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COMBINATORIAL RELATION BETWEEN FREE CUMULANTS AND JACOBI PARAMETERS

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We find a formula which expresses free and conditionally free cumulants in terms of Jacobi parameters. This leads to some necessary conditions for free and conditionally free infinite divisibility. We also express conditionally free cumulants of two measures in terms of their free cumulants.

Keywords: Noncrossing partitions; free cumulants; Jacobi parameters.

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1. Introduction

Free convolution is a binary, associative and commutative operation $\boxplus$ on the class $\mathcal{M}$ of probability measures on $\mathbb{R}$. It corresponds to the notion of free independence (introduced by Voiculescu[25,26]) in the same way as the classical convolution corresponds to the classical independence. There are several ways of describing free convolution.[3,4,9] Here we will use the combinatorial method due to Speicher.[22,23,21] Namely, with every $\mu \in \mathcal{M}$ having all moments, there is associated sequence $\{r_m(\mu)\}_{m=1}^{\infty}$ of real numbers, called *free cumulants*. Then, for two such measures we have: $r_m(\mu_1 \boxplus \mu_2) = r_m(\mu_1) + r_m(\mu_2)$, for every $m \geq 1$, which determines the moments of $\mu_1 \boxplus \mu_2$. For general theory of cumulants we refer to the work of Lehner.[11,12,13,14,15,16]

Conditionally free convolution, in turn, is a binary, associative and commutative operation, introduced by Bożejko, Leinert and Speicher,[7] on pairs of compactly supported probability measures on the real line. In this case, for a pair $(\widetilde{\mu}, \mu)$ of such probability measures there is a sequence $\{R_m(\widetilde{\mu}, \mu)\}_{m=1}^{\infty}$ of real numbers such that if $(\widetilde{\mu}_1, \mu_1) \boxplus (\widetilde{\mu}_2, \mu_2) = (\widetilde{\mu}, \mu)$ then $r_m(\mu) = r_m(\mu_1) + r_m(\mu_2)$ (which means that $\mu = \mu_1 \boxplus \mu_2$) and $R_m(\widetilde{\mu}, \mu) = R_m(\widetilde{\mu}_1, \mu_1) + R_m(\widetilde{\mu}_2, \mu_2)$ for every $m \geq 1$, which determines the pair $(\widetilde{\mu}, \mu)$. It turns out that the conditionally free convolution can also be defined when $\widetilde{\mu}_1, \widetilde{\mu}_2$ are operator measures.[18]

If $\mu \in \mathcal{M}$ has all moments then there is a unique sequence $\{P_m\}_{m=0}^{\infty}$ of monic polynomials, with $\deg P_m = m$, which are orthogonal with respect to μ. They satisfy the recurrence relation: $P_0(x) = 1$ and for $m \geq 0$

$$xP_m(x) = P_{m+1}(x) + \beta_m P_m(x) + \gamma_{m-1} P_{m-1}(x), \tag{1.1}$$

under convention that $P_{-1}(x) = 0$, where the *Jacobi parameters*[8] satisfy: $\beta_m \in \mathbb{R}$, $\gamma_m \geq 0$ and if $\gamma_m = 0$ for some m then $\beta_n = \gamma_n = 0$ for all $n > m$.

The aim of this paper is to express the free and conditionally free cumulants in terms of Jacobi parameters. In Sec. 5 we also express the conditionally free cumulants $R_m(\widetilde{\mu}, \mu)$ in terms of the free cumulants of $\widetilde{\mu}$ and μ.

2. Preliminaries

Throughout the paper, X will denote a finite set of natural numbers. Recall that a *partition* of X is a family π of nonempty, pairwise disjoint subsets of X, called *blocks* of π, such that $\bigcup \pi = X$. The partition π is called *noncrossing* if the following conditions: $x_1 < x_2 < x_3 < x_4$, $x_1, x_3 \in V_1 \in \pi$ and $x_2, x_4 \in V_2 \in \pi$ imply that $V_1 = V_2$. By $\mathrm{NC}(X)$ we will denote the class of all noncrossing partitions of X and $\mathrm{NC}_{1,2}(X)$ will stand for the class of all partitions $\pi \in \mathrm{NC}(X)$ such that $|V| \leq 2$ holds for every $V \in \pi$. We will use the abbreviation "(m)" instead of "$(\{1, 2, \ldots, m\})$", for example $\mathrm{NC}(\{1, 2, \ldots, m\})$ will be denoted by $\mathrm{NC}(m)$.

On every $\pi \in \mathrm{NC}(X)$ there is a natural partial order namely, $U \preceq V$ if there are $r, s \in V$ such that $r \leq k \leq s$ holds for every $k \in U$. Now we can define *depth* of a block $U \in \pi$, namely $d(U, \pi) := |\{V \in \pi : U \preceq V \neq U\}|$. If $d(U, \pi) \geq 1$, then we define *derivative* of U as the unique block $U' \in \pi$ such that $U \preceq U'$ and $d(U', \pi) = d(U, \pi) - 1$. The derivatives of higher orders are defined by putting $V^{(k)} := (V^{(k-1)})'$.

From now on we fix a probability measure μ on $\mathbb{R}$ having all the moments finite

$$s_m := \int_{\mathbb{R}} x^m d\mu(x). \tag{2.1}$$

Then its Jacobi parameters can be obtained from the Accardi–Bożejko[1] formula:

$$s_m = \sum_{\sigma \in \mathrm{NC}_{1,2}(m)} \prod_{\substack{V \in \sigma \\ |V|=1}} \beta_{d(V,\sigma)} \prod_{\substack{V \in \sigma \\ |V|=2}} \gamma_{d(V,\sigma)} \tag{2.2}$$

(cf. Viennot[24]), while for free cumulants we have:

$$s_m = \sum_{\pi \in \mathrm{NC}(m)} \prod_{V \in \pi} r_{|V|} \tag{2.3}$$

(see Speicher[22,23,21]). Both formulas involve noncrossing partitions. The aim of the next section is to find a direct combinatorial relation between free cumulants and Jacobi parameters. For this purpose we will need some additional notions.

By a *labeling* of a partition $\sigma \in \mathrm{NC}(X)$ we will mean a function κ on σ such that for every $V \in \sigma$ we have

$$\kappa(V) \in \{0, 1, \ldots, d(V, \sigma)\}.$$

The family of all labelings of σ will be denoted by $\mathrm{LAB}(\sigma)$ and $\mathrm{NCL}(X)$ (resp. $\mathrm{NCL}_{1,2}(X)$) will stand for the family of all pairs (σ, κ) with $\sigma \in \mathrm{NC}(X)$ (resp. $\sigma \in \mathrm{NC}_{1,2}(X)$) and $\kappa \in \mathrm{LAB}(\sigma)$.

With $(\sigma, \kappa) \in \mathrm{NCL}(X)$ we will associate a partition $\Pi(\sigma, \kappa)$ of X in the following way. First we define a relation on σ:

$$\overrightarrow{\mathcal{R}}_0(\sigma, \kappa) := \{(V^{(k-1)}, V^{(k)}) : V \in \sigma, 1 \le k \le \kappa(V)\}. \tag{2.4}$$

Let $\overset{\kappa}{\sim}$ be the smallest equivalence relation on σ containing $\overrightarrow{\mathcal{R}}_0(\sigma, \kappa)$. We define a partition $\Pi(\sigma, \kappa)$ of X whose blocks are of the form $\bigcup \mathcal{C}$, where $\mathcal{C} \in \sigma/\overset{\kappa}{\sim}$:

$$\Pi(\sigma, \kappa) := \left\{\bigcup \mathcal{C} : \mathcal{C} \in \sigma/\overset{\kappa}{\sim}\right\}. \tag{2.5}$$

Example. Take

$$(\sigma, \kappa) = \{\{1, 7\}_0, \{2, 5\}_0, \{3\}_2, \{4\}_0, \{6\}_1, \{8, 9\}_0\},$$

where we write V_k if $\kappa(V) = k$. Then

$$\overrightarrow{\mathcal{R}}_0(\sigma, \kappa) = \{(\{3\}, \{2, 5\}), (\{2, 5\}, \{1, 7\}), (\{6\}, \{1, 7\})\}$$

and

$$\Pi(\sigma, \kappa) = \{\{1, 2, 3, 5, 6, 7\}, \{4\}, \{8, 9\}\}.$$

It is not an accident that in this example $\Pi(\sigma, \kappa)$ is noncrossing.

Proposition 2.1.

1. *If* $(\sigma, \kappa) \in \mathrm{NCL}(X)$ *then* $\Pi(\sigma, \kappa)$ *is a noncrossing partition of* X*.*
2. *Let* $\pi \in \mathrm{NC}(X)$ *and* $(\sigma, \kappa) \in \mathrm{NCL}(X)$*. Then* $\Pi(\sigma, \kappa) = \pi$ *if and only if* σ *and* κ *admit decompositions*:

$$\sigma = \dot{\bigcup}_{U \in \pi} \sigma_U \quad \text{and} \quad \kappa = \dot{\bigcup}_{U \in \pi} \kappa_U,$$

where $\sigma_U \in \mathrm{NC}(U)$*,* $\kappa_U \in \mathrm{LAB}(\sigma_U)$ *and* $\Pi(\sigma_U, \kappa_U) = \{U\}$*.*

Proof. We will proceed in several steps.

Claim 1. *If* $U, V \in \sigma$*, then* $U \overset{\kappa}{\sim} V$ *holds if and only if there is a sequence* $U = U_0, U_1, \ldots, U_k = V$ *of blocks of* σ *and a number* $0 \le j_0 \le k$ *such that*

$$U_0 \overrightarrow{\mathcal{R}}_0 U_1, \ldots, U_{j_0 - 1} \overrightarrow{\mathcal{R}}_0 U_{j_0}, \quad U_{j_0} \overleftarrow{\mathcal{R}}_0 U_{j_0+1}, \ldots, U_{k-1} \overleftarrow{\mathcal{R}}_0 U_k,$$

where $W_1 \overleftarrow{\mathcal{R}}_0 W_2$ *means that* $W_2 \overrightarrow{\mathcal{R}}_0 W_1$.

Indeed, suppose that $V_0, \ldots, V_n$ is such a sequence of blocks of σ that $V_0 = U$, $V_n = V$ and $(V_{i-1}, V_i) \in \overrightarrow{\mathcal{R}}_0 \cup \overleftarrow{\mathcal{R}}_0$. If for some i we have $V_{i-1} \overleftarrow{\mathcal{R}}_0 V_i$ and $V_i \overrightarrow{\mathcal{R}}_0 V_{i+1}$ then $V_{i-1} = V_{i+1} = V_i'$ and hence we can remove V_i and V_{i+1} from this sequence.

As a consequence we note that the equivalency classes are "convex":

Claim 2. *If* $U \preceq V \preceq W$ *and* $U \overset{\kappa}{\sim} W$*, then* $U \overset{\kappa}{\sim} V$ *and* $V \overset{\kappa}{\sim} W$.

Now observe that every equivalence class possesses the largest block.

Claim 3. *For every* $\mathcal{C} \in \sigma/\overset{\kappa}{\sim}$ *there is* $V_0 \in \mathcal{C}$ *such that if* $V \in \mathcal{C}$ *then* $V \preceq V_0$.

As σ is a finite set, it suffices to notice that if V_0, V_1 are maximal in $\mathcal{C}$ then $V_0 = V_1$, but this is a consequence of Claim 1.

Now note that every class $\mathcal{C} \in \sigma/\overset{\kappa}{\sim}$ is a partition itself: $\mathcal{C} \in \mathrm{NC}_{1,2}(\bigcup \mathcal{C})$.

Claim 4. *For* $V \in \mathcal{C}$ *we have* $d(V, \mathcal{C}) = d(V, \sigma) - d(V_0, \sigma)$*, where* V_0 *is the largest block in* $\mathcal{C}$*. Moreover, the restriction of* κ *to* $\mathcal{C}$ *belongs to* $\mathrm{LAB}(\mathcal{C})$ *and* $\Pi(\mathcal{C}, \kappa|_{\mathcal{C}}) = \{\bigcup \mathcal{C}\}$.

The equality is a consequence of "convexity" of $\mathcal{C}$. Note also that if $V \in \mathcal{C}$ then $\kappa(V) \le d(V, \sigma) - d(V_0, \sigma)$ (for otherwise we would have $V_0' \in \mathcal{C}$) which proves the next statement. The equivalence relation resulting from $(\mathcal{C}, \kappa|_{\mathcal{C}})$ is just the restriction of $\overset{\kappa}{\sim}$ to $\mathcal{C}$, which concludes the proof of Claim 4.

Claim 5. *The partition* $\Pi(\sigma, \kappa)$ *is noncrossing.*

Suppose that $s_1 < s_2 < s_3 < s_4$ and that $s_1, s_3 \in \bigcup \mathcal{C}_1$, $s_2, s_4 \in \bigcup \mathcal{C}_2$, where $\mathcal{C}_1, \mathcal{C}_2 \in \sigma/\overset{\kappa}{\sim}$. Assume that $s_i \in V_i$ and $V_1, V_3 \in \mathcal{C}_1$, $V_2, V_4 \in \mathcal{C}_2$. Let U_i, $i = 1, 2$, be the largest (with respect to $\preceq$) block of $\mathcal{C}_i$ and let k_i (resp. l_i) be the smallest (resp. the largest) element in U_i. Then we have $k_1 \le s_1 < s_2 < s_3 \le l_1$ and $k_2 \le s_2 < s_3 < s_4 \le l_2$. Since σ is noncrossing we have either $k_2 \le k_1 < l_1 \le l_2$ or $k_1 \le k_2 < l_2 \le l_1$. In the former case we get $V_2 \preceq U_1 \preceq U_2$ which, by "convexity", implies $\mathcal{C}_1 = \mathcal{C}_2$, and the same conclusion we get in the latter case. This means that $\Pi(\sigma, \kappa)$ is noncrossing.

Therefore we have proved part 1. For part 2 one implication is a consequence of Claim 4 and the other one is obvious. □

3. Free Cumulants

From now on we fix $\mu \in \mathcal{M}$ as in the previous section. For a block $V \in \sigma \in \mathrm{NC}_{1,2}(X)$, with label k, we define its *weight* by:

$$w(V, k) := \begin{cases} \beta_0 & \text{if } |V| = 1 \text{ and } k = 0, \\ \beta_k - \beta_{k-1} & \text{if } |V| = 1 \text{ and } k \ge 1, \\ \gamma_0 & \text{if } |V| = 2 \text{ and } k = 0, \\ \gamma_k - \gamma_{k-1} & \text{if } |V| = 2 \text{ and } k \ge 1, \end{cases} \tag{3.1}$$

and for $\sigma \in \mathrm{NC}_{1,2}(X)$, $\kappa \in \mathrm{LAB}(\sigma)$ we put

$$w(\sigma, \kappa) := \prod_{V \in \sigma} w(V, \kappa(V)). \tag{3.2}$$

By $\mathrm{NCL}^1_{1,2}(X)$ we will denote the set of all $(\sigma, \kappa) \in \mathrm{NCL}_{1,2}(X)$ for which $\Pi(\sigma, \kappa) = \{X\}$. Now we are ready to present the main result of this section.

Theorem 3.1. *For every $m \geq 1$ we have*

$$r_m = \sum_{(\sigma,\kappa) \in \mathrm{NCL}^1_{1,2}(m)} w(\sigma, \kappa). \tag{3.3}$$

Proof. Denote the right-hand side of (3.3) by c_m. Then, in view of Proposition 2.1, for every $\pi \in \mathrm{NC}(m)$ we have

$$\sum_{\substack{(\sigma,\kappa) \in \mathrm{NC}_{1,2}(m) \\ \Pi(\sigma,\kappa) = \pi}} w(\sigma, \kappa) = \prod_{U \in \pi} \left(\sum_{(\sigma_U, \kappa_U) \in \mathrm{NCL}^1_{1,2}(U)} w(\sigma_U, \kappa_U) \right) = \prod_{U \in \pi} c_{|U|}.$$

Now, let us fix $\sigma \in \mathrm{NC}_{1,2}(m)$. Then expressing every factor a_d, where $a_d = \beta_d$ or $a_d = \gamma_d$, as the sum

$$a_d = (a_d - a_{d-1}) + \cdots + (a_1 - a_0) + a_0$$

and expanding the product, we get

$$\prod_{\substack{V \in \sigma \\ |V| = 1}} \beta_{d(V,\sigma)} \prod_{\substack{V \in \sigma \\ |V| = 2}} \gamma_{d(V,\sigma)} = \sum_{\kappa \in \mathrm{LAB}(\sigma)} w(\sigma, \kappa).$$

Therefore, by (2.2), for every $m \geq 1$

$$s_m = \sum_{\substack{\sigma \in \mathrm{NC}_{1,2}(m) \\ \kappa \in \mathrm{LAB}(\sigma)}} w(\sigma, \kappa) = \sum_{\pi \in \mathrm{NC}(m)} \sum_{\substack{(\sigma,\kappa) \in \mathrm{NCL}_{1,2}(m) \\ \Pi(\sigma,\kappa) = \pi}} w(\sigma, \kappa) = \sum_{\pi \in \mathrm{NC}(m)} \prod_{U \in \pi} c_{|U|}.$$

Since (2.3) defines the free cumulants uniquely, we have $r_m = c_m$ for every m. □

Examples of free cumulants. Using Theorem 3.1 we can give a list of a few free cumulants expressed in terms of Jacobi parameters (cf. Ref. 10 for a special case):

$$r_1 = \beta_0, \tag{3.4}$$

$$r_2 = \gamma_0, \tag{3.5}$$

$$r_3 = \gamma_0(\beta_1 - \beta_0), \tag{3.6}$$

$$r_4 = \gamma_0[(\beta_1 - \beta_0)^2 + (\gamma_1 - \gamma_0)], \tag{3.7}$$

$$r_5 = \gamma_0[(\beta_1 - \beta_0)^3 + 3(\gamma_1 - \gamma_0)(\beta_1 - \beta_0) + \gamma_1(\beta_2 - \beta_1)], \tag{3.8}$$

$$\begin{aligned} r_6 = \gamma_0[&(\beta_1 - \beta_0)^4 + 6(\gamma_1 - \gamma_0)(\beta_1 - \beta_0)^2 + 4\gamma_1(\beta_2 - \beta_1)(\beta_1 - \beta_0) \\ &+ \gamma_1(\beta_2 - \beta_1)^2 + 2(\gamma_1 - \gamma_0)^2 + \gamma_1(\gamma_2 - \gamma_1)]. \end{aligned} \tag{3.9}$$

In (3.8) the last summand comes from two labeled partitions, namely $\{\{1,5\}_0,\{2,4\}_0,\{3\}_2\}$ and $\{\{1,5\}_0,\{2,4\}_1,\{3\}_2\}$. Similar reductions have been done in (3.9).

Remark. One can compare (3.3) with Lehner's formula (Theorem 5.1 in Ref. 12), which in our notation can be expressed as

$$r_m = \sum_{\sigma\in \mathrm{NC}_{1,2}(m)} \frac{(-1)^{|\mathrm{Out}(\sigma)|-1}}{m-1}\binom{m-1}{|\mathrm{Out}(\sigma)|}\prod_{\substack{V\in\sigma\\ |V|=1}}\beta_{d(V,\sigma)}\prod_{\substack{V\in\sigma\\ |V|=2}}\gamma_{d(V,\sigma)}, \tag{3.10}$$

$m\geq 2$, where $|\mathrm{Out}(\sigma)|$ denotes the number of outer blocks in σ. For example

$$3r_4 = -0\beta_0^4+\gamma_0\beta_0^2+\gamma_0\beta_0^2+\gamma_0\beta_0^2-3\gamma_0\beta_0\beta_1-3\gamma_0\beta_0\beta_1+3\gamma_0\beta_1^2-3\gamma_0^2+3\gamma_0\gamma_1,$$

each summand corresponding to one partition $\sigma\in\mathrm{NC}_{1,2}(4)$.

Now let $\beta(t)$, $\gamma(t)$ be the Jacobi parameters of the free power $\mu^{\boxplus t}$. It is known[20] that $\mu^{\boxplus t}$ exists for every $t\geq 1$. Then we have $r_m(\mu^{\boxplus t}) = t\cdot r_m(\mu)$. Using formulas (3.4)–(3.9) one can consecutively check that:

$$\beta_0(t) = t\beta_0, \tag{3.11}$$

$$\gamma_0(t) = t\gamma_0, \tag{3.12}$$

$$\beta_1(t) = \beta_1-\beta_0+t\beta_0, \tag{3.13}$$

$$\gamma_1(t) = \gamma_1-\gamma_0+t\gamma_0, \tag{3.14}$$

$$\beta_2(t) = \beta_1(t)+\frac{\gamma_1(\beta_2-\beta_1)}{\gamma_1(t)}, \tag{3.15}$$

$$\gamma_2(t) = \gamma_1(t)+\frac{\gamma_1(\gamma_2-\gamma_1)\gamma_1(t)-(1-t)\gamma_0\gamma_1(\beta_2-\beta_1)^2}{\gamma_1^2(t)}. \tag{3.16}$$

In particular, if μ is infinitely divisible with respect to $\boxplus$, then we have $\gamma_m(0)\geq 0$ for all $m\geq 0$. Therefore (3.14) and (3.16) lead to the following necessary conditions:

Corollary 3.1. *If μ is $\boxplus$-infinitely divisible, then $\gamma_0\leq\gamma_1$ and*

$$\gamma_0\gamma_1(\beta_2-\beta_1)^2 \leq (\gamma_1-\gamma_0)[\gamma_1(\gamma_2-\gamma_0)-\gamma_0(\gamma_1-\gamma_0)]. \tag{3.17}$$

Corollary 3.2. *If $\gamma_0=\gamma_2\neq\gamma_1$, then μ is not $\boxplus$-infinitely divisible.*

Proof. Note that in this case the right-hand side of (3.17) is negative. □

Example. Let us consider the free Poisson law[26] ρ_λ with parameter $\lambda>0$. For this measure $\gamma_n=\lambda$, $n\geq 0$, $\beta_0=\lambda$, and $\beta_n=\lambda+1$ for $n\geq 1$. The support of ρ_λ is contained in $[0,+\infty)$ and ρ_λ if $\boxplus$-infinite divisible, in fact for any $\lambda_1,\lambda_2>0$ we have $\rho_{\lambda_1}\boxplus\rho_{\lambda_2}=\rho_{\lambda_1+\lambda_2}$. Now consider the symmetric measure ϱ_λ obtained by symmetrization of ρ_λ, so that $\int_{\mathbb{R}} f(x^2)d\varrho_\lambda(x)=\int_{\mathbb{R}} f(x)d\rho_\lambda(x)$ holds for every continuous function on $\mathbb{R}$. One can check (see for example Corollary 3 in Ref. 19) that the Jacobi parameters for ϱ_λ are given by: $\gamma'_n = \lambda$ if n is even, $\gamma'_n = 1$ if n is odd and $\beta'_n = 0$ for all $n \geq 0$. Hence, by Corollary 3.2, ϱ_λ is not $\boxplus$-infinite divisible, except the case $\lambda = 1$ (the Wigner measure).

4. Conditionally Free Cumulants

A block $U \in \pi \in \mathrm{NC}(X)$ will be called *outer* (resp. *inner*) if $d(U,\pi) = 0$ (resp. $d(U,\pi) > 0$). The family of all outer (resp. inner) blocks of π will be denoted by $\mathrm{Out}(\pi)$ (resp. $\mathrm{Inn}(\pi)$).

Suppose we have an additional measure $\widetilde{\mu}$, with moments $\widetilde{s}_m$ and Jacobi parameters $\widetilde{\gamma}_m$, $\widetilde{\beta}_m$. Then the *conditionally free cumulants*[7] $R_m = R_m(\widetilde{\mu}, \mu)$ of the pair $(\widetilde{\mu}, \mu)$ are defined by

$$\widetilde{s}_m = \sum_{\pi \in \mathrm{NC}(m)} \prod_{U \in \mathrm{Out}(\pi)} R_{|U|}(\widetilde{\mu}, \mu) \prod_{U \in \mathrm{Inn}(\pi)} r_{|U|}(\mu), \tag{4.1}$$

where, as before, $r_m = r_m(\mu)$ are the free cumulants of μ.

For $\sigma \in \mathrm{NC}_{1,2}(X)$ and $V \in \sigma$, with label k, we define

$$\widetilde{w}(V,k,\sigma) := \begin{cases} \widetilde{\beta}_k - \beta_{k-1} & \text{if } |V| = 1 \text{ and } k = d(V,\sigma), \\ \beta_k - \beta_{k-1} & \text{if } |V| = 1 \text{ and } k < d(V,\sigma), \\ \widetilde{\gamma}_k - \gamma_{k-1} & \text{if } |V| = 2 \text{ and } k = d(V,\sigma), \\ \gamma_k - \gamma_{k-1} & \text{if } |V| = 2 \text{ and } k < d(V,\sigma), \end{cases} \tag{4.2}$$

under convention that $\beta_{-1} = \gamma_{-1} = 0$. For $(\sigma,\kappa) \in \mathrm{NCL}_{1,2}(X)$ we put

$$\widetilde{w}(\sigma,\kappa) := \prod_{V \in \sigma} \widetilde{w}(V, \kappa(V), \sigma). \tag{4.3}$$

Theorem 4.1. *For every $m \geq 1$ we have*

$$R_m(\widetilde{\mu}, \mu) = \sum_{(\sigma,\kappa) \in \mathrm{NCL}^1_{1,2}(m)} \widetilde{w}(\sigma,\kappa). \tag{4.4}$$

Proof. Fix $\pi \in \mathrm{NC}(m)$. For $(\sigma,\kappa) \in \mathrm{NCL}_{1,2}(m)$, such that $\Pi(\sigma,\kappa) = \pi$, take the decompositions $\sigma = \dot{\bigcup}_{U \in \pi} \sigma_U$ and $\kappa = \dot{\bigcup}_{U \in \pi} \kappa_U$ as in Proposition 2.1. If $U \in \mathrm{Inn}(\pi)$ and $V \in \sigma_U$ then $\kappa(V) \leq d(V,\sigma_U) < d(V,\sigma)$ (see Claim 4 in the proof of Proposition 2.1) and then $\widetilde{w}(V,k,\sigma) = w(V,k)$, see (3.1). Consequently,

$$\widetilde{w}(\sigma,\kappa) = \prod_{V \in \sigma} \widetilde{w}(V,\kappa(V),\sigma) = \prod_{U \in \mathrm{Out}(\pi)} \widetilde{w}(\sigma_U,\kappa_U) \prod_{U \in \mathrm{Inn}(\pi)} w(\sigma_U,\kappa_U)$$

and, denoting the right-hand side of (4.4) by C_m,

$$\sum_{\substack{(\sigma,\kappa) \in \mathrm{NCL}_{1,2}(m) \\ \Pi(\sigma,\kappa) = \pi}} \widetilde{w}(\sigma,\kappa) = \prod_{U \in \mathrm{Out}(\pi)} C_{|U|} \prod_{U \in \mathrm{Inn}(\pi)} r_{|U|}(\mu).$$

Now writing every factor $\widetilde{a}_d$, where $\widetilde{a}_d = \widetilde{\beta}_d$ or $\widetilde{a}_d = \widetilde{\gamma}_d$, as the sum

$$\widetilde{a}_d = (\widetilde{a}_d - a_{d-1}) + \cdots + (a_1 - a_0) + a_0$$

and then expanding the products, we get

$$\begin{aligned}\widetilde{s}_m &= \sum_{\sigma\in\mathrm{NC}_{1,2}(m)} \prod_{\substack{V\in\sigma\\|V|=1}} \widetilde{\beta}_{d(V,\sigma)} \prod_{\substack{V\in\sigma\\|V|=2}} \widetilde{\gamma}_{d(V,\sigma)}\\
&= \sum_{\sigma\in\mathrm{NC}_{1,2}(m)} \sum_{\kappa\in\mathrm{LAB}(\sigma)} \widetilde{w}(\sigma,\kappa) = \sum_{\pi\in\mathrm{NC}(m)} \sum_{\substack{(\sigma,\kappa)\in\mathrm{NCL}_{1,2}(m)\\ \Pi(\sigma,\kappa)=\pi}} \widetilde{w}(\sigma,\kappa)\\
&= \sum_{\pi\in\mathrm{NC}(m)} \prod_{U\in\mathrm{Out}(\pi)} C_{|U|} \prod_{U\in\mathrm{Inn}(\pi)} r_{|U|}(\mu),\end{aligned}$$

which implies that $C_m = R_m(\widetilde{\mu},\mu)$ for every $m \geq 1$. □

Examples of conditionally free cumulants

$$R_1 = \widetilde{\beta}_0, \tag{4.5}$$

$$R_2 = \widetilde{\gamma}_0, \tag{4.6}$$

$$R_3 = \widetilde{\gamma}_0(\widetilde{\beta}_1 - \beta_0), \tag{4.7}$$

$$R_4 = \widetilde{\gamma}_0[(\widetilde{\beta}_1 - \beta_0)^2 + (\widetilde{\gamma}_1 - \gamma_0)], \tag{4.8}$$

$$\begin{aligned}R_5 = \widetilde{\gamma}_0[&(\widetilde{\beta}_1 - \beta_0)^3 + 2(\widetilde{\gamma}_1 - \gamma_0)(\widetilde{\beta}_1 - \beta_0) + (\widetilde{\gamma}_1 - \gamma_0)(\beta_1 - \beta_0)\\
&+ \widetilde{\gamma}_1(\widetilde{\beta}_2 - \beta_1)],\end{aligned} \tag{4.9}$$

$$\begin{aligned}R_6 = \widetilde{\gamma}_0[&(\widetilde{\beta}_1 - \beta_0)^4 + 3(\widetilde{\gamma}_1 - \gamma_0)(\widetilde{\beta}_1 - \beta_0)^2 + 2(\widetilde{\gamma}_1 - \gamma_0)(\widetilde{\beta}_1 - \beta_0)(\beta_1 - \beta_0)\\
&+ (\widetilde{\gamma}_1 - \gamma_0)(\beta_1 - \beta_0)^2 + 2\widetilde{\gamma}_1(\widetilde{\beta}_2 - \beta_1)(\widetilde{\beta}_1 - \beta_0) + 2\widetilde{\gamma}_1(\widetilde{\beta}_2 - \beta_1)(\beta_1 - \beta_0)\\
&+ \widetilde{\gamma}_1(\widetilde{\beta}_2 - \beta_1)^2 + (\widetilde{\gamma}_1 - \gamma_0)^2 + (\widetilde{\gamma}_1 - \gamma_0)(\gamma_1 - \gamma_0) + \widetilde{\gamma}_1(\widetilde{\gamma}_2 - \gamma_1)].\end{aligned} \tag{4.10}$$

The conditionally free power of a pair of measures: $(\widetilde{\mu},\mu)^{\boxplus t} = (\widetilde{\mu}_t, \mu_t)$ is defined by: $\mu_t = \mu^{\boxplus t}$ and $R_m(\widetilde{\mu}_t,\mu_t) = t \cdot R_m(\widetilde{\mu},\mu)$. Denoting by $\widetilde{\beta}_m(t)$, $\widetilde{\gamma}_m(t)$ the Jacobi parameters of $\widetilde{\mu}_t$, and using formulas (4.5)–(4.10) we get

$$\widetilde{\beta}_0(t) = t\widetilde{\beta}_0, \tag{4.11}$$

$$\widetilde{\gamma}_0(t) = t\widetilde{\gamma}_0, \tag{4.12}$$

$$\widetilde{\beta}_1(t) = \widetilde{\beta}_1 - \beta_0 + t\beta_0, \tag{4.13}$$

$$\widetilde{\gamma}_1(t) = \widetilde{\gamma}_1 - \gamma_0 + t\gamma_0, \tag{4.14}$$

$$\widetilde{\beta}_2(t) = \beta_1(t) + \frac{\widetilde{\gamma}_1(\widetilde{\beta}_2 - \beta_1)}{\widetilde{\gamma}_1(t)}, \tag{4.15}$$

$$\widetilde{\gamma}_2(t) = \gamma_1(t) + \frac{\widetilde{\gamma}_1(\widetilde{\gamma}_2 - \gamma_1)\widetilde{\gamma}_1(t) - (1-t)\gamma_0\widetilde{\gamma}_1(\widetilde{\beta}_2 - \beta_1)^2}{\widetilde{\gamma}_1^2(t)}. \tag{4.16}$$

Corollary 4.1. *If the pair $(\widetilde{\mu}, \mu)$ is infinitely divisible with respect to the conditionally free convolution then $\gamma_0 \leq \widetilde{\gamma}_1$ and*

$$\gamma_0\widetilde{\gamma}_1(\widetilde{\beta}_2 - \beta_1)^2 \leq (\widetilde{\gamma}_1 - \gamma_0)[\widetilde{\gamma}_1(\widetilde{\gamma}_2 - \gamma_0) - \gamma_0(\gamma_1 - \gamma_0)]. \tag{4.17}$$

5. Conditionally Free Cumulants in Terms of Free Cumulants

We keep the setting from the previous sections, so we fix a pair $\widetilde{\mu}, \mu$ of probability measures on $\mathbb{R}$, $\widetilde{s}_m$ and s_m are their moments, $\widetilde{r}_m$, r_m their free cumulants and $R_m(\widetilde{\mu}, \mu) = R_m$ the conditionally free cumulants. It turns out that R_m can also be expressed in terms of the free cumulants. Namely, denote by $\mathrm{NC}^1(X)$ the class of all such $\sigma \in \mathrm{NC}(m)$ that σ has only one outer block. Then we have

Theorem 5.1. *For every $m \geq 1$ we have*

$$R_m = \sum_{\sigma \in \mathrm{NC}^1(m)} \prod_{V \in \sigma} c(V, \sigma), \tag{5.1}$$

where for $V \in \sigma \in \mathrm{NC}^1(m)$ we define

$$c(V, \sigma) := \begin{cases} \widetilde{r}_{|V|} - r_{|V|} & \text{if } V \text{ is inner and minimal,} \\ \widetilde{r}_{|V|} & \text{otherwise.} \end{cases} \tag{5.2}$$

The word "minimal" refers to the partial order "$\preceq$" on σ.

Before the proof we introduce some auxiliary notions. For $\sigma \in \mathrm{NC}(X)$ denote by $\mathrm{Sgn}(\sigma)$ the class of all functions $\epsilon : \mathrm{Inn}(\sigma) \to \{0, 1\}$. By $\mathrm{NCS}(X)$ we will denote the class of all pairs (σ, ϵ) such that $\sigma \in \mathrm{NC}(X)$, $\epsilon \in \mathrm{Sgn}(\sigma)$. For fixed $(\sigma, \epsilon) \in \mathrm{NCS}(X)$ we define a partition $\Pi(\sigma, \epsilon)$ of X in the following way. First we define a relation $\overrightarrow{\mathcal{R}}_1$ on σ by

$$\overrightarrow{\mathcal{R}}_1 := \{(V^{(k-1)}, V^{(k)}) : V \in \mathrm{Inn}(\sigma), \epsilon(V) = 1 \text{ and } k = 1, 2, \ldots, d(V, \pi)\} \tag{5.3}$$

and then define $\overset{\epsilon}{\sim}$ as the smallest equivalence relation on σ which contains $\overrightarrow{\mathcal{R}}_1$. Now we put

$$\widetilde{\Pi}(\sigma, \epsilon) := \left\{ \bigcup \mathcal{C} : \mathcal{C} \in \sigma / \overset{\epsilon}{\sim} \right\}. \tag{5.4}$$

Denote by $\mathrm{NCS}^1(X)$ the class of all $(\sigma, \epsilon) \in \mathrm{NCS}(X)$ such that $\widetilde{\Pi}(\sigma, \epsilon) = \{X\}$. In particular, $\sigma \in \mathrm{NC}^1(X)$. For $\sigma \in \mathrm{NC}^1(X)$ we set

$$\mathrm{Sgn}^1(\sigma) := \{\epsilon \in \mathrm{Sgn}(\sigma) : \tilde{\Pi}(\sigma, \epsilon) = \{X\}\}. \tag{5.5}$$

The following lemma can be proved in a similar way as Proposition 2.1.

Lemma 5.1. 1. *If $(\sigma, \epsilon) \in \mathrm{NCS}(X)$ then the partition $\widetilde{\Pi}(\sigma, \epsilon)$ is noncrossing.*
2. *Let $\pi \in \mathrm{NC}(X)$, $(\sigma, \epsilon) \in \mathrm{NCS}(X)$. Then $\pi = \widetilde{\Pi}(\sigma, \epsilon)$ if and only if σ and ϵ admit such decompositions*:

$$\sigma = \dot{\bigcup}_{U \in \pi} \sigma_U \quad \text{and} \quad \epsilon = \dot{\bigcup}_{U \in \pi} \epsilon_U$$

that if $U \in \mathrm{Out}(\pi)$ *then* $(\sigma_U, \epsilon_U) \in \mathrm{NCS}^1(U)$ *and if* $U \in \mathrm{Inn}(\pi)$ *then* $U \in \sigma$, $\sigma_U = \{U\}$ *and* $\epsilon_U(U) = 0$.

For $V \in \sigma \in \mathrm{NC}(X)$, $\epsilon \in \mathrm{Sgn}(\sigma)$ we define:

$$\omega(V,\sigma,\epsilon) := \begin{cases} \widetilde{r}_{|V|} & \text{if } V \in \mathrm{Out}(\sigma), \\ \widetilde{r}_{|V|} - r_{|V|} & \text{if } V \in \mathrm{Inn}(\sigma) \text{ and } \epsilon(V) = 1, \\ r_{|V|} & \text{if } V \in \mathrm{Inn}(\sigma) \text{ and } \epsilon(V) = 0. \end{cases} \tag{5.6}$$

Lemma 5.2. *For* $\sigma \in \mathrm{NC}^1(X)$ *we have*

$$\sum_{\epsilon \in \mathrm{Sgn}^1(\sigma)} \prod_{V \in \sigma} \omega(V,\sigma,\epsilon) = \prod_{V \in \sigma} c(V,\sigma). \tag{5.7}$$

Consequently

$$\sum_{(\sigma,\epsilon) \in \mathrm{NCS}^1(X)} \prod_{V \in \sigma} \omega(V,\sigma,\epsilon) = \sum_{\sigma \in \mathrm{NC}^1(X)} \prod_{V \in \sigma} c(V,\sigma) \tag{5.8}$$

and this quantity depends only on $|X|$.

Proof. Equality (5.7) holds because $\mathrm{Sgn}^1(\sigma)$ consists of all $\epsilon \in \mathrm{Sgn}(\sigma)$ such that $\epsilon(V) = 1$ for those inner blocks $V \in \sigma$ which are minimal. □

Proof of Theorem 5.1. We have

$$\widetilde{s}_m = \sum_{\sigma \in \mathrm{NC}(m)} \prod_{V \in \sigma} \widetilde{r}_{|V|} = \sum_{\sigma \in \mathrm{NC}(m)} \prod_{V \in \mathrm{Out}(\sigma)} \widetilde{r}_{|V|} \cdot \prod_{V \in \mathrm{Inn}(\sigma)} [r_{|V|} + (\widetilde{r}_{|V|} - r_{|V|})]$$
$$= \sum_{\sigma \in \mathrm{NC}(m)} \sum_{\epsilon \in \mathrm{Sgn}(\sigma)} \prod_{V \in \pi} \omega(V,\sigma,\epsilon).$$

Applying first Lemma 5.1, then Lemma 5.2 and denoting by C_m the right-hand side of (5.1) we get

$$\widetilde{s}_m = \sum_{\pi \in \mathrm{NC}(m)} \sum_{\substack{(\sigma,\epsilon) \in \mathrm{NCS}(m) \\ \widetilde{\Pi}(\sigma,\epsilon) = \pi}} \prod_{V \in \sigma} \omega(V,\pi,\epsilon)$$
$$= \sum_{\pi \in \mathrm{NC}(m)} \prod_{U \in \mathrm{Inn}(\pi)} r_{|U|} \cdot \prod_{U \in \mathrm{Out}(\pi)} \left(\sum_{(\sigma_U,\epsilon_U) \in \mathrm{NCS}^1(U)} \prod_{V \in \sigma_U} \omega(V,\sigma_U,\epsilon_U) \right)$$
$$= \sum_{\pi \in \mathrm{NC}(m)} \prod_{U \in \mathrm{Inn}(\pi)} r_{|U|} \cdot \prod_{U \in \mathrm{Out}(\pi)} \left(\sum_{\sigma_U \in \mathrm{NC}^1(U)} \prod_{V \in \sigma_U} c(V,\sigma) \right)$$
$$= \sum_{\pi \in \mathrm{NC}(m)} \prod_{U \in \mathrm{Inn}(\pi)} r_{|U|} \cdot \prod_{U \in \mathrm{Out}(\pi)} C_{|U|},$$

which, by induction, implies that $C_m = R_m$ for every $m \geq 1$. □

Let us give a few examples. The consecutive summands are arranged according to the size of the outer block:

$$R_1 = \widetilde{r}_1, \tag{5.9}$$

$$R_2 = \widetilde{r}_2, \tag{5.10}$$

$$R_3 = \widetilde{r}_3 + \widetilde{r}_2(\widetilde{r}_1 - r_1), \tag{5.11}$$

$$R_4 = \widetilde{r}_4 + 2\widetilde{r}_3(\widetilde{r}_1 - r_1) + \widetilde{r}_2(\widetilde{r}_2 - r_2) + \widetilde{r}_2(\widetilde{r}_1 - r_1)^2, \tag{5.12}$$

$$\begin{aligned} R_5 = {} & \widetilde{r}_5 + 3\widetilde{r}_4(\widetilde{r}_1 - r_1) + \widetilde{r}_3[3(\widetilde{r}_1 - r_1)^2 + 2(\widetilde{r}_2 - r_2)] \\ & + \widetilde{r}_2[(\widetilde{r}_1 - r_1)^3 + 2(\widetilde{r}_2 - r_2)(\widetilde{r}_1 - r_1) + \widetilde{r}_2(\widetilde{r}_1 - r_1) + (\widetilde{r}_3 - r_3)], \end{aligned} \tag{5.13}$$

$$\begin{aligned} R_6 = {} & \widetilde{r}_6 + 4\widetilde{r}_5(\widetilde{r}_1 - r_1) + \widetilde{r}_4[6(\widetilde{r}_1 - r_1)^2 + 3(\widetilde{r}_2 - r_2)] \\ & + \widetilde{r}_3[4(\widetilde{r}_1 - r_1)^3 + 6(\widetilde{r}_2 - r_2)(\widetilde{r}_1 - r_1) + 2\widetilde{r}_2(\widetilde{r}_1 - r_1) + 2(\widetilde{r}_3 - r_3)] \\ & + \widetilde{r}_2[(\widetilde{r}_1 - r_1)^4 + 3(\widetilde{r}_2 - r_2)(\widetilde{r}_1 - r_1)^2 + 3\widetilde{r}_2(\widetilde{r}_1 - r_1)^2 \\ & + 2(\widetilde{r}_3 - r_3)(\widetilde{r}_1 - r_1) + 2\widetilde{r}_3(\widetilde{r}_1 - r_1) + (\widetilde{r}_2 - r_2)^2 \\ & + \widetilde{r}_2(\widetilde{r}_2 - r_2) + (\widetilde{r}_4 - r_4)]. \end{aligned} \tag{5.14}$$

6. Noncommutative Case

In this part we will assume that $\widetilde{\mu}$ is an *operator probability measure*[5,17] on a Hilbert space $\mathcal{H}$, i.e. a function $\widetilde{\mu} : \mathrm{Borel}(\mathbb{R}) \to \mathcal{B}_+(\mathcal{H})$ such that $\widetilde{\mu}(\mathbb{R}) = \mathrm{Id}_{\mathcal{H}}$ and for every $\xi \in \mathcal{H}$ the function $\mathrm{Borel}(\mathbb{R}) \ni A \mapsto \langle \mu(A)\xi, \xi \rangle$ is a usual measure. Here $\mathrm{Borel}(\mathbb{R})$ denotes the family of Borel subsets of $\mathbb{R}$ and $\mathcal{B}_+(\mathcal{H})$ is the class of nonnegative bounded operators on $\mathcal{H}$. We also assume that $\widetilde{\mu}$ has all moments

$$\widetilde{s}_m := \int_{\mathbb{R}} x^m d\widetilde{\mu}(x)$$

finite. Then $\{\widetilde{s}_m\}_{m=0}^{\infty}$ is a *positive definite* sequence of operators, i.e.

$$\sum_{i,j=0}^{n} \langle \widetilde{s}_{i+j}\xi_i, \xi_j \rangle \geq 0 \tag{6.1}$$

whenever $n \geq 0$, $\xi_0, \ldots, \xi_n \in \mathcal{H}$.

Now, if μ is as before a usual probability measure on $\mathbb{R}$ with finite moments, then we define conditionally free cumulants $R_m = R_m(\widetilde{\mu}, \mu)$ as operators on $\mathcal{H}$ satisfying (4.1). This leads to a generalization of the conditionally free convolution.[18]

Assume that $\{\widetilde{\beta}_m\}$, $\{\widetilde{a}^{\mathrm{l}}_m\}$, $\{\widetilde{a}^{\mathrm{r}}_m\}$ are sequences of operators on $\mathcal{H}$ and define $\widetilde{\gamma}_k = \widetilde{a}^{\mathrm{l}}_k \widetilde{a}^{\mathrm{r}}_k$. For $\sigma \in \mathrm{NC}_{1,2}(X)$, $\{i,j\} \in \sigma$, with $d(\{i,j\},\sigma) = d$, we put

$$\widetilde{w}_0(\sigma, i) := \begin{cases} \widetilde{\beta}_d & \text{if } i = j, \\ \widetilde{a}^{\mathrm{l}}_d & \text{if } i < j, \\ \widetilde{a}^{\mathrm{r}}_d & \text{if } i > j. \end{cases} \tag{6.2}$$

We will call $\{\widetilde{\beta}_m\}$, $\{\widetilde{a}^{\mathrm{l}}_m\}$, $\{\widetilde{a}^{\mathrm{r}}_m\}$ *noncommutative Jacobi parameters* of $\widetilde{\mu}$ if for every $m \geq 1$ we have

$$\widetilde{s}_m = \sum_{\sigma \in \mathrm{NC}_{1,2}(m)} \prod_{i=1}^{m} \widetilde{w}_0(\sigma, i), \tag{6.3}$$

where $\prod_{i=1}^{n} a_i$ denotes the ordered product $a_1 a_2 \cdots a_n$. The existence of noncommutative Jacobi parameters, satisfying $\widetilde{a}^{\mathrm{r}}_k = (\widetilde{a}^{\mathrm{l}}_k)^*$, was proved by Mac Nerney.[17]

We will use partitions with *partial labelings.* Fix $\sigma \in \mathrm{NC}_{1,2}(X)$. A subset $\mathcal{D} \subseteq \sigma$ will be called *hereditary* if for every $V_1 \in \sigma$ and $V_2 \in \mathcal{D}$ the relation $V_1 \preceq V_2$ implies $V_1 \in \mathcal{D}$. By $\mathcal{K}(\sigma)$ we will denote the class of all such functions κ that:

(1) the domain $\mathcal{D}(\kappa)$ of κ is a hereditary subset of σ,
(2) if $V \in \mathcal{D}(\kappa)$ then $\kappa(V) \in \{0, 1, \ldots, d(V, \sigma)\}$,
(3) if $V_1 \in \mathcal{D}(\kappa)$, $\kappa(V_1) = d(V_1, \sigma)$, $V_2 \in \sigma$ and $V_1 \preceq V_2 \neq V_1$ then $V_2 \notin \mathcal{D}(\kappa)$.

For $\kappa \in \mathcal{K}(\sigma)$ and $i \in X$, with $i \in \{i, j\} = V \in \sigma$ we define *weight* $\widetilde{w}(\sigma, i, \kappa)$ in the following way. If $V \in \mathcal{D}(\kappa)$, with $\kappa(i) = k$, then we put

$$\widetilde{w}(\sigma, i, \kappa) := \begin{cases} \widetilde{\beta}_k - \beta_{k-1} & \text{if } i = j \text{ and } k = d(V, \sigma), \\ \beta_k - \beta_{k-1} & \text{if } i = j \text{ and } k < d(V, \sigma), \\ \widetilde{\gamma}_k - \gamma_{k-1} & \text{if } i < j \text{ and } k = d(V, \sigma), \\ \gamma_k - \gamma_{k-1} & \text{if } i < j \text{ and } k < d(V, \sigma), \\ 1 & \text{if } i > j, \end{cases} \tag{6.4}$$

with the convention that $\beta_{-1} = \gamma_{-1} = 0$. If, on the other hand, $V \notin \mathcal{D}(\kappa)$ then we put $\widetilde{w}(\sigma, i, \kappa) := \widetilde{w}_0(\sigma, i)$. Finally, we define

$$\widetilde{w}(\sigma, \kappa) := \prod_{i \in X} \widetilde{w}(\sigma, i, \kappa). \tag{6.5}$$

Let $\sigma \in \mathrm{NC}_{1,2}(X)$ and let $\mathcal{D}$ be a hereditary subset of σ. By $\mathrm{PLAB}(\mathcal{D}, \sigma)$ ("partial labelings") we will denote the class of all such functions $\kappa \in \mathcal{K}(\sigma)$ that

(1) $\mathcal{D}(\kappa) \subseteq \mathcal{D}$,
(2) For $V \in \mathcal{D}$ we have: $V \in \mathcal{D}(\kappa)$ if and only if $\kappa(U) < d(U, \sigma)$ holds for every $U \in \mathcal{D}(\kappa)$ such that $U \preceq V$.

Note that if $\mathcal{D}_1 \subseteq \mathcal{D}_2 \subseteq \sigma$, with $\mathcal{D}_1$, $\mathcal{D}_2$ hereditary, and $\kappa \in \mathrm{PLAB}(\mathcal{D}_2, \sigma)$, then $\kappa|_{\mathcal{D}_1} \in \mathrm{PLAB}(\mathcal{D}_1, \sigma)$. The following lemma will be used for $\mathcal{D} = \sigma$.

Lemma 6.1. *Assume that* $\sigma \in \mathrm{NC}_{1,2}(X)$ *and* $\mathcal{D}$ *is a hereditary subset of* σ. *Then*

$$\prod_{i=1}^{m} \widetilde{w}_0(\sigma, i) = \sum_{\kappa \in \mathrm{PLAB}(\mathcal{D}, \sigma)} \widetilde{w}(\sigma, \kappa). \tag{6.6}$$

Proof. Let $X = \{1, \ldots, m\}$. If $\mathcal{D} = \emptyset$ then $\mathrm{PLAB}(\mathcal{D}, \sigma)$ consists only of the empty function and then the statement is obvious.

Assume that $\mathcal{D}_0, \mathcal{D}_1 \subseteq \sigma$ are hereditary, $\mathcal{D}_1 = \mathcal{D}_0 \dot\cup \{V_1\}$ and (6.6) holds for $\mathcal{D}_0$. Then V_1 is a minimal block in $\sigma \backslash \mathcal{D}_0$.

Fix $\kappa_0 \in \mathrm{PLAB}(\mathcal{D}_0)$. We want to describe the class

$$\mathcal{K}(\mathcal{D}_1, \kappa_0) := \{\kappa \in \mathrm{PLAB}(\mathcal{D}_1, \sigma) : \kappa|_{\mathcal{D}_0} = \kappa_0\}.$$

If there is $U \in \mathcal{D}_0$ such that $U \preceq V_1$ and $\kappa(U) = d(U, \sigma)$ then $\mathcal{K}(\mathcal{D}_1, \kappa_0) = \{\kappa_0\}$. In the opposite case we have

$$\mathcal{K}(\mathcal{D}_1, \kappa_0) = \{\kappa_0 \cup \{\langle V_1, k\rangle\} : k \in \{0, 1, \dots, d\}\},$$

where $d := d(V_1, \sigma)$. Assume that $V_1 = \{i_1, j_1\}$ with $i_1 < j_1$. If $i_1 < i < j_1$ then the weight $\widetilde{w}(\sigma, i, \kappa_0)$ is a scalar, so we can write

$$\begin{aligned}
\widetilde{w}(\sigma, \kappa_0) &= \prod_{i=1}^{i_1-1} \widetilde{w}(\sigma, i, \kappa_0) \widetilde{\gamma}_d \prod_{\substack{i=i_1+1 \\ i \neq j_1}}^{m} \widetilde{w}(\sigma, i, \kappa_0) \\
&= \prod_{i=1}^{i_1-1} \widetilde{w}(\sigma, i, \kappa_0)(\gamma_0 + (\gamma_1 - \gamma_0) + \dots + (\widetilde{\gamma}_d - \gamma_{d-1})) \prod_{\substack{i=i_1+1 \\ i \neq j_1}}^{m} \widetilde{w}(\sigma, i, \kappa_0) \\
&= \sum_{k=0}^{d} \widetilde{w}(\sigma, \kappa_0 \cup \{\langle V_1, k\rangle\}) = \sum_{\kappa \in \mathcal{K}(\mathcal{D}_1, \kappa_0)} \widetilde{w}(\sigma, \kappa).
\end{aligned}$$

Similarly we proceed when $|V_1| = 1$. Therefore

$$\sum_{\kappa_0 \in \mathrm{PLAB}(\mathcal{D}_0, \sigma)} \widetilde{w}(\sigma, \kappa_0) = \sum_{\kappa_0 \in \mathrm{PLAB}(\mathcal{D}_0, \sigma)} \sum_{\kappa \in \mathcal{K}(\mathcal{D}_1, \kappa_0)} \widetilde{w}(\sigma, \kappa) = \sum_{\kappa \in \mathrm{PLAB}(\mathcal{D}_1, \sigma)} \widetilde{w}(\sigma, \kappa),$$

so (6.6) holds for $\mathcal{D}_1$. Now we can apply induction to conclude the proof. □

Denote by $\mathrm{NCP}_{1,2}(X)$ the class of all pairs (σ, κ) with $\sigma \in \mathrm{NC}_{1,2}(X)$ and $\kappa \in \mathrm{PLAB}(\sigma, \sigma)$ (i.e. $\mathcal{D} = \sigma$). With $(\sigma, \kappa) \in \mathrm{NCP}_{1,2}(X)$ we associate an equivalence relation $\overset{\kappa}{\sim}$ on σ and the partition $\Pi_0(\sigma, \kappa)$ in similar way as before, defining

$$\overrightarrow{\mathcal{R}}_0(\sigma, \kappa) := \{(V^{(k-1)}, V^{(k)}) : V \in \mathcal{D}(\kappa), 1 \le k \le \kappa(V)\}. \tag{6.7}$$

Then a modified version of Proposition 2.1 remains true, namely

Proposition 6.1. 1. *If $(\sigma, \kappa) \in \mathrm{NCP}_{1,2}(X)$ then $\Pi_0(\sigma, \kappa)$ is noncrossing.*
2. *Let $\pi \in \mathrm{NC}(X)$ and $(\sigma, \kappa) \in \mathrm{NCL}_{1,2}(X)$. Then $\Pi_0(\sigma, \kappa) = \pi$ if and only if σ and κ admit decompositions*:

$$\sigma = \dot{\bigcup}_{U \in \pi} \sigma_U \quad \text{and} \quad \kappa = \dot{\bigcup}_{U \in \pi} \kappa_U,$$

where for every $U \in \pi$ we have $\sigma_U \in \mathrm{NC}_{1,2}(U)$ with $\kappa_U \in \mathrm{PLAB}(\sigma_U, \sigma_U)$, $\Pi_0(\sigma_U, \kappa_U) = \{U\}$ whenever $U \in \mathrm{Out}(\pi)$ and with $\kappa_U \in \mathrm{LAB}(\sigma_U)$, $\Pi(\sigma_U, \kappa_U) = \{U\}$ whenever $U \in \mathrm{Inn}(\pi)$.

Denote by $\mathrm{NCP}^1_{1,2}(X)$ the class of $(\sigma,\tau)\in\mathrm{NCP}_{1,2}(X)$ for which $\Pi_0(\sigma,\tau)=\{X\}$.

Theorem 6.1. *For every $m\geq 1$ we have*

$$R_m(\widetilde{\mu},\mu)=\sum_{(\sigma,\kappa)\in\mathrm{NCP}^1_{1,2}(m)}\widetilde{w}(\sigma,\kappa). \tag{6.8}$$

Proof. Applying Lemma 6.1 to $\mathcal{D}=\sigma$ we have

$$\widetilde{s}_m=\sum_{(\sigma,\tau)\in\mathrm{NCP}_{1,2}(m)}\widetilde{w}(\sigma,\kappa)=\sum_{\pi\in\mathrm{NC}(m)}\sum_{\substack{(\sigma,\kappa)\in\mathrm{NCP}_{1,2}(m)\\ \Pi_0(\sigma,\kappa)=\pi}}\widetilde{w}(\sigma,\kappa).$$

Fix $\pi\in\mathrm{NC}(m)$. Then, denoting the right-hand side of (6.8) by A_m,

$$\begin{aligned}\sum_{\substack{(\sigma,\kappa)\in\mathrm{NCP}_{1,2}(m)\\ \Pi_0(\sigma,\kappa)=\pi}}\widetilde{w}(\sigma,\kappa)&=\prod_{U\in\mathrm{Inn}(\pi)}\left(\sum_{(\sigma_U,\kappa_U)\in\mathrm{NCL}^1_{1,2}(U)}\widetilde{w}(\sigma_U,\kappa_U)\right)\\ &\quad\cdot\prod_{U\in\mathrm{Out}(\pi)}\left(\sum_{(\sigma_U,\kappa_U)\in\mathrm{NCP}^1_{1,2}(U)}\widetilde{w}(\sigma_U,\kappa_U)\right)\\ &=\prod_{U\in\mathrm{Inn}(\pi)}r_{|U|}\cdot\prod_{U\in\mathrm{Out}(\pi)}A_{|U|}\end{aligned}$$

and then we conclude that $A_m=R_m(\widetilde{\mu},\mu)$. □

Examples. Putting $R_m:=R_m(\widetilde{\mu},\mu)$ we have:

$$R_1=\widetilde{\beta}_0, \tag{6.9}$$

$$R_2=\widetilde{\gamma}_0, \tag{6.10}$$

$$R_3=\widetilde{a}^{\mathrm{l}}_0(\widetilde{\beta}_1-\beta_0)\widetilde{a}^{\mathrm{r}}_0, \tag{6.11}$$

$$R_4=\widetilde{a}^{\mathrm{l}}_0[(\widetilde{\beta}_1-\beta_0)^2+(\widetilde{\gamma}_1-\gamma_0)]\widetilde{a}^{\mathrm{r}}_0, \tag{6.12}$$

$$\begin{aligned}R_5=\widetilde{a}^{\mathrm{l}}_0[&(\widetilde{\beta}_1-\beta_0)^3+(\widetilde{\gamma}_1-\gamma_0)(\widetilde{\beta}_1-\beta_0)+(\widetilde{\beta}_1-\beta_0)(\widetilde{\gamma}_1-\gamma_0)\\ &+(\beta_1-\beta_0)(\widetilde{\gamma}_1-\gamma_0)+\widetilde{a}^{\mathrm{l}}_1(\widetilde{\beta}_2-\beta_1)\widetilde{a}^{\mathrm{r}}_1]\widetilde{a}^{\mathrm{r}}_0,\end{aligned} \tag{6.13}$$

$$\begin{aligned}R_6=\widetilde{a}^{\mathrm{l}}_0[&(\widetilde{\beta}_1-\beta_0)^4+(\widetilde{\gamma}_1-\gamma_0)(\widetilde{\beta}_1-\beta_0)^2+(\widetilde{\beta}_1-\beta_0)(\widetilde{\gamma}_1-\gamma_0)(\widetilde{\beta}_1-\beta_0)\\ &+(\widetilde{\beta}_1-\beta_0)^2(\widetilde{\gamma}_1-\gamma_0)+(\beta_1-\beta_0)(\widetilde{\beta}_1-\beta_0)(\widetilde{\gamma}_1-\gamma_0)\\ &+(\beta_1-\beta_0)(\widetilde{\gamma}_1-\gamma_0)(\widetilde{\beta}_1-\beta_0)+(\beta_1-\beta_0)^2(\widetilde{\gamma}_1-\gamma_0)\\ &+(\widetilde{\beta}_1-\beta_0)\widetilde{a}^{\mathrm{l}}_1(\widetilde{\beta}_2-\beta_1)\widetilde{a}^{\mathrm{r}}_1+\widetilde{a}^{\mathrm{l}}_1(\widetilde{\beta}_2-\beta_1)\widetilde{a}^{\mathrm{r}}_1(\widetilde{\beta}_1-\beta_0)\\ &+2(\beta_1-\beta_0)\widetilde{a}^{\mathrm{l}}_1(\widetilde{\beta}_2-\beta_1)\widetilde{a}^{\mathrm{r}}_1+\widetilde{a}^{\mathrm{l}}_1(\widetilde{\beta}_2-\beta_1)^2\widetilde{a}^{\mathrm{r}}_1\\ &+(\widetilde{\gamma}_1-\gamma_0)^2+(\gamma_1-\gamma_0)(\widetilde{\gamma}_1-\gamma_0)+\widetilde{a}^{\mathrm{l}}_1(\widetilde{\gamma}_2-\gamma_1)^2\widetilde{a}^{\mathrm{r}}_1]\widetilde{a}^{\mathrm{r}}_0.\end{aligned} \tag{6.14}$$

For example, the consecutive summands of R_5 correspond to the following partially labeled partitions:

$$\{\{1,5\},\{2\}_1,\{3\}_1,\{4\}_1\}, \quad \{\{1,5\},\{2,3\}_1,\{4\}_1\},$$
$$\{\{1,5\},\{2\}_1,\{3,4\}_1\}, \quad \{\{1,5\},\{2,4\}_1,\{3\}_1\}, \quad \{\{1,5\},\{2,4\},\{3\}_2\}.$$

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