PROBABILITY MEASURES CORRESPONDING TO
AVAL NUMBERS

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Abstract. We describe the class of probability measures whose moments are given in terms of the Aval numbers. They are expressed as the multiplicative free convolution of measures corresponding to the ballot numbers \( \frac{m-k}{m+k} \binom{m+k}{m} \).

Introduction. Aval introduced multivariate Fuss–Catalan numbers, which count \( p \)-Raney sequences (or, equivalently, Dyck paths or trees) of a special type. They decompose the Fuss–Catalan numbers in an analogous way as the ballot numbers
\[
B(m,k) := \frac{m-k}{m+k} \binom{m+k}{k}
\]
decompose the Catalan numbers
\[
\frac{1}{2m+1} \binom{2m+1}{m}.
\]

The aim of this paper is to prove that the generating function (2.1) for these numbers, with nonnegative parameters \( a_0, \ldots, a_{p-1} \), is the moment generating function of a certain probability measure \( \mu \) on \( \mathbb{R} \). It turns out that \( \mu \) can be represented as the multiplicative free convolution
\[
\mu(a_0) \boxtimes \mu(a_1) \boxtimes \cdots \boxtimes \mu(a_{p-1}),
\]
where \( \mu(a) \) is given by (3.4) and corresponds to the classical ballot numbers. It is worth mentioning here that for a real parameter \( p \geq 1 \) the Fuss–Catalan numbers \( \frac{1}{mp+1} \binom{mp+1}{m} \), \( m = 0, 1, \ldots, \), constitute the moment sequence of the probability measure \( \mu(1)^{\boxtimes(p-1)} \), the multiplicative free power of the Marchenko–Pastur distribution (see [4, 9, 10]).

The paper is organized as follows. First we reprove the result of Aval: the formula (1.3) for the number of elements in the class \( \mathcal{F}_m(k_0, \ldots, k_{p-1}) \) of Raney sequences. Next we find the generating function for these numbers, with parameters \( a_0, \ldots, a_{p-1} \in \mathbb{R} \) (Theorem 2.1). In Section 3 we characterize those probability measures which correspond to \( p = 1 \), i.e. the case of the ballot numbers. Finally we briefly recall the notion of multiplicative free convolution and apply it to the proof of our main theorem.

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1. Raney sequences. Let $p \geq 1$ be a fixed integer. A $p$-Raney sequence is a sequence $\mathbf{x} = (x_1, \ldots, x_N)$ such that $x_i \in \{1, -p\}$ and all the partial sums $S_n(\mathbf{x}) = S_n := x_1 + \cdots + x_n$, $0 \leq n \leq N$, are nonnegative. For $0 \leq n \leq N$ define $P_n := (n, S_n) \in \mathbb{Z}^2$. Then $\mathbf{x}$ can be identified with the path $(P_0, P_1, \ldots, P_N)$.

Denote by $\mathcal{F}$ the class of those $p$-Raney sequences $(x_1, \ldots, x_N)$ for which $x_N = 1$ and $p$ divides $S_N$, and by $\mathcal{G}$ the class of those for which $S_N = 0$ (in particular here $p + 1$ divides $N$). By convention we assume that the empty sequence $\emptyset$ belongs to $\mathcal{F}$ as well as to $\mathcal{G}$, so that $\mathcal{F} \cap \mathcal{G} = \{\emptyset\}$. For $\mathbf{x} \in \mathcal{F} \cup \mathcal{G}$ we have $|\{k : x_k = 1\}| = mp$ for some integer $m \geq 0$ and then we put $||\mathbf{x}|| := m$. Define

$$\mathcal{F}_m := \{\mathbf{x} \in \mathcal{F} : ||\mathbf{x}|| = m\} \quad \text{and} \quad \mathcal{G}_m := \{\mathbf{y} \in \mathcal{G} : ||\mathbf{y}|| = m\}.$$  

Then the map $(x_1, \ldots, x_N) \mapsto (x_1, \ldots, x_M)$, where $M < N$ is such that $x_M = 1$, $x_{M+1} = x_{M+2} = \cdots = x_N = -p$ (for $N = 0$ we put $M = 0$) is a bijection $\mathcal{G}_m \rightarrow \mathcal{F}_m$ and elements of these sets are counted by the Fuss–Catalan numbers:

$$(1.1) \quad |\mathcal{F}_m| = |\mathcal{G}_m| = \frac{1}{m(p+1)+1} \binom{m(p+1)+1}{m}$$  

(see $\mathfrak{R}$). Now we are going to decompose $\mathcal{F}_m$ and $\mathcal{G}_m$ into smaller sets.

For $r \in \{0, 1, \ldots, p-1\}$ and for a $p$-Raney sequence $\mathbf{x} = (x_1, \ldots, x_N)$ we define

$$||\mathbf{x}||_r := |\{k : x_k = -p \text{ and } p \text{ divides } S_k - r\}|.$$  

We say that $\mathbf{x}$ is of type $(k_0, \ldots, k_{p-1})$, denoted $\mathbf{t}(\mathbf{x}) := (k_0, \ldots, k_{p-1})$, if $||\mathbf{x}||_0 = k_0, \ldots, ||\mathbf{x}||_{p-1} = k_{p-1}$. By $\mathcal{F}_m(k_0, \ldots, k_{p-1})$ (resp. $\mathcal{G}_m(k_0, \ldots, k_{p-1})$) we will denote the set of all $\mathbf{x}$ in $\mathcal{F}_m$ (resp. $\mathcal{G}_m$) such that $\mathbf{t}(\mathbf{x}) = (k_0, \ldots, k_{p-1})$. In particular, if $k_0 + \cdots + k_{p-1} = m \geq 1$ then $\mathcal{F}_m(k_0, \ldots, k_{p-1}) = \emptyset$. Hence we have

$$\mathcal{F} = \{\emptyset\} \cup \bigcup_{\substack{m \geq 1 \\text{and} \ k_0, \ldots, k_{p-1} \geq 0 \ \text{and} \ k_0 + \cdots + k_{p-1} < m}} \mathcal{F}_m(k_0, \ldots, k_{p-1}),$$  

$$\mathcal{G} = \{\emptyset\} \cup \bigcup_{\substack{m \geq 1 \\text{and} \ k_0, \ldots, k_{p-1} \geq 0 \ \text{and} \ k_0 + \cdots + k_{p-1} = m}} \mathcal{G}_m(k_0, \ldots, k_{p-1}).$$

Now we are going to count the elements in these sets. In particular we will reprove Proposition 2.4 from $\mathfrak{R}$.

**Lemma 1.1.** For $m \geq 0$ and $k_0, \ldots, k_{p-1} \geq 0$ with $k_0 + \cdots + k_{p-1} < m+1$, we have

...
\[ |\mathcal{F}_{m+1}(k_0, \ldots, k_{p-1})| = \sum_{0 \leq j_0 \leq k_0, 0 \leq j_{p-1} \leq k_{p-1}} |\mathcal{F}_m(j_0, \ldots, j_{p-1})|. \]

**Proof.** For \( x = (x_1, \ldots, x_N) \in \mathcal{F}_{m+1}(k_0, \ldots, k_{p-1}) \) put \( T x := (x_1, \ldots, x_M) \) where \( M \) is unique such that \( x_M = 1 \) and \( |\{ j : 1 \leq j \leq M, x_j = 1 \}| = mp. \) For \( m = 0 \) put \( M = 0. \) Then \( T \) is a bijection from \( \mathcal{F}_{m+1}(k_0, \ldots, k_{p-1}) \) onto the disjoint union of the sets \( \mathcal{F}_m(j_0, \ldots, j_{p-1}) \), where \( 0 \leq j_0 \leq k_0, \ldots, 0 \leq j_{p-1} \leq k_{p-1}. \)

We will apply the following elementary identity:

\[
\sum_{i=0}^{k} (c-i) \binom{m-1+i}{i} = \left( c - \frac{mk}{m+1} \right) \binom{m+k}{k},
\]

for \( k \geq 0, m \geq 1 \) and \( c \in \mathbb{R}. \)

**Theorem 1.2.** For \( m \geq 1 \) and for \( k_0, \ldots, k_{p-1} \geq 0 \) with \( k_0 + \cdots + k_{p-1} < m \), we have

\[
|\mathcal{F}_m(k_0, \ldots, k_{p-1})| = \frac{m - \sum_{i=0}^{p-1} k_i \prod_{i=0}^{p-1} \binom{m-1+k_i}{k_i}}{m}.
\]

Theorem 1.2 was proved in [3] by combinatorial means; here we follow Remark 2.6 in [3]. Note that for \( p = 1 \) we get the classical ballot numbers.

**Proof.** Put \( e_0(0, \ldots, 0) := 1, \) and for \( m \geq 1 \) and \( k_0, \ldots, k_{p-1} \geq 0 \) with \( k_0 + \cdots + k_{p-1} \leq m, \) define

\[
e_m(k_0, \ldots, k_{p-1}) := \frac{m - \sum_{i=0}^{p-1} k_i \prod_{i=0}^{p-1} \binom{m-1+k_i}{k_i}}{m}.
\]

In view of Lemma 1.1 it is sufficient to prove that if \( m \geq 0, k_0, \ldots, k_{p-1} \geq 0 \) and \( k_0 + \cdots + k_{p-1} < m + 1 \) then

\[
\sum_{0 \leq j_0 \leq k_0, \ldots, 0 \leq j_{p-1} \leq k_{p-1}} e_m(j_0, \ldots, j_{p-1}) = e_{m+1}(k_0, \ldots, k_{p-1}).
\]

This is true for \( m = 0 \) because

\[
e_1(0, \ldots, 0) = e_0(0, \ldots, 0) = 1.
\]

Now assume that \( m \geq 1. \) Applying (1.2) consecutively \( p \) times we get
whenever
\[ x \sum_{j_{p-1}=0}^{k_{p-1}} \cdots \sum_{j_0=0}^{k_0} e_m(j_0, \ldots, j_{p-1}) = \sum_{j_{p-1}=0}^{k_{p-1}} \cdots \sum_{j_0=0}^{k_1} \frac{m-j_1-\cdots-j_{p-1}-\frac{m k_0}{m+1}}{m} \left( m + k_0 \right)^{p-1} \prod_{i=1}^{j_i} \left( m - 1 + j_i \right) \]
\[ = \frac{m-m k_0}{m+1} - \cdots - \frac{m k_{p-1}}{m+1} \prod_{i=0}^{p-1} \left( m + k_i \right) = e_{m+1}(k_0, \ldots, k_{p-1}), \]
which concludes the proof. □

**Theorem 1.3.** For \( m \geq 1 \) and \( k_0, \ldots, k_{p-1} \geq 0 \) with \( k_0 + \cdots + k_{p-1} = m \), we have
\[ |G_m(k_0, k_1, \ldots, k_{p-1})| = \frac{1}{m} \left( m + k_0 \right)^{p-1} \prod_{i=1}^{k_i} \left( m + k_i - 1 \right). \]

**Proof.** We define \( \Lambda : \mathcal{G} \to \mathcal{F} \) by putting \( \Lambda(x_1, \ldots, x_N) := (x_1, \ldots, x_M) \) whenever \( x_M = 1 \) and \( x_{M+1} = x_{M+2} = \cdots = x_N = -p \). For \( N = 0 \) we put \( M = 0 \). Note that for \( m \geq 1 \) and \( k_0, \ldots, k_{p-1} \geq 0 \) with \( k_0 + k_1 + \cdots + k_{p-1} = m \), \( \Lambda \) establishes a 1-1 correspondence between the class \( \mathcal{G}_m(k_0, k_1, \ldots, k_{p-1}) \) and the disjoint union \( \bigcup_{k=0}^{k_0-1} \mathcal{F}_m(k, k_1, \ldots, k_{p-1}) \). Now it remains to yoke (1.2) together with (1.3). □

**2. Generating functions.** Fix parameters \( a_0, \ldots, a_{p-1} \in \mathbb{R} \). The aim of this section is to describe the generating functions
\[ F(z) = 1 + \sum_{m=1}^{\infty} z^m \sum_{k_0, k_1, \ldots, k_{p-1} \geq 0 \atop k_0 + k_1 + \cdots + k_{p-1} < m} |\mathcal{F}_m(k_0, \ldots, k_{p-1})| a_0^{k_0} a_1^{k_1} \cdots a^{k_{p-1}}, \]
\[ G(z) = 1 + \sum_{m=1}^{\infty} z^m \sum_{k_0, k_1, \ldots, k_{p-1} \geq 0 \atop k_0 + k_1 + \cdots + k_{p-1} = m} |\mathcal{G}_m(k_0, \ldots, k_{p-1})| a_0^{k_0} a_1^{k_1} \cdots a^{k_{p-1}}. \]

We will use ideas from the proof of Proposition 2.7 in [3].

First we introduce some notation. For \( 0 \leq r \leq p-1 \) define
\[ a_{(r)} := (a_r, a_{r+1}, \ldots, a_{p-1}, a_0, \ldots, a_{r-1}) \]
and for \( k = (k_0, \ldots, k_{p-1}) \) put
\[ a_{(r)}^k := a_r^{k_0} a_{r+1}^{k_1} \cdots a_{p-1}^{k_{p-1}} a_0^{k_{p-r}} \cdots a_{r-1}^{k_{p-1}}. \]
Now define auxiliary functions
\[ G_r(z) := \sum_{y \in \mathcal{G}} z^{\|y\|} a_{(r)}^{t(y)}, \]
in particular \( G_0 = G \).
Fix $\mathbf{x} = (x_1, \ldots, x_N) \in \mathcal{F} \setminus \{\emptyset\}$. For $0 \leq r \leq p - 1$ we put $\alpha_r := \max\{n : S_n(\mathbf{x}) = r\}$. Then we decompose

$$\mathbf{x} = (x_0, 1, x_1, 1, \ldots, 1, x_p)$$

where

$$\mathbf{x}_r := (x_{\alpha_r + 1}, x_{\alpha_r + 3}, \ldots, x_{\alpha_r})$$

under the convention that $\alpha_{-1} := -1$, $\alpha_p := N$. Then $x_0, \ldots, x_{p-1} \in \mathcal{G}$, $x_p \in \mathcal{F}$ and we have

$$\|\mathbf{x}\| = 1 + \|x_0\| + \cdots + \|x_{p-1}\| + \|x_p\|$$

and

$$z^{|\mathbf{x}|} a^t(\mathbf{x}) = z \cdot z^{|x_0|} a^t(x_0) \cdots z^{|x_{p-1}|} a^t(x_{p-1}) z^{|x_p|} a^t(x_p),$$

because the level of a given step of $\mathbf{x}_r$ in the whole $\mathbf{x}$ is equal to its level within $\mathbf{x}_r$ plus $r$, i.e.

$$S_k(\mathbf{x}_r) + r = S_{\alpha_r + 1 + k}(\mathbf{x}).$$

Note that the map $\mathbf{x} \mapsto (\mathbf{x}_0, \mathbf{x}_1, \ldots, \mathbf{x}_p)$ is a bijection from $\mathcal{F} \setminus \{\emptyset\}$ onto $\mathcal{G}^p \times \mathcal{F}$.

Similarly, we decompose every $\mathbf{y} \in \mathcal{G} \setminus \{\emptyset\}$ as $\mathbf{y} = (y_0, 1, y_1, 1, \ldots, 1, y_p, -p)$, with $y_0, \ldots, y_p \in \mathcal{G}$, and

$$z^{|\mathbf{y}|} a^t(\mathbf{y}) = z a_0 \cdot z^{|y_0|} a^t(y_0) \cdots z^{|y_{p-1}|} a^t(y_{p-1}) z^{|y_p|} a^t(y_p),$$

which leads to a bijection $\mathcal{G} \setminus \{\emptyset\} \to \mathcal{G}^{p+1}$.

Then (2.4) and (2.5) yield

$$(2.6) \quad F(z) = 1 + \sum_{x \in \mathcal{F} \setminus \{\emptyset\}} z^{|x|} a^t(x) = 1 + z G_0(z) G_1(z) \cdots G_{p-1}(z) F(z),$$

$$(2.7) \quad G(z) = 1 + \sum_{y \in \mathcal{G} \setminus \{\emptyset\}} z^{|y|} a^t(y) = 1 + a_0 z G_0(z) G_1(z) \cdots G_{p-1}(z) G_0(z).$$

In the same way we can proceed with $G_r$, obtaining the equations

$$G_r(z) = 1 + a_r z G_0(z) G_1(z) \cdots G_{p-1}(z) G_r(z).$$

Now we are in a position to describe the functions $F$ and $G$.

**Theorem 2.1.** In a neighborhood of 0 we have $F(z) = 1 + \tilde{F}(z)$ and $G(z) = 1 + \tilde{G}(z)$, where $\tilde{F}$ is the inverse function of

$$w \mapsto \frac{w(1 + w - a_0 w)(1 + w - a_1 w) \cdots (1 + w - a_{p-1} w)}{(1 + w)^{p+1}},$$

and $\tilde{G}$ is the inverse function of
(2.10) \[ w \mapsto \frac{w(a_0 + a_0 w - a_1 w)(a_0 + a_0 w - a_2 w)\ldots(a_0 + a_0 w - a_{p-1} w)}{a_0^p(1 + w)^{p+1}} \]
in a neighborhood of 0.

**Proof.** Put \( H(z) := zG_0(z)G_1(z)\ldots G_{p-1}(z) \). Then from (2.8),
\[ G_r(z) = \frac{1}{1 - a_r H(z)}. \]
Taking the product over \( 0 \leq r \leq p - 1 \) and multiplying by \( z \) we get
(2.11) \[ H(z) = \frac{z}{(1 - a_0 H(z))(1 - a_1 H(z))\ldots(1 - a_{p-1} H(z))}. \]
Now putting \( \tilde{F} := F - 1, \tilde{G} := G - 1 \) we see from (2.6) and (2.7) that
\[ H = \frac{\tilde{F}}{1 + \tilde{F}}, \quad H = \frac{\tilde{G}}{a_0 + a_0 \tilde{G}}. \]
Substituting to (2.11) we get
\[ z = \frac{\tilde{F}(1 + \tilde{F} - a_0 \tilde{F})(1 + \tilde{F} - a_1 \tilde{F})\ldots(1 + \tilde{F} - a_{p-1} \tilde{F})}{(1 + \tilde{F})^{p+1}}, \]
and
\[ z = \frac{\tilde{G}(a_0 + a_0 \tilde{G} - a_1 \tilde{G})(a_0 + a_0 \tilde{G} - a_2 \tilde{G})\ldots(a_0 + a_0 \tilde{G} - a_{p-1} \tilde{G})}{a_0^p(1 + \tilde{G})^{p+1}}, \]
which concludes the proof. \( \square \)

Now it is easy to find \( F \) for \( p = 1 \):

**Corollary 2.2.** For \( p = 1 \) and \( a_0 := a \) we have
\[ F(z) = \frac{1 - 2a + \sqrt{1 - 4az}}{2(1 - a - z)} = \frac{2a}{2a - 1 + \sqrt{1 - 4az}}. \]

**Proof.** Here \( \tilde{F} \) is the inverse function of \( w \mapsto w(1 + w - aw)(1 + w)^{-2} \),
which implies that \( F(z) = \tilde{F}(z) + 1 \) satisfies the quadratic equation
(2.12) \[ (F(z) - 1)(F(z) - aF(z) + a) = zF(z)^2, \]
with \( F(0) = 1 \), from which we find \( F(z) \). \( \square \)

We can also compute the Taylor expansion of the powers \( F(z)^s \):

**Theorem 2.3.** For \( s \in \mathbb{R} \) we have
\[ F(z)^s = 1 + \sum_{m=1}^{\infty} \frac{z^m}{m} \sum_{k=1}^{m} k \binom{s - 1 + k}{k} \sum_{k_0,\ldots,k_{p-1} \geq 0, k_0 + \cdots + k_{p-1} + k = m} \prod_{i=0}^{p-1} \binom{m - 1 + k_i}{k_i} a_0^{k_0} a_1^{k_1} \cdots a_{p-1}^{k_{p-1}}. \]
Proof. For \( x \in \mathcal{F} \), with \( \|x\| = m \) and \( t(x) = (k_0, \ldots, k_{p-1}) \) we define

\[
\phi(x) := m - (k_0 + \cdots + k_{p-1}),
\]
in particular \( \phi(\emptyset) = 0 \). Now we observe that if \( x_1, \ldots, x_s \in \mathcal{F} \) then the concatenation \( x = (x_1, \ldots, x_s) \) is also in \( \mathcal{F} \) and we have \( t(x) = t(x_1) + \cdots + t(x_s) \) (as \( p \)-vectors), and \( \|x\| = \|x_1\| + \cdots + \|x_s\| \).

On the other hand, for given \( x \in \mathcal{F} \) and natural \( s \) there are exactly \((s-1+\phi(x))\) ways of decomposing \( x \) into \( x_1, \ldots, x_s \), with \( x_i \in \mathcal{F} \). Indeed, the beginning and the end points of the pieces can only be the points \( \gamma_i, 0 \leq i \leq \phi(x) \), where \( \gamma_i := \min \{ j \geq 0 : \text{if } k \geq j \text{ then } S_k(x) \geq ip \} \), which, by (1.3), proves the theorem for \( s \in \mathbb{N} \).

Now for \( s \in \mathbb{R} \) define \( c_0(s) := 1 \) and for \( m \geq 1 \) put

\[
c_m(s) := \sum_{k=1}^{m} \frac{k}{m} \binom{s-1+k}{k} \sum_{k_0, \ldots, k_{p-1} \geq 0 \atop k_0 + \cdots + k_{p-1} + k = m} \prod_{i=0}^{p-1} \binom{m-1+k_i}{k_i} a_0 a_1 a_2 \cdots a_{p-1}.
\]

Then for natural \( s \) we have

\[
(2.13) \quad F(z)^s = \sum_{m=0}^{\infty} c_m(s) z^m.
\]

This implies that for \( u, v \in \mathbb{N} \),

\[
(2.14) \quad c_m(u + v) = \sum_{k=0}^{m} c_k(u) c_{m-k}(v).
\]

Since both sides of (2.14) are polynomials on \( u \) and \( v \), (2.14) remains true for all \( u, v \in \mathbb{R} \) and that in turn proves that (2.13) holds for all \( s \in \mathbb{R} \).
3. **Probability measures: the case** $p = 1$. Our purpose now is to prove that if all the parameters $a_j$ are nonnegative then $F$ and $G$ are moment generating functions of certain probability measures. In other words, if $a_0, \ldots, a_{p-1} \geq 0$ then both the sequences

$$f_m(a_0, \ldots, a_{p-1}) := \sum_{k_0, k_1, \ldots, k_{p-1} \geq 0, k_0 + k_1 + \ldots + k_{p-1} \leq m} |F_m(k_0, \ldots, k_{p-1})| a_0^{k_0} a_1^{k_1} \ldots a_{p-1}^{k_{p-1}},$$

$$g_m(a_0, \ldots, a_{p-1}) := \sum_{k_0, k_1, \ldots, k_{p-1} \geq 0, k_0 + k_1 + \ldots + k_{p-1} = m} |G_m(k_0, \ldots, k_{p-1})| a_0^{k_0} a_1^{k_1} \ldots a_{p-1}^{k_{p-1}}$$

(in particular: $f_0(a_0, \ldots, a_{p-1}) = g_0(a_0, \ldots, a_{p-1}) = 1$), $m \geq 0$, are positive definite. We conjecture that this condition is also necessary (apart from the trivial case $g_m(0, a_1, \ldots, a_{p-1}) = \delta_{0,m}$, $a_i \in \mathbb{R}$).

First we start with the case $p = 1$, so that

$$f_m(a) = \sum_{k=0}^{m} B(m, k) a^k,$$

where $B(m, k) := |F_m(k)|$ are the classical ballot numbers: $B(0, 0) := 1$ and $B(m, k) = \binom{m+k}{m} \frac{m-k}{m+k}$ for $m \geq 1$, $0 \leq k \leq m$. In particular $f_m(1) = \binom{2m}{m}/(m+1)$, the $m$th Catalan number. These numbers satisfy the recurrence $B(0, 0) = 1$ and

$$B(m+1, k) = \sum_{i=0}^{k} B(m, i)$$

for $0 \leq k \leq m$, $B(m+1, m+1) = 0$, which is equivalent to the following recurrence for the polynomials $f_m$: $f_0(a) = 1$ and

$$f_{m+1}(a) = \frac{f_m(1) a^{m+1} - f_m(a)}{a - 1}$$

for $m \geq 0$.

Now define a one-parameter family of measures: $\mu(0) := \delta_1$ and for $a > 0$,

$$\mu(a) := \frac{\sqrt{4a x - x^2}}{2\pi x(1 - x + ax)} \chi_{[0,4a]} dx + \text{atom}(a)$$

where

$$\text{atom}(a) = \begin{cases} 
1 - 2a & \text{if } 0 < a < 1/2, \\
\frac{1}{1-a} \delta_{1/(1-a)} & \text{if } a \geq 1/2.
\end{cases}$$

In particular, $\mu(1)$ is called the Marchenko–Pastur distribution.
**Theorem 3.1.** For $a \geq 0$ the moment sequence of $\mu(a)$ is $\{f_m(a)\}_{m=0}^{\infty}$, i.e.

$$\int_{\mathbb{R}} x^m \, d\mu(a)(x) = f_m(a), \quad m = 0, 1, 2, \ldots.$$  

The proof is based on four elementary lemmas; we will skip the proofs of two of them.

**Lemma 3.2.** For $n = 0, 1, 2, \ldots$ and $c \neq 1$ we have

$$\frac{u^2}{(u^2 + 1)^{n+1}(u^2 + c)} = \frac{1}{(1 - c)(u^2 + 1)^{n+1}} - \frac{c}{(1 - c)^{n+1}(u^2 + c)} + \sum_{k=0}^{n-1} \frac{c}{(1 - c)^{n+1-k}(u^2 + 1)^{k+1}}.$$  

**Lemma 3.3.** For $n = 0, 1, 2, \ldots$ we have

$$\int_{0}^{\infty} \frac{dx}{(x^2 + 1)^{n+1}} = \left(\frac{2n}{n}\right) \frac{\pi}{2 \cdot 4^n}.$$  

**Lemma 3.4.** For $0 < a \neq 1$ and $n = 0, 1, 2, \ldots$ we have

$$\frac{4a}{2\pi x(1-x+ax)} \int_{0}^{4a} x^n \sqrt{4ax-x^2} \, dx = \left(\frac{2n}{n}\right) \frac{a^n}{2(1-a)} - \frac{|2a - 1|}{2(1-a)^{n+1}} + \sum_{k=0}^{n-1} \left(\frac{2k}{k}\right) \frac{(2a - 1)^2 a^k}{2(1-a)^{n+1-k}}.$$  

**Proof.** Using the third substitution of Euler: $\sqrt{x(4a-x)} = ux$, we get $x = 4a/(u^2 + 1), dx = -8au \, du/(u^2 + 1)^2$. Putting $c := (2a - 1)^2$, applying the previous lemmas and the formula

$$\int_{0}^{\infty} \frac{dx}{x^2 + c} = \frac{\pi}{2\sqrt{c}}, \quad c > 0,$$

we obtain

$$(3.6) \quad \frac{4a}{2\pi x(1-x+ax)} \int_{0}^{4a} x^n \sqrt{4ax-x^2} \, dx = (4a)^{n+1} \int_{0}^{\infty} \frac{u^2 \, du}{(u^2 + 1)^{n+1}(u^2 + (2a - 1)^2)}$$

$$= (4a)^{n+1} \left[ \left(\frac{2n}{n}\right) \frac{1}{2 \cdot 4^n(1-c)} - \frac{\sqrt{c}}{2(1-c)^{n+1}} + \sum_{k=0}^{n-1} \left(\frac{2k}{k}\right) \frac{(2a - 1)^2 a^k}{2 \cdot 4^k(1-c)^{n+1-k}} \right]$$

$$= \left(\frac{2n}{n}\right) \frac{a^n}{2(1-a)} - \frac{|2a - 1|}{2(1-a)^{n+1}} + \sum_{k=0}^{n-1} \left(\frac{2k}{k}\right) \frac{(2a - 1)^2 a^k}{2(1-a)^{n+1-k}}.$$  

$\blacksquare$
Lemma 3.5.
\[
\binom{2n}{n} \frac{a^n}{2(1-a)} - \frac{2a-1}{2(1-a)^{n+1}} + \sum_{k=0}^{n-1} \binom{2k}{k} \frac{(2a-1)^2 a^k}{2(1-a)^{n+1-k}} = f_n(a).
\]

Proof. We will prove by induction that
\[
\binom{2n}{n} a^n (1-a)^n - (2a-1) + \sum_{k=0}^{n-1} \binom{2k}{k} (2a-1)^2 a^k (1-a)^k = 2(1-a)^{n+1} f_n(a).
\]
It is easy to check this for \(n = 0\). Now suppose it holds for some \(n\). Then the left hand side for \(n + 1\) equals
\[
L_{n+1} := \binom{2n+2}{n+1} a^{n+1} (1-a)^{n+1} - (2a-1) + \sum_{k=0}^{n} \binom{2k}{k} (2a-1)^2 a^k (1-a)^k
\]
\[
= \binom{2n+2}{n+1} a^{n+1} (1-a)^{n+1} + \binom{2n}{n} (2a-1)^2 a^n (1-a)^n
\]
\[
- \binom{2n}{n} a^n (1-a)^n + 2(1-a)^{n+1} f_n(a)
\]
\[
= \binom{2n+2}{n+1} a^{n+1} (1-a)^{n+1} - 4 \binom{2n}{n} a^{n+1} (1-a)^{n+1} + 2(1-a)^{n+1} f_n(a)
\]
\[
= \binom{2n+2}{n+1} a^{n+1} (1-a)^{n+1} + 2(1-a)^{n+1} f_n(a)
\]
\[
= -2(1-a)^{n+1} (f_n(1)a^{n+1} - f_n(a)) = 2(1-a)^{n+2} f_{n+1}(a) = P_{n+1},
\]
which, in view of (3.3), completes the proof. ■

Proof of Theorem 3.1. Comparing Lemmas 3.4 and 3.5 we see that if \(1/2 \leq a \neq 1\) then
\[
\int_0^{4a} x^n \frac{\sqrt{4ax-x^2}}{2\pi x(1-x+ax)} \, dx = f_n(a),
\]
while for \(0 < a < 1/2\),
\[
\int_0^{4a} x^n \frac{\sqrt{4ax-x^2}}{2\pi x(1-x+ax)} + \frac{1-2a}{(1-a)^{n+1}} = f_n(a).
\]
For the remaining case \(a = 1\) we apply (3.6) and Lemma 3.3:
\[
\int_0^4 x^n \frac{\sqrt{4x-x^2}}{2\pi x} \, dx = \frac{4^{n+1}}{\pi} \int_0^\infty \frac{u^2 \, du}{(u^2 + 1)^{n+2}}
\]
\[
= \frac{4^{n+1}}{\pi} \int_0^\infty \frac{(u^2 + 1) - 1}{(u^2 + 1)^{n+2}} \, du = 2 \binom{2n}{n} - \frac{1}{2} \binom{2n+2}{n+1} = \frac{1}{2n+1} \binom{2n+1}{n}. ■
\]
We can also find the Jacobi parameters of $\mu(a)$:

**Theorem 3.6.** The monic orthogonal polynomials for $\mu(a)$ are given by the following recurrence: $P_0(x) = 1$ and for $m \geq 0$,

$$xp_m(x) = P_{m+1}(x) + \beta_m P_m(x) + \gamma_m P_{m-1}(x)$$

(under the convention that $\gamma_{-1} = 0$ and $P_{-1} = 0$), where the Jacobi parameters are

$$\beta_m = \begin{cases} 1 & \text{if } m = 0, \\ 2a & \text{if } m \geq 1, \end{cases} \quad \gamma_m = \begin{cases} a & \text{if } m = 0, \\ a^2 & \text{if } m \geq 1. \end{cases}$$

**Proof.** Using the identity

$$\frac{1 - 2az - \sqrt{1 - 4az}}{2} = \frac{a^2 z^2}{1 - 2az - \frac{1 - 2az - \sqrt{1 - 4az}}{2}}$$

we can express $M_{\mu(a)}(z)$ as a continued fraction:

$$M_{\mu(a)}(z) = \frac{2a}{2a - 1 + \sqrt{1 - 4az}} = \frac{1}{1 - z - \frac{a^2 z^2}{1 - 2az - \frac{1 - 2az - \frac{a^2 z^2}{1 - 2az - \frac{a^2 z^2}{1 - 2az - \ddots}}}}},$$

which implies our statement (see [14] [7]).

Let us mention that these measures $\mu(a)$ belong to the **free Meixner class** which consists of those probability measures on $\mathbb{R}$ for which the Jacobi sequences $\{\beta_n\}_{n=0}^{\infty}$ and $\{\gamma_n\}_{n=0}^{\infty}$ are constant for $n \geq 1$ (see [1] [2] [6] [12]).

4. **Probability measures: the general case.** In this part we will make use of **free probability**, introduced by Voiculescu (for details we refer to [13] [11] [5]). In this theory by a **noncommutative probability space** we mean a pair $(\mathcal{A}, \phi)$, where $\mathcal{A}$ is a unital $C^*$-algebra and $\phi$ is a **tracial state** on $\mathcal{A}$, i.e. a linear map $\phi : \mathcal{A} \to \mathbb{C}$ such that $\phi(1) = 1$ and for every $X, Y \in \mathcal{A}$ we have $\phi(X^*X) \geq 0$ and $\phi(XY) = \phi(YX)$. Every self-adjoint element $X = X^* \in \mathcal{A}$ possesses its **distribution** $\mu_X$, which is a compactly supported probability measure on $\mathbb{R}$ whose moments are $\phi(X^m)$. If in addition $X$ is positive (which means that $X = Z^2$ for some $Z = Z^* \in \mathcal{A}$) then this measure is supported in the positive half-line $[0, \infty)$. A family $\{\mathcal{A}_i\}_{i \in I}$ of unital
subalgebras is said to be freely independent if we have \( \phi(X_1 \ldots X_m) = 0 \) whenever \( X_1 \in A_{i_1}, \ldots, X_m \in A_{i_m}, i_k \in I, i_1 \neq i_2 \neq \cdots \neq i_m \) and \( \phi(X_1) = \cdots = \phi(X_m) = 0 \). Elements \( Y_1, \ldots, Y_n \in A \) are called freely independent if there are free unital subalgebras \( A_1, \ldots, A_n \) such that \( Y_k \in A_k \) for \( 1 \leq k \leq n \).

Now assume that \( X = X^*, Y = Y^* \in A \) are freely independent, with distributions \( \mu_X, \mu_Y \). Then it turns out that the distribution of \( X + Y \) depends only on \( \mu_X \) and \( \mu_Y \) and is denoted by \( \mu_X \boxtimes \mu_Y \). If in addition \( X, Y \) are positive, with \( X = Z^2 \) for some \( Z = Z^* \in A \) then the distribution of \( ZYZ \) depends only on \( \mu_X \) and \( \mu_Y \) and is denoted by \( \mu_X \boxdot \mu_Y \). These operations can be extended to the whole class of probability measures on \( \mathbb{R} \), with support in the positive half-line in the case of \( \boxdot \), and are called additive and multiplicative free convolution respectively (see [5]). Only the latter will be used here.

Denote by \( \mathcal{M}_+^c \) the class of probability measures with compact support contained in \([0, \infty)\), different from \( \delta_0 \). For \( \mu \in \mathcal{M}_+^c \), with moments

\[
s_m(\mu) := \int_{\mathbb{R}} t^m \, d\mu(t),
\]

and with the moment generating function

\[
M_{\mu}(z) := \sum_{m=0}^{\infty} s_m(\mu) z^m = \int_{\mathbb{R}} \frac{d\mu(t)}{1 - tz},
\]

we define the S-transform \( S_{\mu}(z) \) by the equation

\[
M_{\mu} \left( \frac{z}{1 + z} S_{\mu}(z) \right) = 1 + z
\]
on a neighborhood of 0. If \( c > 0 \) and \( D_{c,\mu} \) denotes the dilation of \( \mu \), i.e. \( D_{c,\mu}(X) := \mu(c^{-1}X) \), then \( M_{D_{c,\mu}}(z) = M_{\mu}(cz) \) and \( S_{D_{c,\mu}}(z) = c^{-1} S_{\mu}(z) \).

For \( \mu_1, \mu_2 \in \mathcal{M}_+^c \) the multiplicative free convolution \( \mu_1 \boxdot \mu_2 \) is the unique \( \mu \in \mathcal{M}_+^c \) which satisfies

\[
S_{\mu}(z) = S_{\mu_1}(z) \cdot S_{\mu_2}(z)
\]
(see [13, 5]). The operation \( \boxdot \) can be regarded as a free analog of the Mellin convolution (which expresses the distribution of the product of nonnegative independent random variables), but (just as for \( \boxplus \)) no direct formula is known to compute \( \mu_1 \boxdot \mu_2 \) for general \( \mu_1, \mu_2 \in \mathcal{M}_+^c \).

Now we can prove

**Theorem 4.1.** Assume that \( a_0, \ldots, a_{p-1} \geq 0 \). Then the function \( F \) given by (2.1) is the moment generating function of the probability measure

\[
\mu(a_0) \boxdot \mu(a_1) \boxdot \cdots \boxdot \mu(a_{p-1}).
\]
If moreover $a_0 > 0$ then the function $G$ given by (2.2) is the moment generating function of the probability measure

$$D_{a_0} \mu(1) \boxtimes \mu(a_1/a_0) \boxtimes \mu(a_2/a_0) \boxtimes \cdots \boxtimes \mu(a_{p-1}/a_0).$$

If $a_0 = 0$ then $G(z) = 1$ and it is the moment generating function of $\delta_0$.

Proof. From Corollary 2.2 we have

$$M_{\mu(a)}(z) = \frac{1 - 2a + \sqrt{1 - 4az}}{2(1 - a - z)}$$

and by (2.12) this function satisfies the equation

$$M_{\mu(a)}\left(\frac{z(1 + z - az)}{(1 + z)^2}\right) = 1 + z,$$

which yields

$$S_{\mu(a)}(z) = \frac{1 + z - az}{1 + z}.$$

Now, coming to the general case, in view of Theorem 2.1 and formula (4.4) we have

$$F\left(\frac{z}{1 + z} S_{\mu(a_0)}(z) S_{\mu(a_1)}(z) \cdots S_{\mu(a_{p-1})}(z)\right) = 1 + z$$

and

$$G\left(\frac{z}{(1 + z) a_0} S_{\mu(1)}(z) S_{\mu(a_1/a_0)}(z) \cdots S_{\mu(a_{p-1}/a_0)}(z)\right) = 1 + z,$$

which leads to the desired statement. □

Remark 4.2. One can check (see Theorem 3.7.3 in [13] or Theorem 6.13 in [5]) that for every $a \geq 0$ the measure $\mu(a)$ is infinitely divisible with respect to the multiplicative free convolution $\boxtimes$, which implies that so are the measures (4.2) and (4.3).

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