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LIMIT THEOREMS IN Λ-BOOLEAN PROBABILITY

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We introduce a notion of "Λ-boolean independence", which interpolates between the tensor and the boolean one. The central and the Poisson limit theorem are given in terms of the related orthogonal polynomials.

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1. Preliminaries

There are several notions of independence for a family $\{\mathcal{A}_i\}_{i\in I}$ of subalgebras in a noncommutative probability space $(\mathcal{A}, \phi)$ (see Refs. 22, 15, 19, 6, 9, 12 and 14). In Ref. 12 we provided an interpolation between the tensor and the free independence. The aim of this paper is to present an interpolation between the tensor and the boolean one. The idea is similar: to allow commutations between some of these subalgebras. We prove limit theorems depending on a parameter q and involving boolean crossings between blocks of a partition (cf. Refs. 18, 11 and 8). We do not find the limit measures explicitly, but we give the recurrence relation for the corresponding orthogonal polynomials.

Let X be a set. Then $\mathcal{P}_2(X)$ will denote the family of 2-element subsets of X. By a *partition* of X we mean a family π of nonempty, pairwise disjoint subsets of X satisfying $\bigcup \pi = X$. We will denote by $\mathrm{Pa}(m)$ the family of all partitions of the set $\{1, 2, \ldots, m\}$ and $\mathrm{Pa}_2(m)$ (resp. $\mathrm{Pa}_{1,2}(m)$) will stand for the family of such $\pi \in \mathrm{Pa}(m)$ that $|U| = 2$ (resp. $|U| \le 2$) for every $U \in \pi$.

We say that blocks U, $V \in \pi \in \mathrm{Pa}(m)$, with $U \neq V$, have *boolean crossing* if there are $k < r < l$ such that either $k, l \in U$ and $r \in V$ or $r \in U$ and $k, l \in V$. We denote by $\mathrm{bc}(\pi)$ the number of boolean crossings in π:

$$\mathrm{bc}(\pi) := |\{\{U, V\} : U, V \in \pi, U \neq V \text{ and } U, V \text{ have boolean crossing}\}| .$$

Throughout the paper we fix an index set I and a family $\Lambda \subset \mathcal{P}_2(I)$. We say that words $\mathbf{i} = (i_1, \ldots, i_m)$, $\mathbf{j} = (j_1, \ldots, j_m) \in I^m$ are Λ-*equivalent* (or simply *equivalent*) if $\mathbf{i}$ can be converted to $\mathbf{j}$ by using finitely many operations $(\ldots, i, j, \ldots) \mapsto (\ldots, j, i, \ldots)$ with $\{i, j\} \in \Lambda$. We say that a word $\mathbf{i} = (i_1, \ldots, i_m) \in I^m$ is Λ-*reduced* (or simply *reduced*) if $\mathbf{i}$ is not equivalent to a word of the form $(\ldots, i, i, \ldots)$.

2. Λ-Boolean Independence

The notion of boolean independence originates from the following result, which was proved by Bożejko in 1981 (see Refs. 2–4). Suppose that for every $i \in I$ we are given a group G_i and a complex positive definite function ϕ_i on G_i, with $\phi_i(e) = 1$. Define a function ϕ on the free product group $*_{i\in I} G_i$ by putting

$$\phi(g_1 g_2 \cdots g_m) = \phi_{i_1}(g_1)\phi_{i_2}(g_2)\cdots\phi_{i_m}(g_m)$$

whenever $m \geq 0$, $g_1 \in G_{i_1} \setminus \{e\}$, $g_2 \in G_{i_2} \setminus \{e\}, \ldots, g_m \in G_{i_m} \setminus \{e\}$ and $i_1 \neq i_2 \neq \cdots \neq i_m$. Then ϕ is positive definite on $*_{i\in I} G_i$.

This theorem (see also Refs. 5 and 6), which can be extended to the nonunital free product of $*$-algebras, was a foundation for the study of boolean convolution of probability measures on[20] $\mathbb{R}$ and of boolean independence.[15,19] Here we present a generalization of the latter.

Let $(\mathcal{A}, \phi)$ be a *noncommutative probability space*, i.e. $\mathcal{A}$ is a complex $*$-algebra with a unit $\mathbf{1}$ and ϕ is a state on $\mathcal{A}$.

We will say that a family $\{\mathcal{A}_i\}_{i\in I}$ of $*$-subalgebras is Λ-*boolean independent* if

(a) $ab = ba$ whenever $a \in \mathcal{A}_i$, $b \in \mathcal{A}_j$, $\{i, j\} \in \Lambda$ and
(b) $\phi(a_1 a_2 \cdots a_m) = \phi(a_1)\phi(a_2)\cdots\phi(a_m)$ whenever $a_1 \in \mathcal{A}_{i_1}, \ldots, a_m \in \mathcal{A}_{i_m}$ and the word $(i_1, \ldots, i_m)$ is reduced.

Note that we do not assume that $\mathbf{1} \in \mathcal{A}_i$.

For example, if a, $c \in \mathcal{A}_i$, $b \in \mathcal{A}_j$, $i \neq j$, then

$$\phi(abc) = \begin{cases} \phi(acb) = \phi(ac)\phi(b) & \text{if } \{i, j\} \in \Lambda\,, \\ \phi(a)\phi(b)\phi(c) & \text{otherwise}\,. \end{cases}$$

To give a more general formula we define, for $\mathbf{i} = (i_1, \ldots, i_m) \in I^m$, a partition $\pi_b(\mathbf{i}) \in \mathrm{Pa}(m)$ by the following equivalence relation:

$$k \sim_b l \text{ iff } i_k = i_l \text{ and } (\{i_k, i_p\} \in \Lambda \text{ whenever } (k < p < l \text{ or } l < p < k) \text{ and } i_k \neq i_p)\,.$$

Then we have the following:

Proposition 2.1. *If* $a_1 \in \mathcal{A}_{i_1}, \ldots, a_m \in \mathcal{A}_{i_m}$ *and* $\mathbf{i} = (i_1, \ldots, i_m) \in I^m$ *then*

$$\phi(a_1 a_2 \cdots a_m) = \prod_{V\in\pi_b(\mathbf{i})} \phi\left(\prod_{l\in V} a_l\right).$$

Proof. Let us enumerate the blocks of $\pi_b(\mathbf{i})$, according to the first element of a block: $\pi_b(\mathbf{i}) = \{V_1, V_2, \ldots, V_s\}$ and if k_r denotes the first element of V_r then $k_1 < k_2 < \cdots < k_s$. Put $j_r := i_{k_r}$ and $b_r := \prod_{k \in V_r} a_k$ (ordered product). Then the word $(j_1, \ldots, j_s)$ is reduced and $a_1 \cdots a_m = b_1 \cdots b_s$. Therefore, by definition,

$$\phi(a_1 \cdots a_m) = \phi(b_1 \cdots b_s) = \phi(b_1) \cdots \phi(b_s)\,. \qquad \square$$

A natural construction of Λ-boolean independence is the following. For $i \in I$ let $\mathcal{A}_i$ be a $*$-algebra and let ϕ_i be a positive functional on $\mathcal{A}_i$ satisfying $|\phi_i(a)|^2 \leq \phi(a^*a)$ for $a \in \mathcal{A}_i$. Then ϕ_i can be extended to a state on the unital $*$-algebra $\tilde{\mathcal{A}}_i := \mathbf{C1} \oplus \mathcal{A}_i$ by $\tilde{\phi}_i(\alpha\mathbf{1} + a) := \alpha + \phi_i(a)$. We also put $\tilde{\psi}_i(\alpha\mathbf{1} + a) := \alpha$. Define $\mathcal{A}$ to be the unital Λ-free product $*^{\Lambda}_{i \in I} \tilde{\mathcal{A}}_i$ (see Ref. 12) and ϕ on $\mathcal{A}$ by

$$\phi(a_1 a_2 \cdots a_m) := \phi_{i_1}(a_1)\phi_{i_2}(a_2) \cdots \phi_{i_m}(a_m)$$

whenever $a_1 \in \mathcal{A}_{i_1}, \ldots, a_m \in \mathcal{A}_{i_m}$ (so that $\tilde{\psi}_{i_k}(a_k) = 0$) and the word $(i_1, \ldots, i_m)$ is reduced. Then ϕ is a state on $\mathcal{A}$ (see Refs. 13 and 10) and $\{\mathcal{A}_i\}_{i \in I}$ becomes a Λ-boolean independent family in $(\mathcal{A}, \phi)$.

3. Limit Theorems

Assume that $I = \mathbb{N}$. For $\Lambda \subset \mathcal{P}_2(\mathbb{N})$, $n \in \mathbb{N}$ and $\Gamma \subset \mathcal{P}_2(\{1, \ldots, n\})$ we put

$$t(n, \Gamma, \Lambda) := \lim_{N \to \infty} \frac{|\{(j_1, \ldots, j_n) \in \{1, \ldots, N\}^n_{\neq} : \{j_k, j_l\} \in \Lambda \Leftrightarrow \{k, l\} \in \Gamma\}|}{N(N-1) \cdots (N-n+1)}\,,$$

if it exists, where $\{1, \ldots, N\}^n_{\neq}$ denotes the set of all sequences $(j_1, \ldots, j_n)$ such that $j_k \in \{1, \ldots, N\}$ and $(k \neq l \Rightarrow j_k \neq j_l)$.

Lemma 3.1. *Assume that Λ is a random subset of* $\mathcal{P}_2(\mathbb{N})$, *chosen in such a way that the family of events* $\{\{i, j\} \in \Lambda\}_{1 \leq i < j}$ *is independent, each of them with probability* q. *Then, for almost all* Λ, *we have for all* $n \in \mathbb{N}$ *and for all* $\Gamma \subset \mathcal{P}_2(\{1, \ldots, n\})$

$$t(n, \Gamma, \Lambda) = q^r (1-q)^s\,,$$

where $r = |\Gamma|$, $s = n(n-1)/2 - r$.

Proof. Let us fix n and $\Gamma \subset \mathcal{P}_2(\{1, \ldots, n\})$. For $\mathbf{j} = (j_1, \ldots, j_n) \in \mathbb{N}^n$, with pairwise distinct terms, we put

$$Y_{\mathbf{j}}(\Lambda) := \begin{cases} 1 & \text{if } (\{k, l\} \in \Gamma \Leftrightarrow \{j_k, j_l\} \in \Lambda) \text{ for } 1 \leq k < l\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Then we have $E(Y_{\mathbf{j}}) = q^r(1-q)^s$, where $r = |\Gamma|$, $s = n(n-1)/2 - r$.

Now define

$$X_N(\Lambda) := \frac{1}{N(N-1) \cdots (N-n+1)} \sum_{\mathbf{j} \in \{1, \ldots, N\}^n_{\neq}} Y_{\mathbf{j}}(\Lambda)\,.$$

We need to show that

$$\lim_{N\to\infty} X_N(\Lambda) = q^r(1-q)^s \quad \text{a.e.}$$

If $\mathbf{i} = (i_1, \ldots, i_n)$, $\mathbf{j} = (j_1, \ldots, j_n) \in \{1, \ldots, N\}^n_{\neq}$ are such that $i_k = j_l$ happens at most once then $Y_\mathbf{i}$ and $Y_\mathbf{j}$ are independent, so that $E(Y_\mathbf{i} \cdot Y_\mathbf{j}) = q^{2r}(1-q)^{2s}$, and there are

$$\begin{aligned} &N(N-1)\cdots(N-2n+1) + N(N-1)\cdots(N-2n+2)n^2 \\ &\quad = N(N-1)\cdots(N-2n+2)(N+(n-1)^2) \end{aligned}$$

such cases. The coefficient at N^{2n-1} is $-n(n-1)$, the same as in $N^2(N-1)^2\cdots(N-n+1)^2$. Therefore we can estimate the variance $V(X_N) = E(X_N^2) - E(X_N)^2$ as:

$$V(X_N) = \sum_{\mathbf{i},\mathbf{j}\in\{1,\ldots,N\}^n_{\neq}} \frac{E(Y_\mathbf{i} \cdot Y_\mathbf{j}) - q^{2r}(1-q)^{2s}}{N^2(N-1)^2\cdots(N-n+1)^2} \leq \frac{C}{N^2}\,,$$

where C is independent of N. The rest of the proof is analogous to the proof of Lemma 1 in Ref. 18. □

From now on we fix $\Lambda \subset \mathcal{P}_2(\mathbb{N})$ satisfying $t(n, \Gamma, \Lambda) = q^r(1-q)^s$, where $r = |\Gamma|$, $s = n(n-1)/2 - r$, for every $n \in \mathbb{N}$ and every $\Gamma \subset \mathcal{P}_2(\{1, \ldots, n\})$.

Theorem 3.1. *Suppose that for every $k \in \{1, \ldots, m\}$, N, $i \in \mathbb{N}$ we are given an element $a_{k,N,i} \in \mathcal{A}_i$ and that for every nonempty $U \subset \{1, \ldots, m\}$*

$$\phi\left(\prod_{k\in U} a_{k,N,i}\right) = \phi\left(\prod_{k\in U} a_{k,N,j}\right) := Q(U, N)$$

for N, i, $j \in \mathbb{N}$ and that there exists the limit

$$\lim_{N\to\infty} N \cdot Q(U, N) := Q(U)\,.$$

Then, putting

$$S_{k,N} := a_{k,N,1} + a_{k,N,2} + \cdots + a_{k,N,N}\,,$$

we have

$$\lim_{N\to\infty} \phi(S_{1,N} \cdot S_{2,N} \cdots S_{m,N}) = \sum_{\pi\in\mathrm{Pa}(m)} q^{\mathrm{bc}(\pi)} \prod_{U\in\pi} Q(U)\,.$$

Proof. Put

$$\phi_1(N) := \phi(S_{1,N} \cdot S_{2,N} \cdots S_{m,N})$$

and

$$\phi_2(\mathbf{i}, N) := \phi(a_{1,N,i_1} \cdot a_{2,N,i_2} \cdots a_{m,N,i_m})$$

for $\mathbf{i} = (i_1, \ldots, i_m)$. Then we have

$$\phi_1(N) = \sum_{\mathbf{i} \in \{1,\ldots,N\}^m} \phi_2(\mathbf{i}, N)\,.$$

At this point we introduce the following notions. For $\pi \in \mathrm{Pa}(m)$ and $\Gamma \in \mathcal{P}_2(\pi)$ we say that the pair (π, Γ) is *b-admissible* if $\{U, V\} \in \Gamma$ whenever U and V are distinct blocks of π having boolean crossing. For a sequence $\mathbf{i} = (i_1, \ldots, i_m) \in I^m$ we define a partition $\pi(\mathbf{i})$ of $\{1, \ldots, m\}$ by the equivalence relation: $k \sim l$ iff $i_k = i_l$, and a family $\Gamma(\mathbf{i}) \subset \mathcal{P}_2(\pi(\mathbf{i}))$ by $\{U, V\} \in \Gamma(\mathbf{i})$ iff ($U, V \in \pi(\mathbf{i})$ and $\{i_k, i_l\} \in \Lambda$ for $k \in U$, $l \in V$). We say that the sequence $\mathbf{i}$ is *b-admissible* if the pair $(\pi(\mathbf{i}), \Gamma(\mathbf{i}))$ is b-admissible.

If $\pi(\mathbf{i}) = \pi(\mathbf{j})$ and $\Gamma(\mathbf{i}) = \Gamma(\mathbf{j})$, then $\phi_2(\mathbf{i}, N) = \phi_2(\mathbf{j}, N)$. Denoting this common value by $\phi_3(\pi(\mathbf{i}), \Gamma(\mathbf{i}), N)$ we obtain

$$\phi_1(N) = \sum_{\pi \in \mathrm{Pa}(m)} \sum_{\Gamma \subset \mathcal{P}_2(\pi)} A(\pi, \Gamma, N) \cdot \phi_3(\pi, \Gamma, N)\,,$$

where

$$A(\pi, \Gamma, N) = |\{\mathbf{i} \in \{1, \ldots, N\}^m : \pi(\mathbf{i}) = \pi \text{ and } \Gamma(\mathbf{i}) = \Gamma\}|\,.$$

We have $A(\pi, \Gamma, N) \leq N(N-1)\cdots(N - |\pi| + 1)$ and moreover

$$\lim_{N\to\infty} \frac{A(\pi, \Gamma, N)}{N^{|\pi|}} = t(|\pi|, \Gamma, \Lambda) = q^r (1-q)^s\,,$$

where $r = |\Gamma|$, $s = |\pi|(|\pi| - 1)/2 - r$.

Note that $\pi(\mathbf{i}) = \pi_b(\mathbf{i})$ if and only if $\mathbf{i}$ is b-admissible. Hence, by Proposition 2.1, we have $A(\pi, \Gamma, N)\phi_3(\pi, \Gamma, N) \to 0$ as $N \to \infty$ whenever the pair (π, Γ) is not b-admissible. Therefore

$$\lim_{N\to\infty} \phi_1(N) = \lim_{N\to\infty} \sum_{\pi \in \mathrm{Pa}(m)} B(\pi, N) \prod_{U \in \pi} Q(U, N)\,,$$

where

$$B(\pi, N) := |\{\mathbf{i} \in \{1, \ldots, N\}^m : \pi(\mathbf{i}) = \pi \text{ and } \mathbf{i} \text{ is } b\text{-admissible}\}|\,.$$

Since

$$B(\pi, N) = \sum_{\substack{\Gamma \subset \mathcal{P}_2(\pi) \\ (\pi,\Gamma)\text{admissible}}} A(\pi, \Gamma, N)\,,$$

we have

$$\lim_{N\to\infty} \frac{B(\pi, N)}{N^{|\pi|}} = q^{\mathrm{bc}(\pi)}\,,$$

which concludes the proof. □

Corollary 3.1. (The central limit theorem) *Suppose that* $a_k \in \mathcal{A}_k$, $\phi(a_k) = 0$ *and* $\phi(a_k^2) = \alpha$. *Then, putting*

$$S_N := \frac{a_1 + a_2 + \cdots + a_N}{\sqrt{N}}$$

we have

$$\lim_{N\to\infty}\phi(S_N^m) := \sum_{\pi\in\mathrm{Pa}_2(m)} \alpha^{|\pi|} q^{\mathrm{bc}(\pi)}$$

for $m \geq 0$, *in particular the limit equals* 0 *if* m *is odd.*

Corollary 3.2. (The Poisson limit theorem) *Assume that* $a_{k,N} \in \mathcal{A}_k$ *for* k, $N \in \mathbb{N}$ *and that*

$$\lim_{N\to\infty}\phi(a_{k,N}^s) = \alpha$$

for every $s \geq 1$. *Then, putting*

$$S_N := a_{1,N} + a_{2,N} + \cdots + a_{N,N}\,,$$

we have for $m \geq 0$

$$\lim_{N\to\infty}\phi(S_N^m) = \sum_{\pi\in\mathrm{Pa}(m)} \alpha^{|\pi|} q^{\mathrm{bc}(\pi)}\,.$$

4. Partitions

In this part we prove some elementary facts regarding partitions, which will be used in the final section. A partition $\pi \in \mathrm{Pa}(m)$ is said to be *noncrossing* if $k < r < l < s$, k, $l \in U \in \pi$ and r, $s \in V \in \pi$ implies $U = V$. The class of those $\pi \in \mathrm{Pa}(m)$ which are noncrossing will be denoted by $\mathrm{NC}(m)$ and $\mathrm{NC}_2(m)$ (resp. $\mathrm{NC}_{1,2}(m)$) will denote the family of such $\pi \in \mathrm{NC}(m)$ that $|U| = 2$ (resp. $|U| \leq 2$) for every $U \in \pi$.

There is a natural partial order $\prec$ on the blocks of $\pi \in \mathrm{NC}(m)$. Namely, $U \prec V$ if there are r, $s \in V$ such that $r < k < s$ holds for every $k \in U$. Now we can define *depth* of a block $U \in \pi$, namely $d(U,\pi) := |\{V \in \pi : U \prec V\}|$. Observe the following

Proposition 4.1. *If* $\pi \in \mathrm{NC}(m)$, *then*

$$\mathrm{bc}(\pi) = \sum_{U\in\pi} d(U,\pi)\,.$$

Proof. It holds because two blocks U, V of a noncrossing partition have boolean crossing if and only if either $U \prec V$ or $V \prec U$. ☐

With every $\sigma \in \mathrm{Pa}(m)$ we associate a sequence $\mathcal{E}(\sigma) = (\varepsilon_1, \ldots, \varepsilon_m) \in \{-1, 0, 1\}^m$ by

$$\varepsilon_k := \begin{cases} 1 & \text{if } k \text{ is the first element of a block } U \in \sigma, \text{with } |U| \geq 2\,, \\ -1 & \text{if } k \text{ is the last element of a block } U \in \sigma, \text{with } |U| \geq 2\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Note that

(a) $\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_r \geq 0$ whenever $1 \leq r \leq m$ and
(b) $\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_m = 0$.

Proposition 4.2. *Assume that* $\pi \in \mathrm{NC}(m)$, $\mathcal{E}(\pi) = (\varepsilon_1, \ldots, \varepsilon_m)$ *and* k (*resp.* l) *is the first* (*resp. the last*) *element of a block* $U \in \pi$, *so* $k \leq l$. *Then*

$$d(U, \pi) = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_{k-1} = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_l\,.$$

Proof. If $V \in \pi$, then $U \prec V$ iff the first element of V is less than k and the last element of V is larger than l, which proves the first equality. Moreover, if a block $W \in \pi$ has an element between k and l, then all elements of W are between k and l, which implies that $\varepsilon_k + \varepsilon_{k+1} + \cdots + \varepsilon_l = 0$. □

Let us fix a sequence $(\varepsilon_1, \ldots, \varepsilon_m) \in \{-1, 0, 1\}^m$ satisfying (a) and (b). For k such that $\varepsilon_k = 1$ put

$$f(k) := \min\{l : l > k \text{ and } \varepsilon_k + \varepsilon_{k+1} + \cdots + \varepsilon_l = 0\}$$

(such a number l exists because $\varepsilon_k + \varepsilon_{k+1} + \cdots + \varepsilon_m \leq 0$). Then $\varepsilon_{f(k)} = -1$ and if $k < r < f(k)$, $\varepsilon_r = 1$, then $\varepsilon_k + \varepsilon_{k+1} + \cdots + \varepsilon_{r-1} > 0$, hence $\varepsilon_r + \cdots + \varepsilon_{f(k)} < 0$. This implies that $k < r < f(r) < f(k)$. Therefore if we define a partition

$$\pi := \{\{k, f(k)\} : \varepsilon_k = 1\} \cup \{\{l\} : \varepsilon_l = 0\}\,,$$

then $\pi \in \mathrm{NC}_{1,2}(m)$ and $\mathcal{E}(\pi) = (\varepsilon_1, \ldots, \varepsilon_m)$. If $\sigma \in \mathrm{Pa}(m)$, with $\mathcal{E}(\sigma) = (\varepsilon_1, \ldots, \varepsilon_m)$ then we put $F(\sigma) := \pi$, so that F is a map $\mathrm{Pa}(m) \to \mathrm{NC}_{1,2}(m)$. For $\pi \in \mathrm{NC}_{1,2}(m)$ we have $F(\pi) = \pi$.

Proposition 4.3. *If* $\pi \in \mathrm{NC}_{1,2}(m)$, *then the number of* $\sigma \in \mathrm{Pa}(m)$ *such that* $F(\sigma) = \pi$ *equals*

$$\prod_{U \in \pi} (d(U, \pi) + 1)$$

and the number of $\sigma \in \mathrm{Pa}_{1,2}(m)$ *such that* $F(\sigma) = \pi$ *is*

$$\prod_{\substack{U \in \pi \\ |U| = 2}} (d(U, \pi) + 1)\,.$$

Proof. We are looking for those $\sigma \in \mathrm{Pa}(m)$ for which $\mathcal{E}(\sigma) = \mathcal{E}(\pi) := (\varepsilon_1, \ldots, \varepsilon_m)$. Consider the first number l for which $\varepsilon_l = -1$. It can be joined with one of the preceding k such that $\varepsilon_k = 1$, so the number of these choices is $\varepsilon_1 + \cdots + \varepsilon_{l-1}$. Now assume that $\varepsilon_s = -1$ and that for every $l < s$, with $\varepsilon_l = -1$, we have chosen k such that $k < l$ and $\varepsilon_k = 1$, different k's for different l's. Then the number of $r < s$ such that $\varepsilon_r = 1$ and r has not been joined so far with any l, $r < l < s$, is $\varepsilon_1 + \cdots + \varepsilon_{s-1}$.

In this way we have chosen for every block, which is not a singleton of σ, the first and the last element. Take k with $\varepsilon_k = 0$. Then there is $t := \varepsilon_1 + \cdots + \varepsilon_k$ numbers $r < k$ such that $\varepsilon_r = 1$ and r has been joined with some $s > k$. Hence k

can either belong to one of these t blocks or may form a singleton $\{k\}$. Therefore the number of such $\sigma \in \mathrm{Pa}(m)$ or $\sigma \in \mathrm{Pa}_{1,2}(m)$ that $\mathcal{E}(\sigma) = (\varepsilon_1, \ldots, \varepsilon_m)$ is equal to

$$\prod_{\substack{l=1\\ \varepsilon_l \in \{-1,0\}}}^{m} (\varepsilon_1 + \cdots + \varepsilon_l + 1) \quad \text{or} \quad \prod_{\substack{l=1\\ \varepsilon_l = -1}}^{m} (\varepsilon_1 + \cdots + \varepsilon_l + 1)\,,$$

respectively which, by Proposition 4.2, concludes the proof. □

Proposition 4.4. *If* $\sigma \in \mathrm{Pa}_{1,2}(m)$ *and* $\mathcal{E}(\sigma) = (\varepsilon_1, \ldots, \varepsilon_m)$, *then*

$$\mathrm{bc}(\sigma) = \sum_{\substack{l=1\\ \varepsilon_l \in \{-1,0\}}}^{m} (\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_l)\,.$$

In particular, $\mathrm{bc}(\sigma) = \mathrm{bc}(F(\sigma))$.

Proof. Note that if $\sigma \in \mathrm{Pa}_{1,2}(m)$, then $\mathrm{bc}(\sigma)$ is exactly the number of pairs of blocks $U = \{k, l\} \in \sigma$, $V = \{r, s\} \in \sigma$, $k \le l$, $r < s$, satisfying $r < l < s$. Fix a block $U = \{k, l\} \in \sigma$, $k \le l$. Then there are exactly $\varepsilon_1 + \cdots + \varepsilon_l$ blocks $\{r, s\} \in \sigma$ satisfying $r < l < s$. □

5. Orthogonal Polynomials Corresponding to the Limit Theorems

Assume that μ is an infinitely supported probability measure on the real line with all the moments

$$s(m) := \int_{-\infty}^{\infty} t^m d\mu(t)$$

finite. It is well known (see Ref. 7) that there exists a sequence $\{P_m(x)\}_{m=0}^{\infty}$ of polynomials, with $\deg P_m(x) = m$, which are orthogonal with respect to μ. These polynomials satisfy the recurrence relation

$$xP_m(x) = \alpha_m P_{m+1}(x) + \beta_m P_m(x) + \gamma_{m-1} P_{m-1}(x)\,,$$

$P_0(x) = 1$, with α_m, $\gamma_m > 0$, $\beta_m \in \mathbb{R}$ and under convention that $P_{-1}(x) = 0$. Accardi and Bożejko[1] (see also the work of Viennot[21]) proved the following relation between these coefficients and the moments $s(m)$:

$$s(m) = \sum_{\pi \in \mathrm{NC}_{1,2}(m)} \prod_{\substack{U \in \pi\\ |U|=2}} \alpha_{d(U,\pi)} \gamma_{d(U,\pi)} \cdot \prod_{\substack{V \in \pi\\ |V|=1}} \beta_{d(V,\pi)}\,,$$

for $m \ge 0$. Basing on this, we will prove

Theorem 5.1. *Denote by* $\mu_{q,a}$ *and* $\nu_{q,a}$ *the probability measure corresponding to the central and Poisson limit theorem, respectively, so that*

$$\sum_{\pi \in \mathrm{Pa}_2(m)} a^{|\pi|} q^{\mathrm{bc}(\pi)} = \int_{-\infty}^{\infty} t^m d\mu_{q,a}(t)\,,$$

$$\sum_{\pi \in \mathrm{Pa}(m)} a^{|\pi|} q^{\mathrm{bc}(\pi)} = \int_{-\infty}^{\infty} t^m d\nu_{q,a}(t)\,.$$

Then the families of orthogonal polynomials for $\mu_{q,a}$ and $\nu_{q,a}$ are given by the recurrence

$$xP_m(x) = aq^m P_{m+1}(x) + mP_{m-1}(x)\,,$$

$$xQ_m(x) = aq^m Q_{m+1}(x) + (aq^m + m)Q_m(x) + mQ_{m-1}(x)\,,$$

respectively, with $P_0(x) = Q_0(x) = 1$.

The theorem is an immediate consequence of the Accardi–Bożejko formula and the following two lemmas.

Lemma 5.1. *For $m \geq 0$ and $\pi \in \mathrm{NC}_2(2m)$, we have*

$$\sum_{\substack{\sigma \in \mathrm{Pa}_2(2m)\\ F(\sigma)=\pi}} a^{|\sigma|} q^{\mathrm{bc}(\sigma)} = \prod_{U \in \pi} aq^{d(U,\pi)}(d(U,\pi)+1)\,.$$

Proof. Fix $\pi \in \mathrm{NC}_2(2m)$. Then, by Proposition 4.3, there are $\prod_{U\in\pi}(d(U,\pi)+1)$ partitions $\sigma \in \mathrm{Pa}_2(2m)$ such that $F(\sigma) = \pi$. By Proposition 4.4, these partitions satisfy $\mathrm{bc}(\sigma) = \mathrm{bc}(\pi)$ and using Proposition 4.1 we have

$$\sum_{\substack{\sigma \in \mathrm{Pa}_2(2m)\\ F(\sigma)=\pi}} a^{|\sigma|} q^{\mathrm{bc}(\sigma)} = a^{|\pi|} q^{\mathrm{bc}(\pi)} \prod_{U \in \pi} (d(U,\pi)+1) = \prod_{U \in \pi} aq^{d(U,\pi)}(d(U,\pi)+1)\,.$$

□

Lemma 5.2. *For $m \geq 0$ and $\pi \in \mathrm{NC}_{1,2}(m)$ we have*

$$\sum_{\substack{\sigma \in \mathrm{Pa}(m)\\ F(\sigma)=\pi}} a^{|\sigma|} q^{\mathrm{bc}(\sigma)} = \prod_{\substack{U\in\pi\\ |U|=2}} (aq^{d(U,\pi)}(d(U,\pi)+1)) \cdot \prod_{\substack{V\in\pi\\ |V|=1}} (d(V,\pi) + aq^{d(V,\pi)})\,.$$

Proof. For a block $U = \{k_1, k_2, \ldots, k_s\}$, with $k_1 < k_2 < \cdots < k_s$, define

$$G_0(U) := \{\{k_1, k_s\}, \{k_2\}, \{k_3\}, \ldots, \{k_{s-1}\}\}\,,$$

in particular $G_0(\{k\}) = \{\{k\}\}$. Then, for $\sigma \in \mathrm{Pa}(m)$ define a partition $G(\sigma) \in \mathrm{Pa}_{1,2}(m)$

$$G(\sigma) := \bigcup_{U\in\sigma} G_0(U)\,.$$

Then $\mathcal{E}(G(\sigma)) = \mathcal{E}(\sigma)$.

Fix $\pi \in \mathrm{Pa}_{1,2}(m)$, with $\mathcal{E}(\pi) = (\varepsilon_1, \ldots, \varepsilon_m)$, and suppose that $\{k_1\}, \ldots, \{k_s\}$ are the singletons of π. If $\sigma \in \mathrm{Pa}(m)$ and $G(\sigma) = \pi$, then

$$a^{|\sigma|} q^{\mathrm{bc}(\sigma)} = a^d q^e c_1(\sigma) \cdots c_s(\sigma)\,,$$

where $d = d(\pi)$ is the number of 2-element blocks of π, $e = e(\pi)$ is the number of boolean crossings between 2-element blocks of π and

$$c_r(\sigma) := \begin{cases} aq^{t(r)} & \text{if } \{k_r\} \text{ is a singleton of } \sigma\,, \\ 1 & \text{otherwise}\,, \end{cases}$$

with $t(r) := \varepsilon_1 + \cdots + \varepsilon_{k_r}$. It holds because there are $t(r)$ blocks $\{k, l\} \in \pi$ such that $k < k_r < l$ and we have

$$S(\pi) := \sum_{\substack{\sigma\in\mathrm{Pa}(m)\\ G(\sigma)=\pi}} a^{|\sigma|} q^{\mathrm{bc}(\sigma)} = \sum_{\substack{\sigma\in\mathrm{Pa}(m)\\ G(\sigma)=\pi}} a^d q^e c_1(\sigma)\cdots c_s(\sigma)$$

$$= a^d q^e \prod_{r=1}^{s} (aq^{t(r)} + t(r))\,.$$

Note that if π_1, $\pi_2 \in \mathrm{Pa}_{1,2}(m)$ and $F(\pi_1) = F(\pi_2)$, then $d(\pi_1) = d(\pi_2)$, $e(\pi_1) = e(\pi_2)$, $S(\pi_1) = S(\pi_2)$ and if $\pi \in \mathrm{NC}_{1,2}(m)$, then $t(r) = d(\{k_r\}, \pi)$, $e = \sum_{U\in\pi, |U|=2} d(U, \pi)$ and

$$S(\pi) = \prod_{\substack{U\in\pi\\ |U|=2}} (aq^{d(U,\pi)}) \cdot \prod_{\substack{U\in\pi\\ |U|=1}} (aq^{d(U,\pi)} + d(U,\pi))\,.$$

Now we conclude the proof by applying the second part of Proposition 4.3. □

Remark 5.1. The sequences $\{P_m\}$ and $\{Q_m\}$ belong to the family of octabasic Laguerre polynomials introduced by Simion and Stanton.[16,17]

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