+1}\delta_c)$.

Let us consider three specific cases.

- If $a = 2\sqrt{2\lambda}$ and $c = 1$, then $\rho_{2\sqrt{2\lambda},1,\lambda}$ coincides with the measure $\rho(\mathbf{MP}(\lambda))$ in (4.2).
- If $a = 0$, then the function φ in (4.3) is not the Voiculescu transform of any probability measure on \mathbb{R} .
- If $c = a > 0$, then for $0 < \lambda \leq \frac{1}{4}$, there exists a probability measure $\rho_{a,\lambda} := \rho_{a,a,\lambda} \in \mathcal{P}(\mathbb{R})$ such that $\varphi_{\rho_{a,\lambda}}(z) = -ai - \frac{\lambda az}{z-a}$ by the above discussion. One can see that $\mathbf{D}_a(\rho_{1,\lambda}) = \rho_{a,\lambda}$.

Example 4.2. We consider the Cauchy transform of $\rho_{1,\lambda}$:

$$\begin{aligned} G_{\rho_{1,\lambda}}(z) &= 1/F_{\rho_{1,\lambda}}(z) \\ &= \frac{2}{z + (\lambda + 1) + i + \sqrt{(z + (\lambda + 1) + i)^2 - 4(z + i)}} \\ &= \frac{z + (\lambda + 1) + i - \sqrt{(z + (\lambda + 1) + i)^2 - 4(z + i)}}{2(z + i)} \\ &= \frac{z^2 + (\lambda + 1)(z - i) + 1 - (z - i)\sqrt{(z + (\lambda + 1) + i)^2 - 4(z + i)}}{2(z^2 + 1)}, \end{aligned}$$

for $z \in \mathbb{C}^+$. By the Stieltjes inversion formula, the probability density function $f(x)$ of $\rho_{1,\lambda}$ is

$$\begin{aligned} f(x) &= -\frac{1}{\pi} \lim_{y \downarrow 0} \operatorname{Im}(G_{\rho_{1,\lambda}}(x + iy)) \\ &= \frac{1}{2\pi(x^2 + 1)} \{ \lambda + 1 + xv(x) - u(x) \}, \quad x \in \mathbb{R}, \end{aligned}$$

where

$$u(x) = \operatorname{Re} \left[\sqrt{(x + (\lambda + 1) + i)^2 - 4(x + i)} \right], \quad x \in \mathbb{R}$$

and

$$v(x) = \operatorname{Im} \left[\sqrt{(x + (\lambda + 1) + i)^2 - 4(x + i)} \right], \quad x \in \mathbb{R}.$$

We give an explicit density function for $\lambda = 1/4$ below. Set

$$\begin{aligned} q(x) &:= \left(x + \left(\frac{1}{4} + 1\right) + i\right)^2 - 4(x + i) \\ &= \frac{1}{16}(4x - 3)^2 + \frac{1}{2}(4x - 3)i, \quad x \in \mathbb{R}. \end{aligned}$$

We then obtain $q(x) = re^{i\theta}$, where

$$r = |q(x)| = \frac{1}{16}|4x - 3|\sqrt{16x^2 - 24x + 73}$$

and

$$\theta = \arg(q(x)) = \begin{cases} \arcsin\left(\frac{8}{\sqrt{16x^2 - 24x + 73}}\right), & x \geq 3/4 \\ 2\pi - \arcsin\left(\frac{8}{\sqrt{16x^2 - 24x + 73}}\right), & x < 3/4. \end{cases}$$

Using some calculations, we find that

$$\begin{aligned} u(x) &= \operatorname{Re}\sqrt{q(x)} = \sqrt{r} \cos\left(\frac{1}{2}\theta\right) \\ &= \frac{1}{4}\operatorname{sign}(4x - 3)\sqrt{|4x - 3|}(16x^2 - 24x + 73)^{1/4} \cos\left(\frac{1}{2}\arcsin\left(\frac{8}{\sqrt{16x^2 - 24x + 73}}\right)\right) \\ &= \frac{1}{4\sqrt{2}}\operatorname{sign}(4x - 3)\sqrt{|4x - 3|}\sqrt{\sqrt{16x^2 - 24x + 73} + |4x - 3|} \end{aligned}$$

and

$$\begin{aligned} v(x) &= \operatorname{Im}\sqrt{q(x)} = \sqrt{r} \sin\left(\frac{1}{2}\theta\right) \\ &= \frac{1}{4}\sqrt{|4x - 3|}(16x^2 - 24x + 73)^{1/4} \sin\left(\frac{1}{2}\arcsin\left(\frac{8}{\sqrt{16x^2 - 24x + 73}}\right)\right) \\ &= \frac{1}{4\sqrt{2}}\sqrt{|4x - 3|}\sqrt{\sqrt{16x^2 - 24x + 73} - |4x - 3|}, \end{aligned}$$

where

$$\operatorname{sign}(x) = \begin{cases} 1, & x \geq 0 \\ -1, & x < 0. \end{cases}$$

Finally, we obtain the density function of $\rho_{1,1/4}$ (see Figure 4.1):

$$\begin{aligned} f(x) &= \frac{1}{2\pi(x^2 + 1)} \left(\frac{5}{4} + xv(x) - u(x)\right) \\ &= \frac{5\sqrt{2} + \sqrt{|4x - 3|}(xp_-(x) - \operatorname{sign}(4x - 3)p_+(x))}{8\sqrt{2}\pi(x^2 + 1)}, \quad x \in \mathbb{R}, \end{aligned}$$

where

$$p_{\pm}(x) := \sqrt{\sqrt{16x^2 - 24x + 73} \pm |4x - 3|}, \quad x \in \mathbb{R}.$$

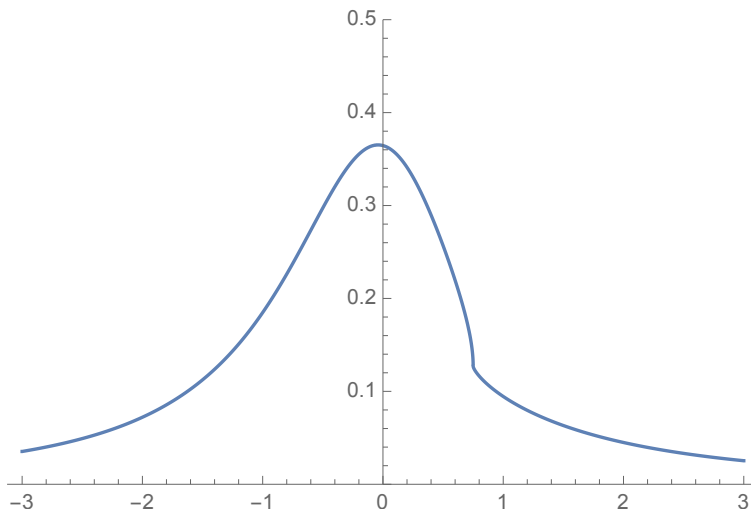


FIGURE 4.1. Probability density function of $\rho_{1,1/4}$

4.1.2. *Case of the semicircle law.* Next, consider

$$\varphi(z) = \varphi_{\mathbf{C}_a}(z) - \varphi_{\mathbf{S}(0,\sigma^2)}(z) = -ai - \frac{\sigma^2}{z}$$

for $a, \sigma > 0$ and $z \in \mathbb{C}^+$. We will show that φ is the Voiculescu transform of some probability measure.

Let $K(z) := z - ai - \frac{\sigma^2}{z}$. Then, a formal computation implies that

$$F(z) := K^{-1}(z) = \frac{z + ai + \sqrt{(z + ai)^2 + 4\sigma^2}}{2}, \quad z \in \mathbb{C}^+.$$

It is easy to see that $F(iy)/iy \rightarrow 1$ as $y \rightarrow \infty$. For $2\sigma \leq a$, $\sqrt{(z + ai)^2 + 4\sigma^2}$ is in \mathbb{C}^+ for all $z \in \mathbb{C}^+$. In a similar manner as for Example 4.1, there is a probability measure γ_{a,σ^2} such that $F = F_{\gamma_{a,\sigma^2}}$, and therefore $\varphi = \varphi_{\gamma_{a,\sigma^2}}$ for all $a, \sigma > 0$ with $2\sigma \leq a$. By definition, it satisfies $\mathbf{S}(0, \sigma^2) \boxplus \gamma_{a,\sigma^2} = \mathbf{C}_a$, where γ_{a,σ^2} is not FID. Moreover, the measure γ_{a,σ^2} is absolutely continuous with respect to the Lebesgue measure. We summarize the above facts as follows.

Theorem 4.3. *Let $a, \sigma > 0$ with $2\sigma \leq a$. Then, there exists some $\gamma_{a,\sigma} \in \mathcal{P}(\mathbb{R})$ for which the Voiculescu transform is*

$$\varphi_{\gamma_{a,\sigma^2}}(z) = -ai - \frac{\sigma^2}{z}, \quad z \in \mathbb{C}^+.$$

It has the following properties:

- (1) $\mathbf{S}(0, \sigma^2) \boxplus \gamma_{a,\sigma^2} = \mathbf{C}_a$. In other words, $\gamma_{a,\sigma^2} = \mathbf{C}_a \boxminus \mathbf{S}(0, \sigma^2)$.
- (2) γ_{a,σ^2} is FQID, rather than FID. Moreover, it is absolutely continuous with respect to the Lebesgue measure.
- (3) $\gamma_{a,\sigma^2} = \gamma_{1,(\sigma/\sqrt{a})^2}^{\boxplus a} = \gamma_{a/\sigma^2,1}^{\boxplus \sigma^2}$ for $a \geq 1$ and $\sigma \geq 1$.
- (4) Its free characteristic triplet is given by $(-\sigma^2, \frac{a du}{\pi u^2}, 0)$.
- (5) Its free characteristic pair is $(0, \frac{a du}{\pi(1+u^2)} - \sigma^2 \delta_0(du))$.

In particular, $\gamma_{2\sqrt{2}\sigma,\sigma^2}$ (the case in which $a = 2\sqrt{2}\sigma$) coincides with the measure $\rho(\mathbf{S}(0, \sigma^2))$ in (4.2).

Remark 4.4.

- (1) The FQID distribution γ_{a,σ^2} has a semicircular component that takes negative values. However, if we consider the case of $a = 0$, then $\varphi(z) = -\frac{\sigma^2}{z}$ is not the Voiculescu transform of any probability measure.
- (2) In general, for a classical characteristic triplet, the Gaussian component must be nonnegative, see [Lindner et al. \(2018, Lemma 2.7\)](#). Therefore the triplet $(-\sigma^2, \frac{adu}{\pi u^2}, 0)$ in [Theorem 4.3](#) is not a classical characteristic triplet.

Example 4.5. Consider $a = 1$ and $\sigma = 1/2$. The Cauchy transform of $\gamma_{1,(1/2)^2}$ is given by

$$G_{\gamma_{1,(1/2)^2}}(z) = \frac{2}{z+i+\sqrt{(z+i)^2+1}} = -2\left(z+i-\sqrt{z^2+2zi}\right), \quad z \in \mathbb{C}^+.$$

By the Stieltjes inversion formula, the probability density function $f(x)$ of $\gamma_{1,(1/2)^2}$ is

$$\begin{aligned} f(x) &= -\frac{1}{\pi} \lim_{y \downarrow 0} \operatorname{Im}(G_{\gamma_{1,(1/2)^2}}(x+iy)) \\ &= \frac{2}{\pi} \left(1 - \operatorname{Im} \left[\sqrt{x^2+2xi}\right]\right) \\ &= \frac{\sqrt{2}}{\pi} \left(\sqrt{2} - \sqrt{|x|\sqrt{x^2+4-x^2}}\right), \quad x \in \mathbb{R}. \end{aligned}$$

[Figure 4.2](#) shows the shape of the probability density function of $\gamma_{1,(1/2)^2}$. Clearly, it is not differentiable at $x = 0$.

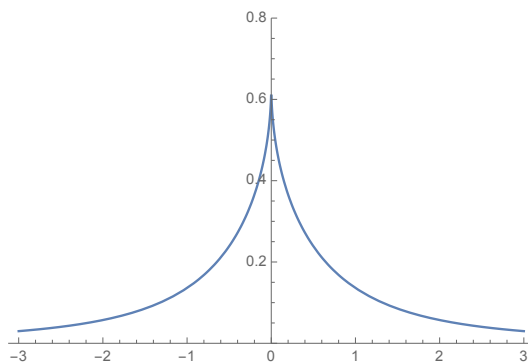


FIGURE 4.2. Probability density function of $\gamma_{1,(1/2)^2}$

4.1.3. *Case of the free Meixner distribution.* We define $\mathbf{FM}_{a,b}$ as the free Meixner distribution:

$$\mathbf{FM}_{a,b}(\mathrm{d}x) := \frac{\sqrt{4(1+b)-(x-a)^2}}{2\pi(bx^2+ax+1)} \mathbf{1}_{[a-2\sqrt{1+b}, a+2\sqrt{1+b}]}(x) \mathrm{d}x + 0, 1, \text{ or } 2 \text{ atoms},$$

for $a \in \mathbb{R}$ and $b \geq -1$. If $b = -1$, then

$$\mathbf{FM}_{a,-1} = \frac{1}{2} \left(1 + \frac{a}{\sqrt{1+a^2}}\right) \delta_{\frac{1}{2}(a-\sqrt{4+a^2})} + \frac{1}{2} \left(1 - \frac{a}{\sqrt{1+a^2}}\right) \delta_{\frac{1}{2}(a+\sqrt{4+a^2})}$$

is not FQID for all $a \in \mathbb{R}$ by [Remark 3.9](#). Moreover, it follows from [Bożejko and Bryc \(2006, Proposition 2.1\)](#) that, for $-1 < b < 0$,

$$\mathbf{FM}_{a,b} = \mathbf{D}_{\sqrt{|b|}} \left(\mathbf{FM}_{\frac{a}{\sqrt{|b|}}, -1}^{\boxplus t} \right), \quad t = -\frac{1}{b}.$$

Therefore, in view of Proposition 3.3(ii), $\mathbf{FM}_{a,b}$ is not FQID for $-1 \leq b < 0$. Note that $\mathbf{FM}_{a,b} \in \text{ID}(\boxplus)$ if and only if $a \in \mathbb{R}$ and $b \geq 0$, see Saitoh and Yoshida (2001, Theorem 3.2). Furthermore, the variance of $\mathbf{FM}_{0,b}$ is equal to 1 for all $b \geq 0$ (in particular, we have $\mathbf{FM}_{0,0} = \mathbf{S}$). Using Arizmendi et al. (2020, Theorem 1.2), for any $b \geq 0$, there exists an FQID distribution $\rho(\mathbf{FM}_{0,b})$ (in (4.2)) such that

$$\mathbf{FM}_{0,b} \boxplus \rho(\mathbf{FM}_{0,b}) = \mathbf{C}_{2\sqrt{2}}.$$

We summarize the above result as follows.

Theorem 4.6. *For any $b \geq 0$, there exists some $\rho(\mathbf{FM}_{0,b})$ such that $\mathbf{FM}_{0,b} \boxplus \rho(\mathbf{FM}_{0,b}) = \mathbf{C}_{2\sqrt{2}}$. For $b \geq 0$, the semicircular component of the measure $\rho(\mathbf{FM}_{0,b})$ is equal to -1 , and so the measure $\rho(\mathbf{FM}_{0,b})$ is not FID. Furthermore, for $b > 0$, the free quasi-Lévy measure ν_b of the probability measure $\rho(\mathbf{FM}_{0,b})$ is given by*

$$\nu_b(dx) = \begin{cases} \frac{1}{\pi x^2} \left(2\sqrt{2} - \frac{\sqrt{4b-x^2}}{2b} \right) dx, & x \in [-2\sqrt{b}, 2\sqrt{b}] \setminus \{0\} \\ \frac{2\sqrt{2}}{\pi x^2} dx, & x \in \mathbb{R} \setminus [-2\sqrt{b}, 2\sqrt{b}]. \end{cases}$$

In particular, we can state the following:

- (1) If $b \geq 1/8$, then the density function $\frac{d\nu_b}{dx}(x)$ is always positive;
- (2) If $0 < b < 1/8$, then there exists some $x \in [-2\sqrt{b}, 2\sqrt{b}] \setminus \{0\}$ such that the density function $\frac{d\nu_b}{dx}(x)$ of the measure ν_b is negative. In this case, we have $\nu_b(\mathbb{R}) = -\infty$. For example, we show the density function of the free quasi-Lévy measure $\nu_{1/16}$ in Figure 4.3;
- (3) If $b = 0$, then the free quasi-Lévy measure of $\rho(\mathbf{FM}_{0,b})$ coincides with the free Lévy measure of the Cauchy distribution $\mathbf{C}_{2\sqrt{2}}$.

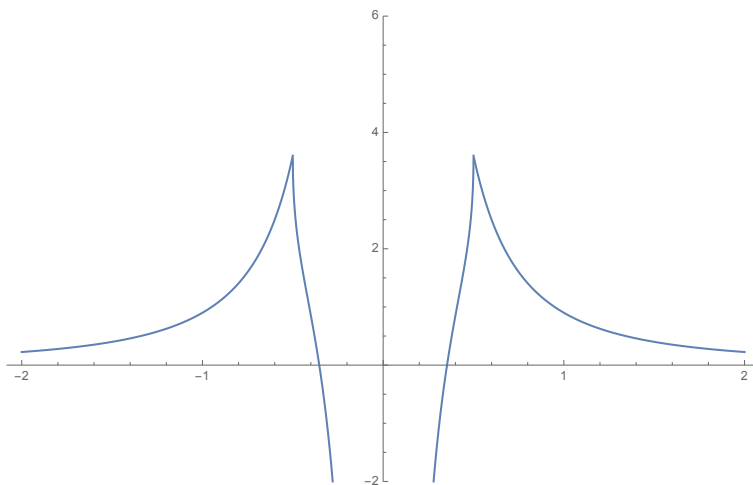


FIGURE 4.3. Density function of the free quasi-Lévy measure $\nu_{1/16}$

4.2. *Free deconvolution of the semicircle law and failure of Cramér’s theorem in free probability.* For $t > 0$, denote

$$R_t^\pm(z) := z^2 \pm tz^4 \sum_{n=0}^\infty t^n z^{2n} = \frac{z^2 - tz^4 \pm tz^4}{1 - tz^2}.$$

Observe that

$$R_t^+(z) = \frac{z^2}{1 - tz^2} = \frac{1}{2\sqrt{t}} \cdot \frac{z\sqrt{t}}{1 - z\sqrt{t}} + \frac{1}{2\sqrt{t}} \cdot \frac{-z\sqrt{t}}{1 + z\sqrt{t}}.$$

This means that $R_t^+(z)$ is the R -transform of a probability distribution $\mu_+(t)$, which can be expressed as

$$\mu_+(t) = \mathbf{MP} \left(\sqrt{t}, \frac{1}{2\sqrt{t}} \right) \boxplus \mathbf{MP} \left(-\sqrt{t}, \frac{1}{2\sqrt{t}} \right).$$

In particular, $\mu_+(t)$ is FID for all $t > 0$.

For the function $R_t^-(z)$, we apply a result by [Bercovici and Voiculescu \(1995, Theorem 2\)](#), which implies that if $t > 0$ is sufficiently small, say $0 < t < t_0$, then $R_t^-(z)$ is the R -transform of some compactly supported probability distribution $\mu_-(t)$ on \mathbb{R} . Then, for $0 < t < t_0$, we have $R_{\mu_+(t)}(z) + R_{\mu_-(t)}(z) = 2z^2$, which means that $\mu_+(t) \boxplus \mu_-(t) = \mathbf{S}(0, 2)$, and consequently,

$$\mu_-(t) = \mathbf{S}(0, 2) \boxminus \mu_+(t).$$

In view of [Bercovici and Voiculescu \(1995, Remark 5\)](#), the measure $\mu_-(t)$ cannot be FID, which is also a consequence of the fact that the fourth free cumulant of $\mu_-(t)$ is $-t < 0$.

This example illustrates the observation of [Bercovici and Voiculescu \(1995\)](#) that the free analog of Cramér’s decomposition theorem [Feller \(1971, Theorem XV.8.1\)](#) (that the Gaussian distribution cannot be expressed as the classical convolution of two non-Gaussian distributions) is not true.

4.3. Free deconvolution with the Marchenko-Pastur law.

4.3.1. *Case of the semicircle law and the Marchenko-Pastur law.* In view of [Młotkowski \(2021, Proposition 5.1\)](#), for every $u, x \in \mathbb{R}$, $u \neq 0$, there exists a probability distribution μ such that

$$R_\mu(z) = x^3 u^2 z^2 + (1 - x)^3 \frac{uz}{1 - uz}.$$

This means that

$$(u^2 x^3, (1 - x)^3 \delta_u, 0)$$

is a free characteristic triplet for the measure μ . If $x < 0$ then we have

$$\mu = \mathbf{MP}(u, (1 - x)^3) \boxminus \mathbf{S}(0, -x^3 u^2),$$

while for $x > 1$

$$\mu = \mathbf{S}(0, x^3 u^2) \boxminus \mathbf{MP}(u, (x - 1)^3 u^2).$$

If $x < 0$, then the Gaussian component $u^2 x^3$ is negative, and therefore $(u^2 x^3, (1 - x)^3 \delta_u, 0)$ is not a classical characteristic triplet (see [Lindner et al. \(2018, Lemma 2.7\)](#)). The following lemma, which was proved within [Lindner et al. \(2018, Example 2.9\)](#), implies that if $x > 1$ then $(u^2 x^3, (1 - x)^3 \delta_u, 0)$ is not a classical characteristic triplet.

Lemma 4.7. *Assume that ν is a quasi-Lévy measure, with the positive and negative part ν^+, ν^- . If $\nu^- \neq 0$ and if either $\nu^+ = 0$ or $\text{supp } \nu^+$ is a one-point set, then (a, ν, γ) is not the characteristic triplet of a QID distribution for any $a, \gamma \in \mathbb{R}$.*

4.3.2. *Case of two Marchenko-Pastur laws.* Assume that $u, v \in \mathbb{R} \setminus \{0\}$, $u < v$. It was proven in [Młotkowski \(2021, Proposition 5.3\)](#) that for every $x \in \mathbb{R}$ there exists a compactly supported probability measure μ such that

$$R_\mu(z) = \frac{(u - x)^3}{u^2(u - v)} \cdot \frac{uz}{1 - uz} + \frac{(v - x)^3}{v^2(v - u)} \cdot \frac{vz}{1 - vz}.$$

Putting

$$a(x) := \frac{(u - x)^3}{u^2(u - v)}, \quad b(x) := \frac{(v - x)^3}{v^2(v - u)}, \tag{4.5}$$

we have $a(x) < 0, b(x) > 0$ for $x < u$, $a(x), b(x) > 0$ for $u < x < v$ and $a(x) > 0, b(x) < 0$ for $x > v$. Therefore

$$\mu = \mathbf{MP}(v, b(x)) \boxminus \mathbf{MP}(u, -a(x))$$

for $x < u$ and

$$\mu = \mathbf{MP}(u, a(x)) \boxplus \mathbf{MP}(v, -b(x))$$

for $x > v$. In other words, the free characteristic triplet for μ is $(0, \nu, 0)$, where

$$\nu := a(x) \cdot \delta_u + b(x) \cdot \delta_v. \tag{4.6}$$

Note that if $u + v \neq 0$ then the sum of weights:

$$a(x) + b(x) = \frac{u^2v^2 - 3uvx^2 + (u + v)x^3}{u^2v^2}$$

can be negative.

Again, by Lemma 4.7, if either $x < u$ or $x > u$, then $(0, \nu, 0)$, with ν defined by (4.5), (4.6), is not a classical characteristic triplet.

4.3.3. *Case of several Marchenko-Pastur laws.* Suppose that $u_1, \dots, u_n \in \mathbb{R} \setminus \{0\}$ are distinct and $n \geq 2$. In view of Liszewska and Młotkowski (2020, Proposition 3.1 and Corollary 2.7), there is a probability distribution μ on \mathbb{R} such that its R -transform is equal to

$$R_\mu(z) = \frac{1 - \prod_{i=1}^n (1 - u_i z)}{\prod_{i=1}^n (1 - u_i z)}. \tag{4.7}$$

If we set

$$t_k := \frac{u_k^{n-1}}{\prod_{i \neq k} (u_k - u_i)},$$

then

$$R_\mu(z) = \sum_{k=1}^n t_k \cdot \frac{u_k z}{1 - u_k z}. \tag{4.8}$$

In the case of $n = 2$, if $u_1 < u_2 < 0$, then $t_1 > 0 > t_2$; if $u_1 < 0 < u_2$, then $t_1, t_2 > 0$; and if $0 < u_1 < u_2$, then $t_1 < 0 < t_2$, see Liszewska and Młotkowski (2020) for details. If $n \geq 3$, then $t_i > 0$ and $t_j < 0$ for some $1 \leq i, j \leq n$, see Liszewska and Młotkowski (2020, Lemma 3.5).

Now, assume that either $n = 2, u_1 u_2 > 0$, or $n \geq 3$. We observe that μ is FQID. Set

$$K_+ := \{k : t_k > 0\} := \{k'_1, \dots, k'_p\}, \quad K_- := \{k : t_k < 0\} := \{k''_1, \dots, k''_q\}$$

and

$$\mu_+ := \mu_{k'_1} \boxplus \dots \boxplus \mu_{k'_p}, \quad \mu_- := \mu_{k''_1} \boxplus \dots \boxplus \mu_{k''_q},$$

where

$$\mu_k := \begin{cases} \mathbf{MP}(u_k, t_k) & \text{if } k \in K_+, \\ \mathbf{MP}(u_k, -t_k) & \text{if } k \in K_-. \end{cases}$$

Then, by (4.8), we have the following deconvolution of μ :

$$\mu_- \boxplus \mu = \mu_+;$$

the characteristic triplet for μ is $(0, \nu, 0)$, where $\nu := \sum_{k=1}^n t_k \cdot \delta_{u_k}$. Note that $\nu(\mathbb{R}) = 1$. Indeed, taking the limit $|z| \rightarrow \infty$ in (4.7) and (4.8), we find that $t_1 + \dots + t_n = 1$. For example, if $n = 4$ and $u_k = k$, then

$$t_1 = -1/6, \quad t_2 = 4, \quad t_3 = -27/2, \quad t_4 = 32/3.$$

One can check that if n is even and the set $\{u_1, \dots, u_n\}$ is symmetric, i.e., $u_k = -u_{n-k+1}, 1 \leq k \leq n$, then ν is symmetric: $t_k = t_{n-k+1}$.

4.4. *Fuss-Catalan distributions.* The Fuss-Catalan numbers are defined by

$$A_n(p, r) := \binom{np+r}{n} \frac{r}{np+r},$$

where p, r are real parameters and $\binom{x}{n} := x(x-1)\dots(x-n+1)/n!$ is the generalized binomial coefficient. The corresponding generating function is of the form

$$\sum_{n=0}^{\infty} A_n(p, r)z^n = \mathcal{B}_p(z)^r,$$

where the function $\mathcal{B}_p(z)$ satisfies the equation $\mathcal{B}_p(z) = 1 + z\mathcal{B}_p(z)^p$ on a neighborhood of $z = 0$, see [Graham et al. \(1994\)](#). Consider $r > 0$. It is known in [Młotkowski \(2010\)](#); [Młotkowski et al. \(2013\)](#); [Młotkowski and Penson \(2014\)](#); [Liu and Pego \(2016\)](#), that the sequence $\{A_n(p, r)\}_{n=0}^{\infty}$ is positive definite if and only if $p \geq 1$ and $r \leq p$. Let $\mu(p, r)$ denote the corresponding probability distribution on \mathbb{R} . The measure $\mu(p, r)$ is called the *Fuss-Catalan distribution*. Then $\mu(p, r)$ is FID if and only if either $1 \leq p = r \leq 2$ or $r \leq \min\{p/2, p-1\}$, see [Młotkowski \(2010\)](#); [Młotkowski et al. \(2020\)](#).

Theorem 4.8. *If $p > 2$ and $p/2 < r < p - 1$ then $\mu(p, r)$ is FQID.*

Proof: For $p > 1$ and $r > 0$ the sequence $A_n(p, r)$ admits the following integral representation

$$A_n(p, r) = \int_0^{c(p)} x^n W_{p,r}(x) dx,$$

where $c(p) := p^p(p-1)^{1-p}$ and $W_{p,r}(x)$ is given by

$$W_{p,r}(\rho_p(\varphi)) = \frac{(\sin((p-1)\varphi))^{p-r-1} \sin(\varphi) \sin(r\varphi)}{\pi(\sin(p\varphi))^{p-r}},$$

where

$$\rho_p(\varphi) := \frac{(\sin(p\varphi))^p}{\sin(\varphi)(\sin((p-1)\varphi))^{p-1}},$$

is a decreasing function on the interval $(0, \pi/p)$, which maps $(0, \pi/p)$ onto $(0, c(p))$, see [Forrester and Liu \(2015\)](#). Note that if $r > p$ then $W_{p,r}(x)$ is bounded on $(0, c(p))$ and admits positive and negative values. Put

$$W_{p,r}^+(x) := \max\{0, W_{p,r}(x)\} \quad \text{and} \quad W_{p,r}^-(x) := -\min\{0, W_{p,r}(x)\}.$$

By [Młotkowski \(2010, Proposition 4.2\)](#), the R -transform of $\mu(p, r)$ is

$$R_{\mu(p,r)}(z) = \mathcal{B}_{p-r}(z)^r - 1$$

Therefore, if $p > 2$ and $p/2 < r < p - 1$ then

$$\begin{aligned} R_{\mu(p,r)}(z) &= rz + \int_0^{c(p)} \frac{z^2}{1-xz} W_{p-r,r}(x) dx \\ &= rz + \int_0^{c(p)} \frac{z^2}{1-xz} W_{p-r,r}^+(x) dx - \int_0^{c(p)} \frac{z^2}{1-xz} W_{p-r,r}^-(x) dx \\ &= R_{\mu_+}(z) - R_{\mu_-}(z), \end{aligned}$$

where μ_+, μ_- are FID distributions such that

$$R_{\mu_{\pm}}(z) = \int_0^{c(p)} \frac{z}{1-xz} W_{p,r}^{\pm}(x) dx$$

and $\mu_- \boxplus \mu(p, r) = \mu_+$, see [Nica and Speicher \(2006, Theorem 13.16\)](#). □

4.5. *Classical characteristic triplets which are not free triplets.* Take $\mu_a := (1 - a)\delta_0 + a\delta_1$, with $0 < a < 1$, $a \neq 1/2$. Then μ_a is QID by Lindner et al. (2018, Theorem 3.9). More precisely, if $0 < a < 1/2$ then the characteristic triplet for μ_a is $\tau_a = (1, \nu_a, 0)$, where

$$\nu_a = - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{a}{a-1} \right)^m \delta_m,$$

while if $1/2 < a < 1$ then the characteristic triplet is $\tau_a = (0, \nu_a, 0)$, with

$$\nu_a = - \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{a-1}{a} \right)^m \delta_{-m}.$$

Proposition 4.9. *If $0 < a < 1$, $a \neq 1/2$, then τ_a is not a free characteristic triplet.*

Proof: The characteristic function of μ_a is given by

$$\widehat{\mu}_a(t) = 1 - a + a \exp(it).$$

Since

$$\begin{aligned} \log \widehat{\mu}_a(-it) &= \log(1 - a + a \exp(t)) \\ &= at + \frac{1}{2}(a - a^2)t^2 + \frac{1}{6}(a - 3a^2 + 2a^3)t^3 + \frac{1}{24}(a - 7a^2 + 12a^3 - 6a^4)t^4 + \dots, \end{aligned}$$

the classical cumulants of μ_a are

$$r_1 = a, \quad r_2 = a - a^2, \quad r_3 = a - 3a^2 + 2a^3, \quad r_4 = a - 7a^2 + 12a^3 - 6a^4, \dots$$

If τ_a was the free characteristic triplet for a probability distribution $\tilde{\mu}_a$, then r_n were the free cumulants for $\tilde{\mu}_a$. This, by the moment-cumulant formula (see Nica and Speicher (2006, Proposition 11.4)), would imply, that the moments of $\tilde{\mu}_a$ are

$$s_0 = 1, \quad s_1 = s_2 = s_3 = a, \quad s_4 = a - a^2 + 2a^3 - a^4, \dots,$$

which leads to a contradiction, because $\det(s_{i+j})_{i,j=0}^2 = a^3(a-1)^3 < 0$. □

5. An extension of the Bercovici-Pata bijection

In the previous section, we gave some examples of free characteristic triplets which are not classical characteristic triplets, and also classical characteristic triplets which are not free characteristic triplets. Here we are going to give some examples of nonpositive triplets, which are at the same time classical and free characteristic triplets.

Define Φ as the set of such pairs (c, ν) , that c is a positive number and ν is a symmetric Lévy measure, which satisfies

$$\int_{\mathbb{R}} (x^2 \vee |x|) \nu(dx) < \infty. \tag{5.1}$$

Note that (5.1) guarantees that the second moment $m_2(\nu)$ of ν is finite. We define three subsets of Φ , namely, Φ^+ is the set of these $(c, \nu) \in \Phi$ that the measure

$$\frac{c}{\pi x^2} dx - \nu(dx)$$

is nonnegative, and Φ^* (respectively, Φ^{\boxplus}) will denote the set of such $(c, \nu) \in \Phi$ that

$$\left(0, \frac{c}{\pi x^2} dx - \nu(dx), 0 \right)$$

is a classical (respectively, free) characteristic triplet. For $(c, \nu) \in \Phi^*$ (resp. Φ^{\boxplus}) we denote by $\mu^*(c, \nu)$ (resp. $\mu^{\boxplus}(c, \nu)$) the corresponding classical (resp. freely) quasi-infinitely divisible distribution. We are going to show that the sets $\Phi^* \setminus \Phi^+$, $\Phi^{\boxplus} \setminus \Phi^+$ are nonempty, and, remarkably, that

the set $\Phi^* \cap \Phi^\boxplus \setminus \Phi^+$ is nonempty. This will allow us to extend the Bercovici-Pata bijection, thus to answer affirmatively a question raised by Bożejko.

Proposition 5.1. *If $(c, \nu) \in \Phi$ and if the function*

$$\phi(t) := \exp \left(-c|t| + 2 \int_0^{+\infty} (1 - \cos tx) \nu(dx) \right) \tag{5.2}$$

is convex on $t \in [0, +\infty)$ then $(c, \nu) \in \Phi^$ and ϕ is the characteristic function of the corresponding distribution $\mu^*(c, \nu)$. In particular, if*

$$2 \int_0^\infty x^2 \cos tx \nu(dx) + \left(-c + 2 \int_0^\infty x \sin tx \nu(dx) \right)^2 \geq 0 \tag{5.3}$$

for all $t > 0$ then $(c, \nu) \in \Phi^$.*

Proof: Since

$$\int_{\mathbb{R}} (e^{itx} - 1 - itx \mathbf{1}_{[-1,1]}(x)) \left(\frac{c}{\pi x^2} dx - \nu(dx) \right) = -c|t| + 2 \int_0^\infty (1 - \cos tx) \nu(dx),$$

the first statement is a consequence of Pólya’s theorem; see Pólya (1949) or Feller (1971, Section XV.3). Denoting the left hand side of (5.3) by $A(t)$ we have $\phi''(t) = A(t)\phi(t)$, which implies the second statement. \square

Proposition 5.2. *If $(c, \nu) \in \Phi$ and $c \geq \sqrt{8m_2(\nu)}$, where $m_2(\nu)$ denotes the second moment of ν , then $(c, \nu) \in \Phi^\boxplus$.*

Proof: Since ν is a Lévy measure, there is an FID distribution μ_ν such that

$$\begin{aligned} R_{\mu_\nu}(z) &= \int_{\mathbb{R}} \left(\frac{1}{1-zx} - 1 - zx \mathbf{1}_{[-1,1]}(x) \right) \nu(dx) \\ &= 2 \int_0^\infty \left(\frac{1}{1-z^2x^2} - 1 \right) \nu(dx). \end{aligned}$$

Consider arbitrary numbers $\alpha, \beta > 0$. As $z \rightarrow 0$ with $z \in \Delta_{\alpha,\beta}$, we get

$$\begin{aligned} R_{\mu_\nu}(z) &= m_2(\nu)z^2 + 2 \int_0^\infty \left(\frac{1}{1-z^2x^2} - 1 - z^2x^2 \right) \nu(dx) \\ &= m_2(\nu)z^2 + o(z^2). \end{aligned}$$

Therefore, by Benaych-Georges (2006, Theorem 1.3), μ_ν has a finite variance equal to $m_2(\nu)$ (note that $R_\mu(z)$ in Benaych-Georges (2006) denotes our $z^{-1}R_\mu(z)$). Hence there is a probability measure $\rho(\mu_\nu)$ on \mathbb{R} such that $\mu_\nu \boxplus \rho(\mu_\nu) = \mathbf{C}_{\sqrt{8m_2(\nu)}}$ by Arizmendi et al. (2020, Theorem 1.2). This implies, that $(c, \nu) \in \Phi^\boxplus$. \square

Now we restrict ourselves to a special case.

Proposition 5.3. *Assume that $\nu = p(\delta_{-\lambda} + \delta_\lambda)$, $p, \lambda, c > 0$, and put*

$$h(p) := \begin{cases} \sqrt{2p(4p+1)} & \text{if } 0 < p \leq \frac{1}{4}, \\ 2p + \sqrt{p} & \text{if } p > \frac{1}{4}. \end{cases}$$

If $c \geq \lambda \cdot h(p)$ then $(c, \nu) \in \Phi^$.*

Proof: Denoting again the left hand side of (5.3) by $A(t)$, and using inequality $\cos \alpha \geq \frac{1}{2} \sin^2 \alpha - 1$, we get in our case

$$\begin{aligned} A(t) &= 2p\lambda^2 \cos \lambda t + (-c + 2p\lambda \sin \lambda t)^2 \\ &\geq (4p+1)p\lambda^2 \sin^2 \lambda t - 4p\lambda c \sin \lambda t - 2p\lambda^2 + c^2. \end{aligned}$$

To conclude it suffices to apply the following elementary observation: if $a > 0$, $|y| \leq 1$ and either $b^2 \leq 4ad$ (applied for $0 < p \leq 1/4$) or $2a \leq |b| \leq a+d$ (applied for $p > 1/4$), then $ay^2 + by + d \geq 0$. \square

Putting $c = \sqrt{8m_2(\nu)} = 4\lambda\sqrt{p}$ and applying Proposition 5.3 we obtain

Corollary 5.4. *Assume that $0 < 4p \leq 9$, $\lambda > 0$, $\nu = p(\delta_{-\lambda} + \delta_\lambda)$ and $c = \sqrt{8m_2(\nu)} = 4\lambda\sqrt{p}$. Then $(c, \nu) \in \Phi^* \cap \Phi^\boxplus \setminus \Phi^+$. Consequently,*

$$\left(0, \frac{c}{\pi x^2} dx - \nu(dx), 0\right)$$

is both a classical and a free characteristic triplet.

Remark 5.5. Let $\mathcal{T}(\ast)$ and $\mathcal{T}(\boxplus)$ denote the set of classical and free characteristic triplets, respectively, and let \mathcal{T} denote the set of characteristic triplets corresponding to the (classically or freely) infinitely divisible distributions. From Corollary 5.4 we see that $\mathcal{T} \subsetneq \mathcal{T}(\ast) \cap \mathcal{T}(\boxplus)$. We have also seen that the set $\mathcal{T}(\boxplus) \setminus \mathcal{T}(\ast)$ is nonempty (Remark 4.4, subsections 4.3.1 and 4.3.2) and the set $\mathcal{T}(\ast) \setminus \mathcal{T}(\boxplus)$ is nonempty (Proposition 4.9).

Corollary 5.4 allows us to extend the Bercovici-Pata bijection. Define two families of distributions:

$$\begin{aligned} \mathcal{Q}(\ast) &:= \{\mu \ast \mu^\ast(c, \nu) : \mu \in \text{ID}(\ast), (c, \nu) \in \Phi^* \cap \Phi^\boxplus\}, \\ \mathcal{Q}(\boxplus) &:= \{\mu \boxplus \mu^\boxplus(c, \nu) : \mu \in \text{ID}(\boxplus), (c, \nu) \in \Phi^* \cap \Phi^\boxplus\}. \end{aligned}$$

We have $\text{ID}(\ast) \subsetneq \mathcal{Q}(\ast)$ and $\text{ID}(\boxplus) \subsetneq \mathcal{Q}(\boxplus)$. Note that the set $\Phi^* \cap \Phi^\boxplus$ is closed under addition, which implies that $\mathcal{Q}(\ast)$ and $\mathcal{Q}(\boxplus)$ are closed under the classical and the free convolution respectively. Now we can define an extension of the Bercovici-Pata bijection as $\tilde{\Lambda} : \mathcal{Q}(\ast) \rightarrow \mathcal{Q}(\boxplus)$ by

$$\tilde{\Lambda}(\mu \ast \mu^\ast(c, \nu)) := \Lambda(\mu) \boxplus \mu^\boxplus(c, \nu),$$

for $\mu \in \text{ID}(\ast)$, $(c, \nu) \in \Phi^* \cap \Phi^\boxplus$. This map is well defined, as the characteristic function and the R -transform characterize the corresponding distribution uniquely. One can see that $\tilde{\Lambda}$ is an isomorphism between $(\mathcal{Q}(\ast), \ast)$ to $(\mathcal{Q}(\boxplus), \boxplus)$. The bijectivity is clear by the definition of $\tilde{\Lambda}$. Moreover, $\tilde{\Lambda}$ is a homomorphism relative to \ast and \boxplus . Indeed, if $\rho_i = \mu_i \ast \mu^\ast(c_i, \nu_i) \in \mathcal{Q}(\ast)$ for $i = 1, 2$, then

$$\begin{aligned} \tilde{\Lambda}(\rho_1 \ast \rho_2) &= \tilde{\Lambda}((\mu_1 \ast \mu_2) \ast \mu^\ast(c_1 + c_2, \nu_1 + \nu_2)) \\ &= \Lambda(\mu_1 \ast \mu_2) \boxplus \mu^\boxplus(c_1 + c_2, \nu_1 + \nu_2) \\ &= (\Lambda(\mu_1) \boxplus \Lambda(\mu_2)) \boxplus (\mu^\boxplus(c_1, \nu_1) \boxplus \mu^\boxplus(c_2, \nu_2)) \\ &= \tilde{\Lambda}(\rho_1) \boxplus \tilde{\Lambda}(\rho_2). \end{aligned}$$

Recall that, the original Bercovici-Pata bijection Λ is weakly continuous. Unfortunately, we still do not know whether the extension $\tilde{\Lambda}$ is weakly continuous or not.

Problem 5.6. *Is the map $\tilde{\Lambda}$ weakly continuous? Further, are the two classes $\mathcal{Q}(\ast)$ and $\mathcal{Q}(\boxplus)$ closed with respect to weak topology?*

Acknowledgements

The authors would also like to thank the referees who gave useful comments to improve this paper. IH was supported by JSPS KAKENHI Grant Numbers JP20K03632. NS was supported by JSPS KAKENHI Grant Numbers JP19K03515, JP23K03133, JP19H01791, JPJSBP120209921, JPJSBP120203202. YU was supported by JSPS KAKENHI Grant Numbers JP22K13925, JP19H01791, JPJSBP120209921, JPJSBP120203202.

References

- Arizmendi, O. and Hasebe, T. Classical scale mixtures of Boolean stable laws. *Trans. Amer. Math. Soc.*, **368** (7), 4873–4905 (2016). [MR3456164](#).
- Arizmendi, O., Tarrago, P., and Vargas, C. Subordination methods for free deconvolution. *Ann. Inst. Henri Poincaré Probab. Stat.*, **56** (4), 2565–2594 (2020). [MR4164848](#).
- Barndorff-Nielsen, O. E. and Thorbjørnsen, S. Self-decomposability and Lévy processes in free probability. *Bernoulli*, **8** (3), 323–366 (2002). [MR1913111](#).
- Barndorff-Nielsen, O. E. and Thorbjørnsen, S. Classical and free infinite divisibility and Lévy processes. In *Quantum independent increment processes. II*, volume 1866 of *Lecture Notes in Math.*, pp. 33–159. Springer, Berlin (2006). [MR2213448](#).
- Belinschi, S. T. and Bercovici, H. Atoms and regularity for measures in a partially defined free convolution semigroup. *Math. Z.*, **248** (4), 665–674 (2004). [MR2103535](#).
- Belinschi, S. T., Bożejko, M., Lehner, F., and Speicher, R. The normal distribution is \boxplus -infinitely divisible. *Adv. Math.*, **226** (4), 3677–3698 (2011). [MR2764902](#).
- Benaych-Georges, F. Taylor expansions of R -transforms: application to supports and moments. *Indiana Univ. Math. J.*, **55** (2), 465–481 (2006). [MR2225442](#).
- Bercovici, H. and Pata, V. The law of large numbers for free identically distributed random variables. *Ann. Probab.*, **24** (1), 453–465 (1996). [MR1387645](#).
- Bercovici, H. and Pata, V. Stable laws and domains of attraction in free probability theory. *Ann. of Math. (2)*, **149** (3), 1023–1060 (1999). [MR1709310](#).
- Bercovici, H. and Voiculescu, D. Lévy-Hinčin type theorems for multiplicative and additive free convolution. *Pacific J. Math.*, **153** (2), 217–248 (1992). [MR1151559](#).
- Bercovici, H. and Voiculescu, D. Free convolution of measures with unbounded support. *Indiana Univ. Math. J.*, **42** (3), 733–773 (1993). [MR1254116](#).
- Bercovici, H. and Voiculescu, D. Superconvergence to the central limit and failure of the Cramér theorem for free random variables. *Probab. Theory Related Fields*, **103** (2), 215–222 (1995). [MR1355057](#).
- Biane, P. Processes with free increments. *Math. Z.*, **227** (1), 143–174 (1998). [MR1605393](#).
- Bogachev, V. I. *Measure theory. Vol. I, II*. Springer-Verlag, Berlin (2007). ISBN 978-3-540-34513-8; 3-540-34513-2. [MR2267655](#).
- Bożejko, M. and Bryc, W. On a class of free Lévy laws related to a regression problem. *J. Funct. Anal.*, **236** (1), 59–77 (2006). [MR2227129](#).
- Bożejko, M. and Hasebe, T. On free infinite divisibility for classical Meixner distributions. *Probab. Math. Statist.*, **33** (2), 363–375 (2013). [MR3158562](#).
- Feller, W. *An introduction to probability theory and its applications. Vol. II*. John Wiley & Sons, Inc., New York-London-Sydney, second edition (1971). [MR0270403](#).
- Forrester, P. J. and Liu, D.-Z. Raney distributions and random matrix theory. *J. Stat. Phys.*, **158** (5), 1051–1082 (2015). [MR3313617](#).
- Graham, R. L., Knuth, D. E., and Patashnik, O. *Concrete mathematics. A foundation for computer science*. Addison-Wesley Publishing Company, Reading, MA, second edition (1994). ISBN 0-201-55802-5. [MR1397498](#).
- Hasebe, T. Free infinite divisibility for beta distributions and related ones. *Electron. J. Probab.*, **19**, no. 81, 33 (2014). [MR3256881](#).
- Hasebe, T. Free infinite divisibility for powers of random variables. *ALEA Lat. Am. J. Probab. Math. Stat.*, **13** (1), 309–336 (2016). [MR3481440](#).
- Hasebe, T., Simon, T., and Wang, M. Some properties of the free stable distributions. *Ann. Inst. Henri Poincaré Probab. Stat.*, **56** (1), 296–325 (2020). [MR4058989](#).
- Lindner, A., Pan, L., and Sato, K.-i. On quasi-infinitely divisible distributions. *Trans. Amer. Math. Soc.*, **370** (12), 8483–8520 (2018). [MR3864385](#).

- Linnik, J. V. and Ostrovskiĭ, u. V. *Decomposition of random variables and vectors*. Translations of Mathematical Monographs, Vol. 48. American Mathematical Society, Providence, R.I. (1977). [MR0428382](#).
- Liszewska, E. and Młotkowski, W. Some relatives of the Catalan sequence. *Adv. in Appl. Math.*, **121**, 102105, 29 (2020). [MR4149609](#).
- Liu, J.-G. and Pego, R. L. On generating functions of Hausdorff moment sequences. *Trans. Amer. Math. Soc.*, **368** (12), 8499–8518 (2016). [MR3551579](#).
- Maassen, H. Addition of freely independent random variables. *J. Funct. Anal.*, **106** (2), 409–438 (1992). [MR1165862](#).
- Mingo, J. A. and Speicher, R. *Free probability and random matrices*, volume 35 of *Fields Institute Monographs*. Springer, New York; Fields Institute for Research in Mathematical Sciences, Toronto, ON (2017). ISBN 978-1-4939-6941-8; 978-1-4939-6942-5. [MR3585560](#).
- Młotkowski, W. Fuss-Catalan numbers in noncommutative probability. *Doc. Math.*, **15**, 939–955 (2010). [MR2745687](#).
- Młotkowski, W. Probability distributions with rational free R -transform. *ArXiv Mathematics e-prints* (2021). [arXiv: 2111.10150](#).
- Młotkowski, W. and Penson, K. A. Probability distributions with binomial moments. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, **17** (2), 1450014, 32 (2014). [MR3212684](#).
- Młotkowski, W., Penson, K. A., and Życzkowski, K. Densities of the Raney distributions. *Doc. Math.*, **18**, 1573–1596 (2013). [MR3158243](#).
- Młotkowski, W., Sakuma, N., and Ueda, Y. Free self-decomposability and unimodality of the Fuss-Catalan distributions. *J. Stat. Phys.*, **178** (5), 1055–1075 (2020). [MR4081219](#).
- Morishita, J. and Ueda, Y. Free infinite divisibility for generalized power distributions with free Poisson term. *Probab. Math. Statist.*, **40** (2), 245–267 (2020). [MR4206414](#).
- Nica, A. and Speicher, R. On the multiplication of free N -tuples of noncommutative random variables. *Amer. J. Math.*, **118** (4), 799–837 (1996). [MR1400060](#).
- Nica, A. and Speicher, R. *Lectures on the combinatorics of free probability*, volume 335 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge (2006). ISBN 978-0-521-85852-6; 0-521-85852-6. [MR2266879](#).
- Pólya, G. Remarks on characteristic functions. In *Proceedings of the Berkeley Symposium on Mathematical Statistics and Probability, 1945, 1946*, pp. 115–123. University of California Press, Berkeley-Los Angeles, Calif. (1949). [MR0028541](#).
- Rudin, W. *Real and complex analysis*. McGraw-Hill Book Co., New York, third edition (1987). ISBN 0-07-054234-1. [MR924157](#).
- Saitoh, N. and Yoshida, H. The infinite divisibility and orthogonal polynomials with a constant recursion formula in free probability theory. *Probab. Math. Statist.*, **21** (1, Acta Univ. Wratislav. No. 2298), 159–170 (2001). [MR1869728](#).
- Sato, K.-i. *Lévy processes and infinitely divisible distributions*, volume 68 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge (1999). ISBN 0-521-55302-4. [MR1739520](#).
- Voiculescu, D. Addition of certain noncommuting random variables. *J. Funct. Anal.*, **66** (3), 323–346 (1986). [MR839105](#).