Infinite Dimensional Analysis, Quantum Probability
and Related Topics
Vol. 7, No. 1 (2004) 27–41

Λ-FREE PROBABILITY

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Received 2 April 2002
Revised 27 October 2003
Communicated by M. Bożejko

We introduce and study a notion of Λ-freeness, which generalises both freeness and independence in the context of noncommutative probability. In particular, we extend Voiculescu's construction of the free product of representations of unital $*$-algebras. A central limit theorem is also proved.

Keywords: Star-algebra; state; noncommutative probability space; free product; star-representation.

AMS Subject Classification: Primary 46L54, Secondary 46L09, 43A65

1. Introduction

The idea of noncommutative probability is to deal with a complex unital $*$-algebra $\mathcal{A}$, elements of which are called *random variables*, equipped with a *state* ϕ (i.e. a linear function $\phi : \mathcal{A} \to \mathbb{C}$ satisfying $\phi(\mathbf{1}) = 1$ and $\phi(a^*a) \geq 0$ for every $a \in \mathcal{A}$) which plays the role of the expectation. The pair $(\mathcal{A}, \phi)$ is called a *noncommutative probability space*. Then the classical notion of independence can be extended to this setup in the following way:

Definition 1. A family $\{\mathcal{A}_i\}_{i\in I}$ of unital subalgebras is said to be *independent* if

(1) they commute, i.e. $ab = ba$ whenever $a \in \mathcal{A}_i$, $b \in \mathcal{A}_j$, $i, j \in I$ and $i \neq j$,
(2) $\phi(a_1 \cdots a_m) = \phi(a_1) \cdots \phi(a_m)$ whenever $a_1 \in \mathcal{A}_{i_1}, \ldots, a_m \in \mathcal{A}_{i_m}$, $i_k \in I$ and $i_k \neq i_l$ for $k \neq l$.

(If $\mathcal{A}$ is an algebra with a unit $\mathbf{1}$ then by a *unital subalgebra* we will mean a subalgebra containing $\mathbf{1}$.) It is easy to check that the second condition can be replaced by the following one:

(2′) $\phi(a_1 a_2 \cdots a_m) = 0$ holds whenever $m \geq 1$, $a_1 \in \mathcal{A}_{i_1}, \ldots, a_m \in \mathcal{A}_{i_m}$, $i_k \in I$, $\phi(a_1) = \cdots = \phi(a_m) = 0$ and $i_k \neq i_l$ for $k \neq l$.

On the other hand, there is a notion of freeness, due to Voiculescu[13,14]:

Definition 2. A family $\{\mathcal{A}_i\}_{i\in I}$ of unital subalgebras is called *free* if $\phi(a_1\cdots a_m) = 0$ holds whenever $m \geq 1$, $a_1 \in \mathcal{A}_{i_1}, \ldots, a_m \in \mathcal{A}_{i_m}$, $\phi(a_1) = \cdots = \phi(a_m) = 0$, $i_k \in I$, and $i_1 \neq i_2 \neq \cdots \neq i_m$.

It is worth pointing out that independence is related to the tensor product, while freeness is related to the unital free product of algebras. The purpose of this paper is to investigate a notion of Λ-freeness, which depends on a family Λ of two-element subsets of the index set I. This notion generalises both the previous ones and is related to the Λ-free products: $*^{\Lambda}_{i\in I}\mathcal{A}_i$ of unital algebras and $*^{\Lambda}_{i\in I}\phi_i$ of unital functionals on them, which will be defined and constructed in Sec. 3. Next, assuming that $\mathcal{A}_i$ are complex $*$-algebras and that every ϕ_i is a state related to a $*$-representation π_i of $\mathcal{A}_i$, we construct the Λ-free product representation $*^{\Lambda}_{i\in I}\pi_i$ of $*^{\Lambda}_{i\in I}\mathcal{A}_i$, which is related to $*^{\Lambda}_{i\in I}\phi_i$. Then we discuss Λ-freeness in the context of C^*- and W^*-algebras. Finally we prove a central limit theorem, which involves the q-deformed Gaussian measure (see Ref. 4), the parameter q depending on the asymptotical properties of the family Λ.

2. Λ-Freeness

In this section we introduce and study the notion of Λ-freeness from a purely algebraic point of view. Most of results given here extend those which are known in free probability (see Ref. 14).

From now on we fix an index set I and a family $\Lambda \subset \mathcal{P}_2(I)$, where $\mathcal{P}_2(I)$ denotes the family of two-element subsets of I. It will be convenient for us to introduce an auxiliary group $S(I,\Lambda)$ by the following presentation:

$$S(I,\Lambda) := \langle I | i^2 = e \text{ for all } i \in I \text{ and } ij = ji \text{ whenever } \{i,j\} \in \Lambda\rangle\,.$$

Note that $S(I,\Lambda)$ is an example of Coxeter group (see Refs. 6 and 10). A formal word $\mathbf{i} = (i_1,\ldots,i_m) \in I^m$ is said to be Λ-*reduced* (or simply: *reduced*) if there is no shorter word representing the element $u := i_1 i_2 \cdots i_m \in S(I,\Lambda)$. Note that if $(i_1,\ldots,i_m)$ is reduced then so is $(i_m,\ldots,i_1)$ and we have $i_m \cdots i_2 i_1 = u^{-1}$. We define the *length* of the element u putting $\ell(u) := m$. Recall that in view of the Tits theorem (see p. 50 of Ref. 6) every reduced word $\mathbf{j}$ representing u can be converted to $\mathbf{i}$ by the application of finitely many operations of the kind $(\ldots,i,j,\ldots) \mapsto (\ldots,j,i,\ldots)$ for $\{i,j\} \in \Lambda$. In this situation we say that $\mathbf{i}$ and $\mathbf{j}$ are Λ-*equivalent*, or simply: *equivalent*.

Let $\mathcal{A}$ be a unital algebra over a field $\mathcal{K}$ and let ϕ be a fixed linear function $\phi : \mathcal{A} \to \mathcal{K}$ satisfying $\phi(\mathbf{1}) = 1$. For a subalgebra $\mathcal{B} \subset \mathcal{A}$ we denote $\mathcal{B}^0 := \{b \in \mathcal{B} : \phi(b) = 0\}$.

Definition 3. Let I be a set and let Λ be a family of two-element subsets of I. Then we say that a family $\{\mathcal{A}_i\}_{i\in I}$ of unital subalgebras of $\mathcal{A}$ is Λ-*free* in $(\mathcal{A},\phi)$ if

(1) $ab = ba$ holds whenever $a \in \mathcal{A}_i$, $b \in \mathcal{A}_j$ and $\{i, j\} \in \Lambda$,
(2) $\phi(a_1 a_2 \cdots a_m) = 0$ holds whenever $m \geq 1$, $a_1 \in \mathcal{A}^0_{i_1}, \ldots, a_m \in \mathcal{A}^0_{i_m}$ and the word $(i_1, i_2, \ldots, i_m)$ is Λ-reduced.

One can see that this notion coincides with freeness if $\Lambda = \emptyset$ and with independence if $\Lambda = \mathcal{P}_2(I)$.

Now let us fix, for the rest of this section, a Λ-free family $\{\mathcal{A}_i\}_{i \in I}$ of unital subalgebras of $\mathcal{A}$.

Lemma 1. *Every product of the form $a_1 a_2 \cdots a_m$, where $a_1 \in \mathcal{A}_{i_1}, \ldots, a_m \in \mathcal{A}_{i_m}$, with $i_k \in I$, can be expressed as a scalar multiple of* $\mathbf{1}$ *plus a finite sum of products of the form $b_1 b_2 \cdots b_n$, where $n \geq 1$, $b_1 \in \mathcal{A}^0_{j_1}, \ldots, b_n \in \mathcal{A}^0_{j_n}$ and the word $(j_1, \ldots, j_n)$ is reduced.*

Proof. We proceed by induction on m. If the word $(i_1, \ldots, i_m)$ is not reduced, then after some rearrangements allowed by Λ, we may assume that $i_p = i_{p+1}$ for some p, $1 \leq p < m$. Then we can treat the product $a_p a_{p+1}$ as a single factor and apply the induction.

Now assume that the word $(i_1, \ldots, i_m)$ is reduced. Then we decompose every factor as $a_k = \beta_k \cdot \mathbf{1} + b_k$ putting $\beta_k := \phi(a_k)$ and $b_k := a_k - \beta_k \cdot \mathbf{1}$, so that $\phi(b_k) = 0$. Hence expanding the product

$$a_1 a_2 \cdots a_m = (\beta_1 \cdot \mathbf{1} + b_1)(\beta_2 \cdot \mathbf{1} + b_2) \cdots (\beta_m \cdot \mathbf{1} + b_m)$$

we get $b_1 b_2 \cdots b_m$ as one summand and every other summand is a scalar times a shorter product. □

Lemma 2. *Suppose that $a_1 \in \mathcal{A}^0_{i_1}, \ldots, a_m \in \mathcal{A}^0_{i_m}$, $b_1 \in \mathcal{A}^0_{j_1}, \ldots, b_n \in \mathcal{A}^0_{j_n}$ and that the words $(i_1, \ldots, i_m)$ and $(j_1, \ldots, j_n)$ are reduced.*

If $m = n$ and $i_1 = j_1, \ldots, i_m = j_m$, then

$$\phi(a_1 a_2 \cdots a_m b_m \cdots b_2 b_1) = \phi(a_1 b_1) \phi(a_2 b_2) \cdots \phi(a_m b_m) .$$

On the other hand, if $i_1 i_2 \cdots i_m \neq j_1 j_2 \cdots j_n$ (i.e. if the words $(i_1, \ldots, i_m)$ and $(j_1, \ldots, j_n)$ are not equivalent), then

$$\phi(a_1 a_2 \cdots a_m b_n \cdots b_2 b_1) = 0 .$$

Proof. We use induction on n. The statement is obvious if $n = 0$. Suppose that $n \geq 1$. If the word $(i_1, \ldots, i_m, j_n, \ldots, j_1)$ is reduced, then $i_1 i_2 \cdots i_m \neq j_1 j_2 \cdots j_n$ and by definition of Λ-freeness $\phi(a_1 \cdots a_m b_n \cdots b_1) = 0$, which proves the statement in this case. If, on the other hand, the word $(i_1, \ldots, i_m, j_n, \ldots, j_1)$ is not reduced, then after some rearrangements, we can assume that $i_m = j_n$. Now put $a_m b_n = \phi(a_m b_n) \cdot \mathbf{1} + c$, where $c \in \mathcal{A}^0_{i_m}$. Then

$$\begin{aligned}&\phi(a_1 \cdots a_m b_n \cdots b_1) \\ &\quad = \phi(a_m b_n) \phi(a_1 \cdots a_{m-1} b_{n-1} \cdots b_1) + \phi((a_1 \cdots a_{m-1} c)(b_{n-1} \cdots b_1)) .\end{aligned}$$

Now the second summand equals 0 by induction, because we cannot rearrange the word $(j_1, \ldots, j_{n-1})$ so that the last letter is $i_m = j_n$. Hence, we conclude the proof by applying induction to the first summand. □

Theorem 1. *Suppose that $\mathcal{A}$ is generated by the family $\{\mathcal{A}_i\}_{i\in I}$ and assume that for every $i \in I$ the restriction of ϕ to $\mathcal{A}_i$ is tracial. Then ϕ is tracial on whole $\mathcal{A}$.*

Proof. We have to prove that $\phi(ab) = \phi(ba)$ holds for every $a, b \in \mathcal{A}$. By the first lemma we can assume that $a = a_1 \cdots a_m$, $b = b_n \cdots b_1$, where $a_1 \in \mathcal{A}^0_{i_1}, \ldots, a_m \in \mathcal{A}^0_{i_m}$, $b_1 \in \mathcal{A}^0_{j_1}, \ldots, b_n \in \mathcal{A}^0_{j_n}$ and the words $(i_1, \ldots, i_m)$, $(j_1, \ldots, j_n)$ are reduced. Now we conclude the proof by applying the second lemma. □

The following result, proved by Speicher[11] in the free case, remains true for Λ-freeness:

Proposition 1. *Suppose that $a_1 \in \mathcal{A}_{i_1}, \ldots, a_m \in \mathcal{A}_{i_m}$ and that there is p, $1 \le p \le m$, such that $i_k \ne i_p$ for every $k \ne p$. Then*

$$\phi(a_1 \cdots a_m) = \phi(a_p)\phi(a_1 \cdots a_{p-1}a_{p+1} \cdots a_m)\,.$$

Proof. Similarly as before we decompose every factor as $a_k = \beta_k \cdot \mathbf{1} + b_k$, with $\beta_k = \phi(a_k)$, $\phi(b_k) = 0$. First we prove the statement assuming that $\beta_p = 0$. We have

$$\phi((\beta_1 \cdot \mathbf{1} + b_1)(\beta_2 \cdot \mathbf{1} + b_2) \cdots (\beta_m \cdot \mathbf{1} + b_m)) = \sum_{V \subset \{1,\ldots,m\}} \left(\prod_{k\in V} \beta_k\right) \cdot \phi\left(\prod_{l \notin V} b_l\right).$$

We will show that every summand is equal to 0. It is obvious if $p \in V$. If $p \notin V$ and V is nonempty, then it holds by induction on m. Now let us consider the summand $\phi(b_1 b_2 \cdots b_m)$ related to $V = \emptyset$. If the word $(i_1, i_2, \ldots, i_m)$ is reduced, then this summand is equal 0 by definition. On the other hand, if $(i_1, \ldots, i_m)$ is not reduced then we can assume that there are two neighbouring factors which come from the same subalgebra. Then replacing them by their product, we can apply the induction.

Now, in the general case we have

$$\begin{aligned}\phi(a_1 \cdots a_m) &= \phi(a_1 \cdots a_{p-1}(\phi_{i_p}(a_p) \cdot \mathbf{1} + b_p)a_{p+1} \cdots a_m) \\ &= \phi_{i_p}(a_p) \cdot \phi(a_1 \cdots a_{p-1}a_{p+1} \cdots a_m) + \phi(a_1 \cdots a_{p-1}b_p a_{p+1} \cdots a_m)\,,\end{aligned}$$

and the second summand is 0 by the first part of the proof. □

Corollary 1. *If $a_1 \in \mathcal{A}_{i_1}, \ldots, a_m \in \mathcal{A}_{i_m}$ and if the indices $i_1, \ldots, i_m$ are pairwise distinct then $\phi(a_1 a_2 \cdots a_m) = \phi(a_1)\phi(a_2) \cdots \phi(a_m)$.*

Note that the next proposition appears in Ref. 14 for the free case, i.e. when $\Lambda = \emptyset$, under a superfluous assumption that the word $(i_1, \ldots, i_m)$ is reduced.

Proposition 2. *Assume that* $a_1 \in \mathcal{A}_{i_1}, \ldots, a_m \in \mathcal{A}_{i_m}$, *and that these subalgebras are Abelian. Then* $\phi(a_1 a_2 \cdots a_m) = \phi(a_m \cdots a_2 a_1)$.

Proof. If the word $(i_1, \ldots, i_m)$ is not reduced, then we can assume that $i_p = i_{p+1}$ for some p, $1 \leq p < m$, and apply induction on m, treating $a_p a_{p+1} = a_{p+1} a_p$ as a single factor. On the other hand, if the word $(i_1, \ldots, i_m)$ is reduced then, using the previous decompositions $a_k = \beta_k \cdot \mathbf{1} + b_k$, we can expand $\phi(a_1 \cdots a_m)$ as in the proof of Proposition 1. Now we can apply the induction to every summand related to a nonempty subset $V \subset \{1, \ldots, m\}$, while the one related to $V = \emptyset$ is equal to $\phi(b_1 \cdots b_m) = \phi(b_m \cdots b_1) = 0$. □

3. Λ-Free Product of Algebras

In this section we define and construct the Λ-free product of unital algebras and unital functionals.

Definition 4. Let $\{\mathcal{A}_i\}_{i \in I}$ be a family of unital algebras over $\mathcal{K}$. The *unital* Λ-*free product* $*^{\Lambda}_{i \in I} \mathcal{A}_i$ is the unique unital algebra $\mathcal{A}$ over $\mathcal{K}$, together with unital homomorphisms $h_i : \mathcal{A}_i \to \mathcal{A}$, satisfying $h_i(a) h_j(b) = h_j(b) h_i(a)$ whenever $a \in \mathcal{A}_i$, $b \in \mathcal{A}_j$, $\{i, j\} \in \Lambda$, such that given any unital $\mathcal{B}$ over $\mathcal{K}$ and unital homomorphisms $f_i : \mathcal{A}_i \to \mathcal{B}$, such that $f_i(a) f_j(b) = f_j(b) f_i(a)$ holds whenever $a \in \mathcal{A}_i$, $b \in \mathcal{A}_j$, $\{i, j\} \in \Lambda$, there exists a unique $f := *^{\Lambda}_{i \in I} f_i : \mathcal{A} \to \mathcal{B}$ such that $f \circ h_i = f_i$ for every $i \in I$.

Now we provide a construction of Λ-free product from which we will see that the homomorphisms h_i are in fact injections.

Let $\{\mathcal{A}_i\}_{i \in I}$ be a family of unital algebras over $\mathcal{K}$ and assume that for each $i \in I$ we have a fixed linear function $\phi_i : \mathcal{A}_i \to \mathcal{K}$ satisfying $\phi_i(\mathbf{1}) = 1$. Denote $\mathcal{A}_i^0 := \operatorname{Ker} \phi_i = \{a \in \mathcal{A}_i : \phi_i(a) = 0\}$. First take the unital free product $*_{i \in I} \mathcal{A}_i$. It can be represented as the following direct sum of the tensor products:

$$*_{i \in I} \mathcal{A}_i = \mathcal{K} \cdot \mathbf{1} \oplus \bigoplus_{\substack{m \geq 1, i_k \in I \\ i_1 \neq \cdots \neq i_m}} \mathcal{A}^0_{i_1} \otimes \mathcal{A}^0_{i_2} \otimes \cdots \otimes \mathcal{A}^0_{i_m}$$

(see Ref. 14). Define $\mathcal{A}$ to be the quotient of $*_{i \in I} \mathcal{A}_i$ over the ideal generated by the set

$$\{ab - ba : a \in \mathcal{A}_i, b \in \mathcal{A}_j, \{i, j\} \in \Lambda\}.$$

Now $\mathcal{A}$ can be identified as the direct sum

$$\mathcal{A} = \bigoplus_{u \in S(I, \Lambda)} \mathcal{A}(u),$$

where for $u \in S(I, \Lambda)$ and for a reduced word $(i_1, \ldots, i_m)$ representing u we put

$$\mathcal{A}(u) := \mathcal{A}^0_{i_1} \otimes \cdots \otimes \mathcal{A}^0_{i_m},$$

in particular $\mathcal{A}(e) := \mathcal{K} \cdot \mathbf{1}$. Due to the Λ-commutation relations, we identify the tensor products

$$\mathcal{A}^0_{i_1} \otimes \cdots \otimes \mathcal{A}^0_{i_p} \otimes \mathcal{A}^0_{i_{p+1}} \otimes \cdots \otimes \mathcal{A}^0_{i_m}$$

and

$$\mathcal{A}^0_{i_1} \otimes \cdots \otimes \mathcal{A}^0_{i_{p+1}} \otimes \mathcal{A}^0_{i_p} \otimes \cdots \otimes \mathcal{A}^0_{i_m} ,$$

whenever $\{i_p, i_{p+1}\} \in \Lambda$, in the obvious way.

Now we define $\phi : \mathcal{A} \to \mathcal{K}$ by putting

$$\phi(a) := \begin{cases} \alpha & \text{if } a = \alpha \cdot \mathbf{1} \in \mathcal{A}(e) , \\ 0 & \text{if } a \in \mathcal{A}(u), u \neq e . \end{cases}$$

Denote $\mathcal{A} = *^{\Lambda}_{i\in I}\mathcal{A}_i$, $\phi = *^{\Lambda}_{i\in I}\phi_i$ and $(\mathcal{A}, \phi) = *^{\Lambda}_{i\in I}(\mathcal{A}_i, \phi_i)$. Then it is clear from our construction that $\{\mathcal{A}_i\}_{i\in I}$ is a Λ-free family in $(\mathcal{A}, \phi)$. As a consequence of Theorem 1 we get

Corollary 2. *If all ϕ_i's are tracial then so is $\phi = *^{\Lambda}_{i\in I}\phi_i$.*

4. The Λ-Free Product of Representations

From now on we assume that $\mathcal{K} = \mathbb{C}$. Let $\{\mathcal{A}_i\}_{i\in I}$ be a family of unital $*$-algebras. Then their Λ-free product $\mathcal{A} = *^{\Lambda}_{i\in I}\mathcal{A}_i$ admits a unique involution extending those on $\mathcal{A}_i$'s. Assume that for every $i \in I$ we are given a $*$-representation $\pi_i : \mathcal{A}_i \to \mathcal{B}(\mathcal{H}_i)$ acting on a Hilbert space $\mathcal{H}_i = \mathbb{C}\xi_i \oplus \mathcal{H}^0_i$ endowed with a distinguished unit vector ξ_i. Denote by ϕ_i the corresponding *state* on $\mathcal{A}_i$, i.e. $\phi_i(a) = \langle \pi_i(a)\xi_i, \xi_i\rangle$ for $a \in \mathcal{A}_i$. We are going to construct Λ-free product representation $\pi = *^{\Lambda}_{i\in I}\pi_i$ of $\mathcal{A} = *^{\Lambda}_{i\in I}\mathcal{A}_i$, which generalises that given by Avitzour[1] and Voiculescu[13,14] in the purely free case (i.e. when $\Lambda = \emptyset$).

For $u \in S(I, \Lambda)$ and for a reduced word $(i_1, \ldots, i_m)$ representing u we denote

$$\mathcal{H}(u) := \mathcal{H}^0_{i_1} \otimes \mathcal{H}^0_{i_2} \otimes \cdots \otimes \mathcal{H}^0_{i_m} ,$$

in particular we put $\mathcal{H}(e) := \mathbb{C}\xi_0$, where ξ_0 is a distinguished unit vector. We will identify

$$\mathcal{H}^0_{i_1} \otimes \cdots \otimes \mathcal{H}^0_{i_p} \otimes \mathcal{H}^0_{i_{p+1}} \otimes \cdots \otimes \mathcal{H}^0_{i_m}$$

and

$$\mathcal{H}^0_{i_1} \otimes \cdots \otimes \mathcal{H}^0_{i_{p+1}} \otimes \mathcal{H}^0_{i_p} \otimes \cdots \otimes \mathcal{H}^0_{i_m}$$

whenever $\{i_p, i_{p+1}\} \in \Lambda$ so that $\mathcal{H}(u)$ depends only upon u but not upon its particular representation.

Now we define

$$\mathcal{H} := \bigoplus_{u\in S(I,\Lambda)} \mathcal{H}(u)$$

and denote $(\mathcal{H}, \xi_0) = *_{i\in I}^{\Lambda}(\mathcal{H}_i, \xi_i)$. Note that $\mathcal{H}_i^0 \otimes \mathcal{H}(u) = \mathcal{H}(iu)$ whenever $\ell(iu) > \ell(u)$. Therefore, for every $i \in I$, we have the following decomposition:

$$\mathcal{H} = (\mathbb{C}\xi_i \oplus \mathcal{H}_i^0) \otimes \left(\bigoplus_{\substack{u\in S(I,\Lambda) \\ \ell(iu)>\ell(u)}} \mathcal{H}(u) \right) ,$$

where we identify ξ_i with ξ_0. Now we let $\mathcal{A}_i$ act on $\mathcal{H}$ by $\tilde{\pi}_i(a) := \pi_i(a) \otimes \mathrm{Id}(i)$, $a \in \mathcal{A}_i$, where $\mathrm{Id}(i)$ denotes the identity operator acting on $\bigoplus \mathcal{H}(u)$, the sum being taken over all $u \in S(I, \Lambda)$ satisfying $\ell(iu) > \ell(u)$. Obviously $\tilde{\pi}_i$ is a $*$-representation of $\mathcal{A}_i$. In order to extend this to a $*$-representation of whole $\mathcal{A} = *_{i\in I}^{\Lambda}\mathcal{A}_i$ we need to prove

Lemma 3. *If $a \in \mathcal{A}_i$, $b \in \mathcal{A}_j$ and $\{i, j\} \in \Lambda$, then $\tilde{\pi}_i(a)\tilde{\pi}_j(b) = \tilde{\pi}_j(b)\tilde{\pi}_i(a)$.*

Proof. Let

$$\pi_i(a)\xi_i = \alpha \cdot \xi_i + \zeta \quad \text{and} \quad \pi_j(b)\xi_j = \beta \cdot \xi_j + \eta\,,$$

where α, $\beta \in \mathbb{C}$, $\zeta \in \mathcal{H}_i^0$, $\eta \in \mathcal{H}_j^0$. Take an element $v \in S$ and a vector $\mathbf{x} \in \mathcal{H}(v)$, which for a reduced word $(j_1, \ldots, j_n)$ representing v has the form $\mathbf{x} = x_1 \otimes \cdots \otimes x_n$, with $x_k \in \mathcal{H}_{j_k}^0$. Now we consider four cases

(1) Assume that $\ell(iv) > \ell(v)$ and $\ell(jv) > \ell(v)$. Then

$$\begin{aligned}\tilde{\pi}_i(a)\tilde{\pi}_j(b)\mathbf{x} &= \tilde{\pi}_i(a)(\beta \cdot \mathbf{x} + \eta \otimes \mathbf{x}) \\ &= \alpha\beta \cdot \mathbf{x} + \beta \cdot \zeta \otimes \mathbf{x} + \alpha \cdot \eta \otimes \mathbf{x} + \zeta \otimes \eta \otimes \mathbf{x} = \tilde{\pi}_j(b)\tilde{\pi}_i(a)\mathbf{x}\end{aligned}$$

as we identify $\zeta \otimes \eta$ with $\eta \otimes \zeta$.

(2) If $\ell(iv) < \ell(v)$, and $\ell(jv) < \ell(v)$, then we can assume that $i = j_1$ and $j = j_2$. Denote

$$\pi_i(a)x_1 = \alpha_1 \cdot \xi_i + \zeta_1 \quad \text{and} \quad \pi_j(b)x_2 = \beta_1 \cdot \xi_j + \eta_1\,,$$

with α_1, $\beta_1 \in \mathbb{C}$, $\zeta_1 \in \mathcal{H}_i^0$, $\eta_1 \in \mathcal{H}_j^0$. Then

$$\begin{aligned}\tilde{\pi}_i(a)\tilde{\pi}_j(b)\mathbf{x} &= \tilde{\pi}_i(a)(\beta_1 \cdot x_1 \otimes x_3 \otimes \cdots \otimes x_n + x_1 \otimes \eta_1 \otimes x_3 \otimes \cdots \otimes x_n) \\ &= \alpha_1\beta_1 \cdot x_3 \otimes \cdots \otimes x_n + \beta_1 \cdot \zeta_1 \otimes x_3 \otimes \cdots \otimes x_n \\ &\quad + \alpha_1 \cdot \eta_1 \otimes x_3 \otimes \cdots \otimes x_n + \zeta_1 \otimes \eta_1 \otimes x_3 \otimes \cdots \otimes x_n = \tilde{\pi}_j(b)\tilde{\pi}_i(a)\mathbf{x}\,.\end{aligned}$$

(3) Now if $\ell(iu) < \ell(u)$ but $\ell(jv) > \ell(v)$, then assuming that $i = j_1$ and using the previous notation,

$$\begin{aligned}\tilde{\pi}_i(a)\tilde{\pi}_j(b)\mathbf{x} &= \tilde{\pi}_i(a)(\beta \cdot \mathbf{x} + \eta \otimes \mathbf{x}) \\ &= \alpha_1\beta \cdot x_2 \otimes \cdots \otimes x_n + \beta \cdot \zeta_1 \otimes x_2 \otimes \cdots \otimes x_n \\ &\quad + \alpha_1 \cdot \eta \otimes x_2 \otimes \cdots \otimes x_n + \eta \otimes \zeta_1 \otimes x_2 \otimes \cdots \otimes x_n = \tilde{\pi}_j(b)\tilde{\pi}_i(a)\mathbf{x}\,.\end{aligned}$$

The remaining case, when $\ell(iu) > \ell(u)$ and $\ell(jv) < \ell(v)$, can be checked similarly.

□

Now, by definition of the Λ-free product of unital algebras, we can extend the family $\{\tilde{\pi}_i\}_{i\in I}$ to a $*$-representation π of $\mathcal{A}$, so that

$$\pi(a_1 a_2 \cdots a_m) := \tilde{\pi}_{i_1}(a_1)\tilde{\pi}_{i_2}(a_2)\cdots\tilde{\pi}_{i_m}(a_m)$$

whenever $a_k \in \mathcal{A}_{i_k}$ and $(i_1, \ldots, i_m)$ is a reduced word. We show that the state $\phi(a) = \langle \pi(a)\xi_0, \xi_0\rangle$ on $\mathcal{A}$ is equal to $*^{\Lambda}_{i\in I}\phi_i$.

Theorem 2. *Assume that* $a_1 \in \mathcal{A}^0_{i_1}, \ldots, a_m \in \mathcal{A}^0_{i_m}$, $m \geq 1$ *and that* $(i_1, \ldots, i_m)$ *is a reduced word. Then* $\phi(a_1 a_2 \cdots a_m) = 0$.

Proof. Put $\zeta_k := \pi_{i_k}(a_k)\xi_{i_k}$. Then

$$\langle \zeta_k, \xi_{i_k}\rangle = \langle \pi_{i_k}(a_k)\xi_{i_k}, \xi_{i_k}\rangle = \phi_{i_k}(a_k) = 0$$

hence $\zeta_k \in \mathcal{H}^0_{i_k}$ and consequently

$$\pi(a_1 \cdots a_m)\xi_0 = \tilde{\pi}_{i_1}(a_1)\cdots\tilde{\pi}_{i_m}(a_m)\xi_0 = \zeta_1 \otimes \cdots \otimes \zeta_m \in \mathcal{H}(i_1 i_2 \cdots i_m)\,,$$

so that $\phi(a_1 \cdots a_m) = \langle \pi(a_1 \cdots a_m)\xi_0, \xi_0\rangle = 0$. □

5. C^*- and W^*-Probability Spaces

The results from the previous section allow us to study Λ-freeness and Λ-free products in the context of C^*- and W^*-algebras.

Definition 5. A *C^*-probability space* is a pair $(\mathcal{A}, \phi)$, where $\mathcal{A}$ is a unital C^*-algebra and ϕ is a state on $\mathcal{A}$.

A *W^*-proability space* is a pair $(\mathcal{A}, \phi)$, where $\mathcal{A}$ is a W^*-algebra and ϕ is a normal state on $\mathcal{A}$.

Using the same methods as in the purely free case (see Ref. 14) one can prove

Proposition 3. *If* $(\mathcal{A}, \phi)$ *is a C^*-probability space and* $\{\mathcal{A}_i\}_{i\in I}$ *is a Λ-free family of unital $*$-subalgebras, then the family* $\{C^*(\mathcal{A}_i)\}_{i\in I}$ *is Λ-free.*

If $(\mathcal{A}, \phi)$ *is a W^*-probability space and* $\{\mathcal{A}_i\}_{i\in I}$ *is a Λ-free family of unital $*$-subalgebras, then the family of bicommutants* $\{\mathcal{A}''_i\}_{i\in I}$ *is Λ-free.*

We are also able to define the Λ-free product of C^*- and of W^*-algebras.

Definition 6. Let $\{\mathcal{A}_i\}_{i\in I}$ be a family of unital C^*-algebras. The *unital C^*-algebra Λ-free product* $*^{\Lambda}_{i\in I}\mathcal{A}_i$ is the unique unital C^*-algebra $\mathcal{A}$, together with unital homomorphisms $h_i : \mathcal{A}_i \to \mathcal{A}$, satisfying $h_i(a)h_j(b) = h_j(b)h_i(a)$ whenever $a \in \mathcal{A}_i$, $b \in \mathcal{A}_j$, $\{i,j\} \in \Lambda$, such that given any unital C^*-algebra $\mathcal{B}$ and unital homomorphisms $f_i : \mathcal{A}_i \to \mathcal{B}$, such that $f_i(a)f_j(b) = f_j(b)f_i(a)$ holds whenever $a \in \mathcal{A}_i$, $b \in \mathcal{A}_j$, $\{i,j\} \in \Lambda$, there exists a unique $f := *^{\Lambda}_{i\in I} f_i : \mathcal{A} \to \mathcal{B}$ such that $f \circ h_i = f_i$ for every $i \in I$.

This algebra $\mathcal{A}$ is the enveloping C^* algebra of the $*$-algebra Λ-free product of the $\mathcal{A}_i$'s. Due to the construction from the previous section, the homomorphims h_i are in fact injections.

Definition 7. Let $\{\mathcal{A}_i\}_{i\in I}$ be a family of W^* algebras, each $\mathcal{A}_i$ acting on a Hilbert space $\mathcal{H}_i$, endowed with a distinguished unit vector ξ_i. Their *W^*-algebra Λ-free product* $*_{i\in I}^{\Lambda}\mathcal{A}_i = \mathcal{A}$, acting on $(\mathcal{H}, \xi_0) = *_{i\in I}^{\Lambda}(\mathcal{H}_i, \xi_i)$ is defined as

$$\mathcal{A} := \left(\bigcup_{i\in I} \tilde{\pi}_i(\mathcal{A}_i)\right)'' ,$$

where the representations $\tilde{\pi}_i$ are those defined in the previous section.

Let $(\mathcal{A}, \phi)$ be a C^*-probability space. For an element $a \in \mathcal{A}$ we will denote by $\langle a \rangle$ the unital C^*-subalgebra generated by a. We will say that a family $\{a_i\}_{i\in I}$ of elements of $\mathcal{A}$ is *Λ-free* if the family of subalgebras $\{\langle a_i \rangle\}_{i\in I}$ is Λ-free. Now assume that $I = \{1, 2, \ldots, m\}$ and that $\{a_i\}_{i\in I}$ is a Λ-free family of self-adjoint elements of $\mathcal{A}$ and let μ_i denote the *distribution* of a_i, i.e. the unique compactly supported probability measure on the real line satisfying

$$\phi(a_i^k) = \int t^k \, d\mu_i(t)\,, \quad k = 0, 1, 2, \ldots .$$

Then the distribution μ of $a = a_1 + \cdots + a_m$ depends only on the measures μ_i. Denoting

$$\mu := \mu_1 \oplus \cdots \oplus \mu_m(\Lambda)$$

we obtain the Λ-convolution of m compactly supported probability measures on $\mathbb{R}$ (which, of course, for $m = 2$ corresponds either to the classical or to the free convolution). Using the methods of Bercovici and Voiculescu[2] one can extend this to an m-ary operation on the whole class of probability measures on $\mathbb{R}$.

6. Limit Theorems

This part is devoted to limit theorems in Λ-free probability. As the main tool we use the notion of admissibility (see Ref. 11) adapted to Λ-freeness. We go on assuming that $\{\mathcal{A}_i\}_{i\in I}$ is a Λ-free family of unital subalgebras in our noncommutative probability space $(\mathcal{A}, \phi)$.

For $m \geq 1$ we denote by $\mathrm{Pa}(m)$ the class of all partition of the set $\{1, 2, \ldots, m\}$. If U, V are distinct blocks of a partition $\pi \in \mathrm{Pa}(m)$, then we say that they *cross* if there are $1 \leq k < r < l < s \leq m$ such that $k,\, l \in U,\ r,\, s \in V$. Set

$$c(\pi) := |\{\{U, V\} \in \mathcal{P}_2(\pi) : U \quad \text{and} \quad V \text{ cross}\}|\,.$$

Note that in general $c(\pi)$ is different from the left-reduced number of crossings introduced by Nica[9] and from the number of restricted crossings, studied by Biane[3] but these three numbers do coincide if π is a pair partition.

With a word $\mathbf{i} = (i_1, \ldots, i_m) \in I^m$ we associate a partition $\pi(\mathbf{i}) \in \mathrm{Pa}(m)$ by the following equivalence relation:

$$k \sim l \text{ if and only if } i_k = i_l \,.$$

A block $U = \{k(1), \ldots, k(t)\} \in \pi(\mathbf{i})$, $k(1) < k(2) < \cdots < k(t)$, is said to be a *segment* if $\{i_{k(1)}, i_l\} \in \Lambda$ holds for every l satisfying $k(1) < l < k(t)$ and $l \notin U$. In particular, every singleton $\{k\} \in \pi(\mathbf{i})$ is a segment. Note that if $a_1 \in \mathcal{A}_{i_1}, \ldots, a_m \in \mathcal{A}_{i_m}$ and U is a segment of $\pi(i_1, \ldots, i_m)$ then, in view of Proposition 1,

$$\phi(a_1 \cdots a_m) = \phi\left(\prod_{k \in U} a_k\right) \phi\left(\prod_{l \notin U} a_l\right)$$

(writing $\prod_{l \in V} a_l$ we mean the *ordered product*, i.e. $\prod_{l \in V} a_l := a_{l(1)} a_{l(2)} \cdots a_{l(s)}$ if $V = \{l(1), l(2), \ldots, l(s)\}$ and $l(1) < l(2) < \cdots < l(s)$).

Definition 8. A sequence $\mathbf{i} = (i_1, \ldots, i_m)$ is said to be *admissible* if either $|\pi(\mathbf{i})| = 1$ or $\pi(\mathbf{i})$ has a segment $U = \{k(1), \ldots, k(t)\}$, $k(1) < \cdots < k(t)$, and the subsequence

$$(i_1, \ldots, \check{i}_{k(1)}, \ldots, \check{i}_{k(t)}, \ldots, i_m)\,,$$

of length $m - t$ obtained from $\mathbf{i}$ by removing terms i_k for $k \in U$ is also admissible.

Note that if $a_1 \in \mathcal{A}_{i_1}, \ldots, a_m \in \mathcal{A}_{i_m}$ and if $\mathbf{i} = (i_1, \ldots, i_m)$ is admissible, then

$$\phi(a_1 \cdots a_m) = \prod_{U \in \pi(\mathbf{i})} \phi\left(\prod_{k \in U} a_k\right).$$

Proposition 4. *A word* $\mathbf{i} = (i_1, \ldots, i_m) \in I^m$ *is admissible if and only if*

$$1 \le p < r < q < s \le m \quad \text{and} \quad i_p = i_q \ne i_r = i_s \text{ implies } \{i_p, i_r\} \in \Lambda\,. \qquad (*)$$

Proof. Assume that $\mathbf{i}$ satisfies $(*)$. Since this property is inherited by subsequences, it is enough to show that $\pi(\mathbf{i})$ has a segment. Define a partial order on blocks of $\pi(\mathbf{i})$ by $V \prec U = \{k(1), \ldots, k(t)\}$, $k(1) < k(2) < \cdots < k(t)$ if and only if there exists s such that $k(s) < l < k(s+1)$ for every $l \in V$. Then it is easy to see that every block which is minimal with respect to "$\prec$" is a segment.

On the other hand, if $k < r < l < s$, $i_k = i_l \ne i_r = i_s$ and $\{i_k, i_r\} \notin \Lambda$ then none of the blocks containing k, l or r, s can become a segment. □

Definition 9. Suppose that $a_1 \in \mathcal{A}_{i_1}, \ldots, a_m \in \mathcal{A}_{i_m}$. By an *elementary moment* we will mean a term of the form $\phi(a_{k(1)} a_{k(2)} \cdots a_{k(t)})$, where $1 \le k(1) < k(2) < \cdots < k(t) \le m$ and $i_{k(1)} = i_{k(2)} = \cdots = i_{k(t)}$.

The following two lemmas in the free case were proved by Speicher.[11]

Lemma 4. *Assume that* $a_1 \in \mathcal{A}_{i_1}, \ldots, a_m \in \mathcal{A}_{i_m}$. *Then* $\phi(a_1 \cdots a_m)$ *is a sum of products of elementary moments, with sign "+" or "−", where every summand consists of at least* M *factors, where* $M = |\{i_1, \ldots, i_m\}|$.

Proof. We proceed by induction on m. If the word $(i_1, \ldots, i_m)$ is not reduced then we can assume that $i_p = i_{p+1}$ for some $1 \le p < m$ and apply induction, treating $a_p a_{p+1}$ as a single factor.

If $(i_1, \ldots, i_m)$ is reduced, then we have

$$\phi((a_1 - \phi(a_1)\mathbf{1})(a_2 - \phi(a_2)\mathbf{1}) \cdots (a_m - \phi(a_m)\mathbf{1})) = 0$$

hence

$$\phi(a_1 \cdots a_m) = \sum_{\emptyset \neq U \subset \{1,\ldots,m\}} (-1)^{|U|+1} \prod_{k \in U} \phi(a_k) \phi\left(\prod_{l \notin U} a_l\right)$$

and we apply the induction to every summand. □

Lemma 5. *Assume that the word $(i_1, \ldots, i_m)$ is not admissible. Then $\phi(a_1 \cdots a_m)$ is a sum of products of elementary moments, with signs "+" or "−", where every summand consists of more than M factors, where $M = |\{i_1, \ldots, i_m\}|$.*

Proof. First observe that if $\{i_u, i_{u+1}\} \in \Lambda$ and $i_v = i_{v+1}$ then the operations

$$(\ldots, i_u, i_{u+1}, \ldots) \mapsto (\ldots, i_{u+1}, i_u, \ldots)$$

and

$$(\ldots, i_v, i_{v+1}, \ldots) \mapsto (\ldots, i_v, \ldots)$$

preserve admissibility (and nonadmissibility) of a word. Therefore we can assume that $(i_1, \ldots, i_m)$ is reduced.

Now we proceed by induction on m. If a_1, $a_2 \in \mathcal{A}_i$, b_1, $b_2 \in \mathcal{A}_j$, $i \neq j$ and $\{i, j\} \notin \Lambda$, then

$$\phi(a_1 b_1 a_2 b_2) = \phi(a_1)\phi(a_2)\phi(b_1 b_2) + \phi(a_1 a_2)\phi(b_1)\phi(b_2) - \phi(a_1)\phi(b_1)\phi(a_2)\phi(b_2)\,,$$

which proves the assertion for $m = 4$.

Assume that $\mathbf{i}$ is nonadmissible, reduced and that $p < r < q < s$, $i_p = i_q \neq i_r = i_s$, $\{i_p, i_r\} \neq \Lambda$. Now we expand as before:

$$\phi(a_1 \cdots a_m) = \sum_{\emptyset \neq U \subset \{1,\ldots,m\}} (-1)^{|U|+1} \prod_{k \in U} \phi(a_k) \phi\left(\prod_{l \notin U} a_l\right).$$

By the previous lemma the summand related to U is a sum of at least $|U| + |\{i_l : l \notin U\}|$ elementary moments. This number is obviously greater than M if $U \cap \{p, r, q, s\} \neq \emptyset$, and for the other summands the induction is applicable. □

From now on we assume that $I = \mathbb{N}$ and that for every $k \in \{1, \ldots, m\}$ and N, $i \in \mathbb{N}$ we are given elements $a(k, N, i) \in \mathcal{A}_i$ such that for every $U \subset \{1, \ldots, m\}$, N, i, $j \in \mathbb{N}$:

$$Q(U, N) := \phi\left(\prod_{k \in U} a(k, N, i)\right) = \phi\left(\prod_{k \in U} a(k, N, j)\right).$$

Put

$$S(k,N) := a(k,N,1) + a(k,N,2) + \cdots + a(k,N,N)\,.$$

We want to study limit of

$$\phi_1(N) := \phi(S(1,N)\cdots S(m,N))$$

when $N\to\infty$. We have

$$\phi_1(N) = \sum_{i_1,\ldots,i_m\in\{1,\ldots,N\}} \phi_2((i_1\ldots,i_m),N)\,,$$

where

$$\phi_2((i_1,\ldots,i_m),N) := \phi(a(1,N,i_1)a(2,N,i_2)\cdots a(m,N,i_m))\,.$$

With a word $\mathbf{i}\in I^m$ we associate $\pi(\mathbf{i})\in \mathrm{Pa}(m)$ as before and $\Gamma(\mathbf{i})\subset \mathcal{P}_2(\pi(\mathbf{i}))$ by

$$\Gamma(\mathbf{i}) = \{\{U,V\} : U,V\in\pi(\mathbf{i}) \quad \text{and} \quad \{i_p,i_r\}\in\Lambda \text{ for } p\in U, r\in V\}\,.$$

Then we have $\phi(\mathbf{i},N) = \phi(\mathbf{j},N)$ whenever $\mathbf{i}$, $\mathbf{j}\in\{1,\ldots,N\}^m$, $\pi(\mathbf{i}) = \pi(\mathbf{j})$ and $\Gamma(\mathbf{i}) = \Gamma(\mathbf{j})$, so we can define

$$\phi_3(\pi(\mathbf{i}),\Gamma(\mathbf{i}),N) := \phi_2(\mathbf{i},N)\,.$$

Now we obtain

$$\phi_1(N) = \sum_{\pi\in\mathrm{Pa}(m)}\sum_{\Gamma\subset\mathcal{P}_2(\pi)} A(\pi,\Gamma,N)\cdot\phi_3(\pi,\Gamma,N)\,,$$

where

$$A(\pi,\Gamma,N) = |\{\mathbf{i}\in\{1,\ldots,N\}^m : \pi(\mathbf{i}) = \pi \quad \text{and} \quad \Gamma(\mathbf{i}) = \Gamma\}|\,.$$

We will call a pair (π,Γ), with $\pi\in\mathrm{Pa}(m)$, $\Gamma\in\mathcal{P}_2(\pi)$, *admissible* if $k<r<l<s$, k, $l\in U\in\pi$, r, $s\in V\in\pi$ and $U\neq V$ implies $\{U,V\}\in\Gamma$.

At this point we impose the following assumptions:

Assumption 1. For every nonempty $U\subset\{1,\ldots,m\}$, $i\in\mathbb{N}$:

$$\lim_{N\to\infty} N\cdot Q(U,N) = Q(U)\,.$$

Note that by Lemma 5 this assumption excludes all nonadmissible pairs in the limit, i.e.

$$\lim_{n\to\infty} A(\pi,\Gamma,N)\cdot\phi_3(\pi,\Gamma,N) = 0\,,$$

whenever the pair (π,Γ) is nonadmissible, because

$$A(\pi,\Gamma,N)\leq N(N-1)\cdots(N-|\pi|+1)\,.$$

Therefore

$$\lim_{n\to\infty}\phi_1(N) = \lim_{N\to\infty}\sum_{\pi\in\mathrm{Pa}(m)} B(\pi,N)\cdot\prod_{U\in\pi} Q(U,N)\,,$$

where

$$B(\pi, N) = |\{\mathbf{i} \in \{1, \ldots, N\}^m : \pi(\mathbf{i}) = \pi \quad \text{and} \quad \mathbf{i} \text{ is admissible}\}| \,.$$

Secondly we assume that probability that a pair $\{i, j\}$ belongs to Λ tends to a fixed number q and that for every n the family of events: $\{j_k, j_l\} \in \Lambda$, $1 \leq k < l \leq n$, becomes independent as N tends to infinity. Namely, for n, $N \in \mathbb{N}$ define $\Omega(n, N)$ as the set of all such sequences $(j_1, \ldots, j_n)$ that $j_1, \ldots, j_n$ are pairwise distinct elements of $\{1, \ldots, N\}$ and for every $\Gamma \subset \mathcal{P}_2(\{1, \ldots, n\})$ define $\Omega(n, \Gamma, N)$ to be the set of those $(j_1, \ldots, j_n) \in \Omega(n, N)$ which satisfy:

$$\{j_k, j_l\} \in \Lambda \Leftrightarrow \{k, l\} \in \Gamma \,.$$

Assumption 2. (cf. Ref. 8) For every $n \in \mathbb{N}$, $\Gamma \subset \mathcal{P}_2(\{1, \ldots, n\})$

$$\lim_{N \to \infty} \frac{|\Omega(n, \Gamma, N)|}{|\Omega(n, N)|} = q^r (1 - q)^s \,,$$

where $r = |\Gamma|$ and $s = |\mathcal{P}_2(\{1, \ldots, n\}) \setminus \Gamma| = n(n-1)/2 - r$.

Note that $|\Omega(n, N)| = N(N-1) \cdots (N - n + 1)$, so we can replace $|\Omega(n, N)|$ by N^n in the denominator. As a consequence the probability that a given $\pi \in \mathrm{Pa}(m)$ is admissible tends to $q^{c(\pi)}$, i.e.

$$\lim_{N \to \infty} \frac{B(\pi, N)}{N^{|\pi|}} = q^{c(\pi)} \,.$$

Therefore we have proved the following limit theorem:

Theorem 3. *Under Assumptions* 1 *and* 2 *we have*

$$\lim_{N \to \infty} \phi(S(1, N) \cdots S(m, N)) = \sum_{\pi \in \mathrm{Pa}(m)} q^{c(\pi)} \prod_{U \in \pi} Q(U) \,.$$

Now we consider the central limit theorem. Denote by $\mathrm{Pa}_2(m)$ the family of all *pair partitions* of $\{1, \ldots, m\}$, i.e. partitions satisfying $|U| = 2$ for every $U \in \pi$. Of course $\mathrm{Pa}_2(m) = \emptyset$ if m is odd. Then under Assumption 2 we have

Theorem 4. (Central Limit Theorem) *Suppose that* $m \geq 1$ *and that for every* $i \in \mathbb{N}$ *we are given elements* $a(1, i), \ldots, a(m, i) \in \mathcal{A}_i$, *with* $\phi(a(k, i)) = 0$, *and satisfying*

$$\phi(a(k, i) \cdot a(l, i)) = \alpha(\{k, l\})$$

for every $1 \leq k < l \leq m$, $i \in \mathbb{N}$. *Then putting*

$$S(k, N) := \frac{a(k, 1) + a(k, 2) + \cdots + a(k, N)}{\sqrt{N}}$$

we have

$$\lim_{N\to\infty} \phi(S(1,N)\cdots S(m,N)) = \sum_{\pi\in\mathrm{Pa}_2(m)} q^{c(\pi)} \prod_{U\in\pi} \alpha(U)$$

if m is even, and this limit equals 0 if m is odd.

Proof. It is enough to put $a(k,N,i) := \frac{1}{\sqrt{N}} a(k,i)$ and apply the last theorem. □

Finally we get a version of the central limit theorem, which is analogous to those obtained by Bożejko, Kümmerer and Speicher[4,5,12] in the context of quantum probability and by Fendler[7] in the context of Coxeter groups. It involves the so-called q-deformed Gaussian measure ν_q, which is supported on the interval $[-2/\sqrt{1-q}, 2/\sqrt{1-q}]$ and has the following form:

$$\nu_q(dx) = \frac{\sin\theta}{\pi}\sqrt{1-q}\prod_{n=1}^{\infty}(1-q^n)|1-q^n e^{2i\theta}|^2 dx\,,$$

where $\theta \in [0,\pi]$ is such that $x\sqrt{1-q} = 2\cos\theta$. The corresponding family of orthogonal polynomials are the q-Hermite polynomials, for which the recurrence relation is the following: $H_0(x) = 1$, $H_1(x) = x$ and

$$x \cdot H_m(x;q) = H_{m+1}(x;q) + (1 + q + \cdots + q^{m-1})H_{m-1}(x;q)\,, \quad m \geq 1\,.$$

Corollary 3. *Suppose that for every $i \in \mathbb{N}$ we have an element $a_i \in \mathcal{A}_i$ satisfying $\phi(a_i) = 0$ and $\phi(a_i^2) = 1$ and define*

$$S_N := \frac{a_1 + a_2 + \cdots + a_N}{\sqrt{N}}\,.$$

Then, under Assumption 2, the distribution of S_N tends to the q-deformed Gaussian measure, i.e. we have

$$\lim_{N\to\infty} \phi(S_N^m) = \sum_{\pi\in\mathrm{Pa}_2(m)} q^{c(\pi)}$$

if m is even and this limit equals 0 if m is odd.

Acknowledgments

This research, supported by the Korea Science and Engineering Foundation, was carried out during a visit of the author at the Sogang University in Seoul from 1 March 2000 to 31 August 2000 under the KOSEF postdoctor fellowship programme. My special thanks go to professor Dong Myung Chung for his warm hospitality. I am also greatly indebted to Waldemar Hebisch who has suggested to me the study of Λ-free products.

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