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THE PROBABILITY MEASURE CORRESPONDING TO 2-PLANE TREES

BY

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Abstract. We study the probability measure μ_0 for which the moment sequence is $\binom{3n}{n}\frac{1}{n+1}$. We prove that μ_0 is absolutely continuous, find the density function and prove that μ_0 is infinitely divisible with respect to the additive free convolution.

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1. INTRODUCTION

A 2-*plane tree* is a planted plane tree such that each vertex is colored black or white and for each edge at least one of its ends is white. Gu and Prodinger [3] proved that the number of 2-plane trees on $n+1$ vertices with black (white) root is $\binom{3n+1}{n}\frac{1}{3n+1}$ (Fuss–Catalan number of order three, sequence A001764 in OEIS [10]) and $\binom{3n+2}{n}\frac{2}{3n+2}$ (sequence A006013 in OEIS) respectively (see also [4]). We will study the sequence

$$\binom{3n}{n}\frac{2}{n+1} = \binom{3n+1}{n}\frac{1}{3n+1} + \binom{3n+2}{n}\frac{2}{3n+2}, \tag{1.1}$$

which begins with

$$2, 3, 10, 42, 198, 1001, 5304, 29070, 163438, \ldots,$$

of total numbers of such trees (A007226 in OEIS).

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Both the sequences on the right-hand side of (1.1) are positive definite (see [5] and [6]), therefore so is the sequence $\binom{3n}{n}\frac{2}{n+1}$ itself. In this paper we study the corresponding probability measure μ_0, i.e. such that the numbers $\binom{3n}{n}\frac{1}{n+1}$ are moments of μ_0. First we prove that μ_0 is Mellin convolution of two beta distributions, in particular μ_0 is absolutely continuous. Then we find the density function of μ_0. In the last section we prove that μ_0 can be decomposed as additive free convolution $\mu_1 \boxplus \mu_2$ of two measures μ_1 and μ_2, which are both infinitely divisible with respect to $\boxplus$ and are related to the Marchenko–Pastur distribution. In particular, the measure μ_0 itself is $\boxplus$-infinitely divisible.

2. THE GENERATING FUNCTION

Let us consider the generating function

$$G(z) = \sum_{n=0}^{\infty} \binom{3n}{n} \frac{2z^n}{n+1}.$$

According to (1.1), G can be represented as a sum of two generating functions. The former is usually denoted by $\mathcal{B}_3$,

$$\mathcal{B}_3(z) = \sum_{n=0}^{\infty} \binom{3n+1}{n} \frac{z^n}{3n+1},$$

and satisfies the equation

$$\mathcal{B}_3(z) = 1 + z \cdot \mathcal{B}_3(z)^3. \tag{2.1}$$

Lambert's formula (see (5.60) in [2]) implies that the latter is just square of $\mathcal{B}_3$,

$$\mathcal{B}_3(z)^2 = \sum_{n=0}^{\infty} \binom{3n+2}{n} \frac{2z^n}{3n+2},$$

so we have

$$G(z) = \mathcal{B}_3(z) + \mathcal{B}_3(z)^2. \tag{2.2}$$

Combining (2.1) and (2.2), we obtain the following equation for G:

$$2 - z - (1+2z)G(z) + 2zG(z)^2 - z^2G(z)^3 = 0, \tag{2.3}$$

which will be applied later on.

Now we will give a formula for $G(z)$.

PROPOSITION 2.1. *For the generating function of the sequence* (1.1) *we have*

$$G(z) = \frac{12\cos^2\alpha + 6}{(4\cos^2\alpha - 1)^2}, \tag{2.4}$$

where $\alpha = \frac{1}{3}\arcsin\left(\sqrt{27z/4}\right)$.

P r o o f. Defining $(a)_n := a(a+1)\dots(a+n-1)$ we have

$$\frac{2(3n)!}{(n+1)!(2n)!} = \frac{-2\left(\frac{-2}{3}\right)_{n+1}\left(\frac{-1}{3}\right)_{n+1} 27^{n+1}}{3(n+1)!\left(\frac{-1}{2}\right)_{n+1} 4^{n+1}}.$$

Therefore

$$G(z) = \frac{2 - 2\cdot {}_2F_1\left(\frac{-2}{3}, \frac{-1}{3}; \frac{1}{2}\middle| \frac{27z}{4}\right)}{3z}.$$

Now we apply the formula

$$\begin{aligned}&{}_2F_1\left(\frac{-2}{3}, \frac{-1}{3}; \frac{-1}{2}\middle| u\right)\\ &\qquad = \frac{1}{3}\sqrt{u}\sin\left(\frac{1}{3}\arcsin\left(\sqrt{u}\right)\right) + \sqrt{1-u}\cos\left(\frac{1}{3}\arcsin\left(\sqrt{u}\right)\right),\end{aligned}$$

which can be checked by verifying the hypergeometric equation (note that both the functions $w \mapsto w\sin\left(\frac{1}{3}\arcsin(w)\right)$ and $w \mapsto \cos\left(\frac{1}{3}\arcsin(w)\right)$ are even, so the right-hand side is well defined for $|u| < 1$). Putting $\alpha = \frac{1}{3}\arcsin\left(\sqrt{u}\right)$, $u = 27z/4$, we have $\sqrt{u} = \sin 3\alpha$, $\sqrt{1-u} = \cos 3\alpha$, which leads to (2.4). ∎

3. THE MEASURE

Now we want to study the (unique) measure μ_0 for which $\left\{\binom{3n}{n}\frac{1}{n+1}\right\}_{n=0}^{\infty}$ is the moment sequence. We will show that μ_0 can be expressed as the Mellin convolution of two beta distributions. Then we will provide an explicit formula for the density function $V(x)$ of μ_0.

Recall (see [1]) that for $\alpha, \beta > 0$, the *beta distribution* $\mathrm{Beta}(\alpha,\beta)$ is the absolutely continuous probability measure defined by the density function

$$f_{\alpha,\beta}(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\cdot x^{\alpha-1}(1-x)^{\beta-1}$$

for $x \in (0,1)$. The moments of $\mathrm{Beta}(\alpha,\beta)$ are

$$\int_0^1 x^n f_{\alpha,\beta}(x)\,dx = \frac{\Gamma(\alpha+\beta)\Gamma(\alpha+n)}{\Gamma(\alpha)\Gamma(\alpha+\beta+n)} = \prod_{i=0}^{n-1}\frac{\alpha+i}{\alpha+\beta+i}.$$

For probability measures ν_1, ν_2 on the positive half-line $[0,\infty)$ the *Mellin convolution* is defined by

$$(\nu_1 \circ \nu_2)(A) := \int_0^\infty\int_0^\infty \chi_A(xy)\,d\nu_1(x)\,d\nu_2(y) \tag{3.1}$$

for every Borel set $A \subseteq [0,\infty)$ (χ_A denotes the indicator function of the set A). This is the distribution of the product $X_1 \cdot X_2$ of two independent nonnegative random variables with $X_i \sim \nu_i$. In particular, if $c > 0$ then $\nu \circ \delta_c$ is the *dilation* of the measure ν:

$$(\nu \circ \delta_c)(A) = \mathbf{D}_c\nu(A) := \nu\left(\frac{1}{c}A\right),$$

where δ_c denotes the Dirac delta measure at c.

If both the measures ν_1, ν_2 have all *moments*

$$s_n(\nu_i) := \int_0^\infty x^n \, d\nu_i(x)$$

finite, then so has $\nu_1 \circ \nu_2$ and

$$s_n(\nu_1 \circ \nu_2) = s_n(\nu_1) \cdot s_n(\nu_2)$$

for all n. The method of Mellin convolution has been recently applied to a number of related problems, see for example [6] and [8].

From now on we will study the probability measure corresponding to the sequence $\binom{3n}{n}\frac{1}{n+1}$.

PROPOSITION 3.1. *Define* μ_0 *as the Mellin convolution*:

$$\mu_0 = \mathrm{Beta}(1/3, 1/6) \circ \mathrm{Beta}(2/3, 4/3) \circ \delta_{27/4}. \tag{3.2}$$

Then the numbers $\binom{3n}{n}\frac{1}{n+1}$ *are moments of* μ_0:

$$\int_0^{27/4} x^n \, d\mu_0(x) = \binom{3n}{n}\frac{1}{n+1}.$$

P r o o f. It is sufficient to check that

$$\frac{(3n)!}{(n+1)!(2n)!} = \prod_{i=0}^{n-1} \frac{1/3+i}{1/2+i} \cdot \prod_{i=0}^{n-1} \frac{2/3+i}{2+i} \cdot \left(\frac{27}{4}\right)^n. \quad \blacksquare$$

In view of formula (3.2), the measure μ_0 is absolutely continuous and its support is the interval $[0, 27/4]$. Now we want to find the density function $V(x)$ of the measure μ_0.

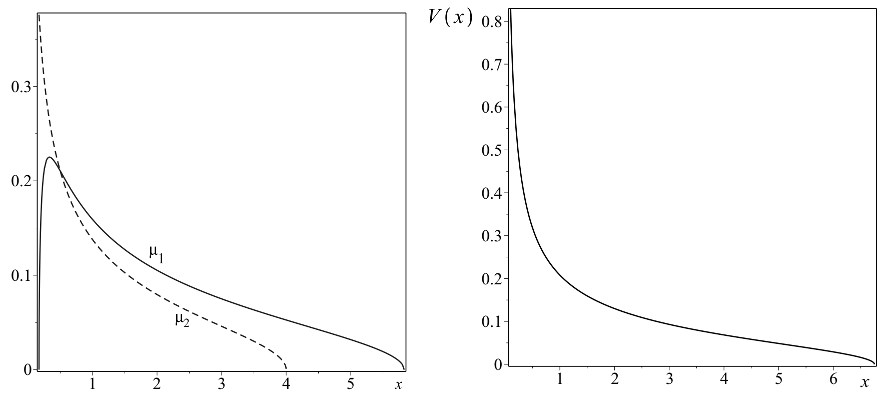
THEOREM 3.1. *Let*

$$V(x) = \frac{\sqrt{3}}{2^{10/3}\pi x^{2/3}}\left(3\sqrt{1-4x/27}-1\right)\left(1+\sqrt{1-4x/27}\right)^{1/3} + \frac{1}{2^{8/3}\pi x^{1/3}\sqrt{3}}\left(3\sqrt{1-4x/27}+1\right)\left(1+\sqrt{1-4x/27}\right)^{-1/3},$$

$x \in (0, 27/4)$. *Then* V *is the density function of* μ_0, *i.e.*

$$\int_0^{27/4} x^n \, V(x) \, dx = \binom{3n}{n}\frac{1}{n+1}$$

for $n = 0, 1, 2, \ldots$



(a) The densities of μ_1, μ_2 (b) The density of $\mu_0 = \mu_1 \boxplus \mu_2$

FIGURE 1. The densities of μ_1, μ_2 and $\mu_0 = \mu_1 \boxplus \mu_2$

The density $V(x)$ of μ_0 is represented in Figure 1 (b).

Proof. Putting $n = s - 1$ and applying the Gauss–Legendre multiplication formula

$$\Gamma(mz) = (2\pi)^{(1-m)/2} m^{mz-1/2} \Gamma(z)\Gamma\left(z + \frac{1}{m}\right)\Gamma\left(z + \frac{2}{m}\right)\dots\Gamma\left(z + \frac{m-1}{m}\right)$$

we obtain

$$\begin{aligned}\binom{3n}{n}\frac{1}{n+1} &= \frac{\Gamma(3n+1)}{\Gamma(n+2)\Gamma(2n+1)} = \frac{\Gamma(3s-2)}{\Gamma(s+1)\Gamma(2s-1)} \\ &= \frac{2}{27}\sqrt{\frac{3}{\pi}}\left(\frac{27}{4}\right)^s \frac{\Gamma(s-2/3)\Gamma(s-1/3)}{\Gamma(s-1/2)\Gamma(s+1)} := \psi(s).\end{aligned}$$

Then ψ can be extended to an analytic function on the complex plane, except for the points $1/3 - n, 2/3 - n, n = 0, 1, 2, \dots$

Now we want to apply a particular type of the Meijer G-function, see [9] for details. Let $\widetilde{V}$ denote the inverse Mellin transform of ψ. Then we have

$$\begin{aligned}\widetilde{V}(x) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\psi(s)\,ds \\ &= \frac{2}{27}\sqrt{\frac{3}{\pi}}\,\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(s-2/3)\Gamma(s-1/3)}{\Gamma(s-1/2)\Gamma(s+1)}\left(\frac{4x}{27}\right)^{-s} ds \\ &= \frac{2}{27}\sqrt{\frac{3}{\pi}}\,G^{2,0}_{2,2}\left(\frac{4x}{27}\,\middle|\,\begin{matrix}-1/2, & 1\\ -2/3, & -1/3\end{matrix}\right),\end{aligned}$$

where $x \in (0, 27/4)$ (see [11] for the role of c in the integrals). On the other hand, for the parameters of the G-function we have

$$(-2/3 - 1/3) - (-1/2 + 1) = -3/2 < 0,$$

and hence the assumptions of formula 2.24.2.1 in [9] are satisfied. Therefore we can apply the Mellin transform on $\widetilde{V}(x)$:

$$\int_0^{27/4} x^{s-1}\widetilde{V}(x)\,dx = \frac{2}{27}\sqrt{\frac{3}{\pi}}\int_0^{27/4} x^{s-1} G^{2,0}_{2,2}\left(\frac{4x}{27}\middle|\begin{matrix}-1/2, & 1\\ -2/3, & -1/3\end{matrix}\right)dx$$
$$= \frac{2}{27}\sqrt{\frac{3}{\pi}}\left(\frac{27}{4}\right)^s \int_0^1 u^{s-1} G^{2,0}_{2,2}\left(u\middle|\begin{matrix}-1/2, & 1\\ -2/3, & -1/3\end{matrix}\right)du = \psi(s)$$

whenever $\Re s > 2/3$. Consequently, $\widetilde{V} = V$.

Now we use Slater's formula (see [9], formula 8.2.2.3) and express V in terms of the hypergeometric functions:

$$V(x) = \frac{2}{27}\sqrt{\frac{3}{\pi}}\, G^{2,0}_{2,2}\left(\frac{4x}{27}\middle|\begin{matrix}-1/2, & 1\\ -2/3, & -1/3\end{matrix}\right)$$
$$= \frac{2}{27}\sqrt{\frac{3}{\pi}}\,\frac{\Gamma(1/3)}{\Gamma(1/6)\Gamma(5/3)}\left(\frac{4x}{27}\right)^{-2/3} {}_2F_1\left(\frac{-2}{3}, \frac{5}{6}; \frac{2}{3}\middle| \frac{4x}{27}\right)$$
$$+ \frac{2}{27}\sqrt{\frac{3}{\pi}}\,\frac{\Gamma(-1/3)}{\Gamma(-1/6)\Gamma(4/3)}\left(\frac{4x}{27}\right)^{-1/3} {}_2F_1\left(\frac{-1}{3}, \frac{7}{6}; \frac{4}{3}\middle| \frac{4x}{27}\right)$$
$$= \frac{\sqrt{3}}{4\pi x^{2/3}}\, {}_2F_1\left(\frac{-2}{3}, \frac{5}{6}; \frac{2}{3}\middle| \frac{4x}{27}\right) + \frac{1}{2\pi\sqrt{3}x^{1/3}}\, {}_2F_1\left(\frac{-1}{3}, \frac{7}{6}; \frac{4}{3}\middle| \frac{4x}{27}\right).$$

Applying the formula

$${}_2F_1\left(\frac{t-2}{2}, \frac{t+1}{2}; t\middle| z\right) = \frac{2^t}{2t}\left(t - 1 + \sqrt{1-z}\right)\left(1 + \sqrt{1-z}\right)^{1-t}$$

(see [6]) for $t = 2/3$ and $t = 4/3$ we complete the proof. ■

4. RELATIONS WITH FREE PROBABILITY

In this part we describe relations of μ_0 with free probability. In particular, we will show that μ_0 is infinitely divisible with respect to the additive free convolution. Let us briefly describe the additive and multiplicative free convolutions. For details we refer to [12] and [7].

Denote by $\mathcal{M}^c$ the class of probability measures on $\mathbb{R}$ with compact support. For $\mu \in \mathcal{M}^c$, with moments

$$s_m(\mu) := \int_{\mathbb{R}} t^m\, d\mu(t)$$

and the *moment generating function*

$$M_\mu(z) := \sum_{m=0}^{\infty} s_m(\mu) z^m = \int_{\mathbb{R}} \frac{d\mu(t)}{1-tz},$$

we define its *R-transform* $R_\mu(z)$ by the equation

$$R_\mu\big(zM_\mu(z)\big) + 1 = M_\mu(z). \tag{4.1}$$

Then the *additive free convolution* of $\mu', \mu'' \in \mathcal{M}^c$ is defined as the unique measure $\mu' \boxplus \mu'' \in \mathcal{M}^c$ which satisfies

$$R_{\mu' \boxplus \mu''}(z) = R_{\mu'}(z) + R_{\mu''}(z).$$

If the support of $\mu \in \mathcal{M}^c$ is contained in the positive half-line $[0, +\infty)$ then we define its *S-transform* $S_\mu(z)$ by

$$M_\mu\left(\frac{z}{1+z} S_\mu(z)\right) = 1 + z \quad \text{or} \quad R_\mu\big(zS_\mu(z)\big) = z \tag{4.2}$$

on a neighborhood of zero. If μ', μ'' are such measures then their *multiplicative free convolution* $\mu' \boxtimes \mu''$ is defined by

$$S_{\mu' \boxtimes \mu''}(z) = S_{\mu'}(z) \cdot S_{\mu''}(z).$$

Recall that for dilated measure $\mathbf{D}_c\mu$ we have

$$M_{\mathbf{D}_c\mu}(z) = M_\mu(cz), \quad R_{\mathbf{D}_c\mu}(z) = R_\mu(cz), \quad \text{and} \quad S_{\mathbf{D}_c\mu}(z) = S_\mu(z)/c.$$

The operations $\boxplus$ and $\boxtimes$ can be regarded as free analogs of the classical and Mellin convolution.

For $t > 0$ let ϖ_t denote the *Marchenko–Pastur distribution* with parameter t,

$$\varpi_t = \max\{1-t, 0\}\delta_0 + \frac{\sqrt{4t - (x-1-t)^2}}{2\pi x}\, dx, \tag{4.3}$$

with the absolutely continuous part supported on $[(1-\sqrt{t})^2, (1+\sqrt{t})^2]$. Then

$$\begin{aligned} M_{\varpi_t}(z) &= \frac{2}{1 + z - tz + \sqrt{(1-z-tz)^2 - 4tz^2}} \\ &= 1 + \sum_{n=1}^{\infty} z^n \sum_{k=1}^{n} \binom{n}{k}\binom{n}{k-1}\frac{t^k}{n}, \end{aligned} \tag{4.4}$$

$$R_{\varpi_t}(z) = \frac{tz}{1-z}, \qquad S_{\varpi_t}(z) = \frac{1}{t+z}. \tag{4.5}$$

In free probability the measures ϖ_t play the role of the Poisson distributions. Note that by (4.5) the family $\{\varpi_t\}_{t>0}$ constitutes a semigroup with respect to $\boxplus$, i.e. we have $\varpi_s \boxplus \varpi_t = \varpi_{s+t}$ for $s, t > 0$.

THEOREM 4.1. *The measure μ_0 can be decomposed as the additive free convolution $\mu_0 = \mu_1 \boxplus \mu_2$, where $\mu_1 = \mathbf{D}_2\varpi_{1/2}$, so that*

$$\mu_1 = \frac{1}{2}\delta_0 + \frac{\sqrt{8-(x-3)^2}}{4\pi x}\chi_{(3-\sqrt{8},3+\sqrt{8})}(x)\,dx, \tag{4.6}$$

and $\mu_2 = \frac{1}{2}\delta_0 + \frac{1}{2}\varpi_1$, i.e.

$$\mu_2 = \frac{1}{2}\delta_0 + \frac{\sqrt{4x-x^2}}{4\pi x}\chi_{(0,4)}(x)\,dx. \tag{4.7}$$

The measures μ_1, μ_2 are infinitely divisible with respect to the additive free convolution $\boxplus$, and, consequently, so is μ_0.

The absolutely continuous parts of the measures μ_1 and μ_2 are represented in Figure 1 (a).

Proof. The moment generating function of μ_0 is $M_{\mu_0}(z) = G(z)/2$. Then we have $M_{\mu_0}(0) = 1$ and, by (2.3),

$$2 - z - 2(1+2z)M_{\mu_0}(z) + 8zM_{\mu_0}(z)^2 - 8z^2M_{\mu_0}(z)^3 = 0.$$

Let $T(z)$ be the inverse function for $M_{\mu_0}(z) - 1$, so that we have $T(0) = 0$ and $M_{\mu_0}\big(T(z)\big) = 1 + z$. Then

$$2 - T(z) + \big(-1 - 2T(z)\big)2(1+z) + 8T(z)(1+z)^2 - 8T(z)^2(1+z)^3 = 0,$$

which gives

$$8(1+z)^3T(z)^2 - (8z^2+12z+3)T(z) + 2z = 0,$$

and finally

$$T(z) = \frac{8z^2+12z+3-\sqrt{9+8z}}{16(1+z)^3} = \frac{4z}{8z^2+12z+3+\sqrt{9+8z}}.$$

Therefore we can find the S-transform of μ_0:

$$S_{\mu_0}(z) = \frac{1+z}{z}T(z) = \frac{8z^2+12z+3-\sqrt{9+8z}}{16z(1+z)^2} = \frac{4(1+z)}{8z^2+12z+3+\sqrt{9+8z}};$$

consequently, from (4.2) we get the R-transform

$$R_{\mu_0}(z) = \frac{4z-1+\sqrt{1-2z}}{2(1-2z)}.$$

Now we observe that $R_{\mu_0}(z)$ can be decomposed as follows:

$$R_{\mu_0}(z) = \frac{z}{1-2z} + \frac{1-\sqrt{1-2z}}{2\sqrt{1-2z}} = R_1(z) + R_2(z).$$

Comparing this formula with (4.5) we observe that $R_1(z)$ is the R-transform of $\mu_1 = \mathbf{D}_2\varpi_{1/2}$, which implies that μ_1 is $\boxplus$-infinitely divisible.

Consider the Taylor expansion of $R_2(z)$:

$$R_2(z) = \sum_{n=1}^{\infty} \binom{2n}{n} 2^{-n-1} z^n = \frac{z}{2} + z^2 \sum_{n=0}^{\infty} \binom{2(n+2)}{n+2} 2^{-n-3} z^n.$$

Since the numbers $\binom{2n}{n}$ are moments of the *arcsine distribution*

$$\frac{1}{\pi\sqrt{x(4-x)}}\chi_{(0,4)}(x)\,dx,$$

the coefficients of the last sum constitute a positive definite sequence. So $R_2(z)$ is the R-transform of a probability measure μ_2, which is $\boxplus$-infinitely divisible (see Theorem 13.16 in [7]). Now using (4.1) we obtain

$$M_{\mu_2}(z) = \frac{1+2z-\sqrt{1-4z}}{4z} = \frac{1}{2} + \frac{1-\sqrt{1-4z}}{4z} = \frac{1}{2} + \frac{1}{1+\sqrt{1-4z}}.$$

Comparing this formula with (4.4) for $t=1$ we see that $\mu_2 = \frac{1}{2}\delta_0 + \frac{1}{2}\varpi_1$. ■

Let us now consider the measures μ_1, μ_2 separately. For $\mu_1 = \mathbf{D}_2\varpi_{1/2}$ the moment generating function is

$$M_{\mu_1}(z) = \frac{2}{1+z+\sqrt{1-6z+z^2}} = 1 + \sum_{n=1}^{\infty} z^n \sum_{k=1}^{n} \binom{n}{k}\binom{n}{k-1}\frac{2^{n-k}}{n},$$

so the moments are

$$1, 1, 3, 11, 45, 197, 903, 4279, 20793, 103049, 518859, \ldots$$

This is the A001003 sequence in OEIS (little Schroeder numbers), $s_n(\mu_1)$ is the number of ways to insert parentheses in product of $n+1$ symbols. There is no restriction on the number of pairs of parentheses. The number of objects inside a pair of parentheses must be at least two.

On the subject of μ_2, applying (4.2) we can find the S-transform:

$$S_{\mu_2}(z) = \frac{2(1+z)}{(1+2z)^2} = \frac{1+z}{1/2+z} \cdot \frac{1}{1+2z}.$$

One can check that $(1+z)/(1/2+z)$ is the S-transform of $\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1$, which yields

$$\mu_2 = \left(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1\right) \boxtimes \mu_1. \tag{4.8}$$

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