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CONSTRUCTIVE APPROXIMATION

Nonnegative Linearization for Polynomials Orthogonal with Respect to Discrete Measures

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Abstract. We give conditions for the coefficients in three term recurrence relations implying nonnegative linearization for polynomials orthogonal with respect to measures supported on convergent sequences of points. The previous methods were unable to cover this case.

1. Introduction

If $P_n(x)$ is a sequence of orthogonal polynomials, then the *linearization coefficients* are defined by

$$P_n(x)P_m(x) = \sum_k g(n, m, k)P_k(x). \tag{1}$$

By the orthogonality relation each coefficient $g(n, m, k)$ can be computed as the integral of the triple product $P_n P_m P_k$ with respect to the orthogonality measure. For many polynomials, like ultraspherical polynomials and their q-analogues, the coefficients $g(n, m, k)$ can be calculated explicitly. However, there are many orthogonal systems, including nonsymmetric Jacobi polynomials, for which explicit formulas are not available.

The problem of determining if a given orthogonal polynomial system admits *nonnegative product linearization* (i.e., if all $g(n, m, k)$ are nonnegative) is one of the most important in the theory of orthogonal polynomials. The main reason is that the property has many important consequences. Nonnegativity of linearization coefficients gives rise to a convolution structure associated with the polynomials P_n.

There are general criteria, stated in terms of the recurrence relation (2), that the polynomials P_n always satisfy (see [1], [4], [5], [6]). For example, the following criterion have been shown in [5] and [6].

Let the polynomials P_n satisfy the recurrence relation

$$xP_n(x) = \alpha_n P_{n+1}(x) + \beta_n P_n(x) + \gamma_{n-1} P_{n-1}(x), \tag{2}$$

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where $\alpha_n, \gamma_n > 0$ and $P_0 = 1$ and $P_{-1} = 0$. If the sequences $\{\gamma_n\}$, $\{\alpha_n + \gamma_{n-1}\}$, and $\{\beta_n\}$ are all nondecreasing and $\alpha_n \geq \gamma_{n-1}$ for every n, then the polynomials P_n admit nonnegative product linearization. This criterion is strong enough to include many classical polynomial systems; however, it cannot cover the case when the orthogonality measure is supported on a sequence of points accumulating at some point b. In that case, it can be shown that

$$\alpha_n \gamma_n \to 0, \qquad \beta_n \to b.$$

One can easily check that no sequences α_n and γ_n can satisfy all the mentioned assumptions simultaneously.

An example of such a system, not covered by the results of this paper, are the so-called little q-Legendre polynomials, $0 < q < 1$, orthogonal with respect to $\mu = \sum_{k=0}^{\infty} q^k \delta_{q^k}$, for which nonnegative linearization was proved by Koornwinder [3].

The aim of this paper is to give new criteria that can handle the case of coefficients satisfying $\alpha_n \gamma_n \to 0$. In doing this, we will apply two different methods. Following [4] we will use a combinatorial interpretation of linearization coefficients $g(n, m, k)$ as a weighted sum over special paths connecting points $(0, m)$ and (k, n) in the plane lattice. Next, following [5], we will use the method of maximum principle for a discrete boundary problem associated with the recurrence relation. This method will also be used to cover the case of measures which are symmetric about the origin and supported on a set $\{\pm a_n\}$ with $a_n \nearrow a$. This is done in Section 5.

The main result of this paper is the following:

Theorem 1. *Let the polynomials P_n satisfy* (2). *Assume that β_n is increasing and that for every n the sequence*

$$\frac{\alpha_m \gamma_m}{(\beta_n - \beta_m)(\beta_n - \beta_{m+1})}, \qquad m = 0, 1, 2, \dots, n-2,$$

is a chain sequence. Then the polynomials P_n admit nonnegative product linearization.

In particular, if β_n is increasing and $\alpha_n \gamma_n \leq (\beta_{n+2} - \beta_{n+1})^2$ for every n then, in view of Corollary 1, P_n admits nonnegative product linearization.

2. Preliminary Results

The main results of this paper can be stated in terms of chain sequences or positive definite matrices. Recall that a sequence $(u_0, u_1, \dots, u_n)$ is called a *chain sequence* if there exist numbers $(g_0, g_1, \dots, g_{n+1})$, $0 \leq g_i \leq 1$, satisfying $u_i = (1 - g_i) g_{i+1}$ for $0 \leq i \leq n$. A complex matrix $(a_{i,j})_{i,j=1}^{n}$ is said to be *positive definite* if the inequality $\sum a_{i,j} z_i \bar{z}_j \geq 0$ holds for every collection $z_1, \dots, z_n$ of complex numbers. For the sake of completeness we prove the following equivalence which can be derived from the Wall monograph [8].

Lemma 1. *Let $(u_0, u_1, \dots, u_{t-2})$ be a sequence of positive numbers. Then the following conditions are equivalent*:

(i) $(u_0, u_1, \dots, u_{t-2})$ *is a chain sequence.*

(ii) *The numbers* $f(r,s)$, *where* $0 \le r \le s \le t$, *defined by*

$$\begin{aligned} f(r,r) &= 1, \\ f(r,r+1) &= 1, \\ f(r,s+1) &= f(r,s) - f(r,s-1)u_{s-1}, \end{aligned}$$

are all nonnegative.

(iii) *The matrix* $A = (a_{i,j})_{i,j=1}^{t}$, *where*

$$\begin{aligned} a_{i,i} &= 1, \\ a_{i,i+1} = a_{i+1,i} &= \sqrt{u_{i-1}}, \\ a_{i,j} &= 0 \qquad \text{for} \quad |i-j| > 1, \end{aligned}$$

is positive definite.

Proof. Assume that $u_i = (1-g_i)g_{i+1}$, where $0 \le g_i \le 1$ and $i = 0, 1, \dots, t-2$.

By induction on $k \ge 0$, we get

$$f(r, r+2k) = \sum_{j=0}^{k} g_r g_{r+1} \dots g_{r+2j-1}(1-g_{r+2j+1})(1-g_{r+2j+2}) \dots (1-g_{r+2k-1}),$$

$$f(r, r+2k+1) = \sum_{j=0}^{k} g_r g_{r+1} \dots g_{r+2j-1}(1-g_{r+2j+1})(1-g_{r+2j+2}) \dots (1-g_{r+2k}).$$

For complex numbers $z_1, \dots, z_t$ we recall the formula 16.10 from [8] to obtain

$$\begin{aligned} \sum_{i,j=1}^{t} a_{i,j} z_i \bar{z}_j &= \sum_{i=1}^{t} |z_i|^2 + \sum_{i=1}^{t-1} \sqrt{u_{i-1}}(z_i \bar{z}_{i+1} + z_{i+1}\bar{z}_i) \\ &= g_0|z_1|^2 + \sum_{i=1}^{t-1} |z_i\sqrt{1-g_{i-1}} + z_{i+1}\sqrt{g_i}|^2 + (1-g_{t-1})|z_t|^2 \ge 0. \end{aligned}$$

This shows that (i) implies (ii) and (iii).

Now assume that all the numbers $f(r,s)$ are nonnegative. Note that $f(r,s) = 0$ can occur only for $s = t$. Otherwise we can fix r and take the smallest possible s for which $f(r,s) = 0$. Then $r+1 < s < t$ and $f(r,s-1) > 0$. Next $f(r,s+1) = -f(r,s-1)u_{s-1} < 0$, which is a contradiction.

Put $g_0 = 0$ and $g_1 = u_0$. Now assume $s < t$ and that we have defined $0 < g_1, g_2, \dots, g_{s-1} < 1$ and that $f(0,k) = (1-g_1)\cdots(1-g_{k-1})$ holds for $k = 1, 2, \dots, s$. Then

$$\begin{aligned} f(0,s+1) &= (1-g_1)\cdots(1-g_{s-1}) - (1-g_1)\cdots(1-g_{s-2})u_{s-1} \\ &= (1-g_1)\cdots(1-g_{s-2})(1-g_{s-1}-u_{s-1}) > 0, \end{aligned}$$

(≥ 0 can occur if $s+1 = t$). So $u_{s-1} = (1-g_{s-1})g_s$ for a unique g_s satisfying $0 < g_s < 1$ if $s+1 < t$ or $0 < g_s \le 1$ if $s+1 = t$. Hence $(u_0, u_1, \dots, u_{t-2})$ is a chain sequence. This shows that (ii) implies (i).

Finally, if A is positive definite, then one can easily check that

$$0 \le \det(a_{i,j})_{i,j=r}^{s} \;=\; f(r-1,s). \qquad \blacksquare$$

Assume that we are given two infinite sequences $(\beta_0, \beta_1, \ldots)$, $(c_0, c_1, \ldots)$ of real numbers, the first one strictly increasing and the second positive. We will associate with them a collection of numbers $F(r,s,t)$, $0 \le r \le s \le t$, defined by the rule:

$$\begin{aligned} F(r,r,t) &= 1, && 0 \le r \le t,\\ F(r,r+1,t) &= \beta_t - \beta_r, && 0 \le r < t,\\ F(r,s+1,t) &= F(r,s,t)(\beta_t - \beta_s) - F(r,s-1,t)c_{s-1}, && 0 \le r < s < t, \end{aligned}$$

and a sequence of matrices $A(t) = (a(t)_{i,j})_{i,j=1}^{t}$ by putting

$$\begin{aligned} a(t)_{i,i} &= \beta_t - \beta_{i-1},\\ a(t)_{i,i+1} = a(t)_{i+1,i} &= \sqrt{c_{i-1}},\\ a(t)_{i,j} &= 0 \qquad \text{for} \quad |i-j| > 1. \end{aligned}$$

Proposition 1 provides five equivalent forms of the hypothesis of Theorem 1.

Proposition 1. *The following conditions are equivalent*:

(i) *For every $t \ge 2$ the sequence*

$$\frac{c_s}{(\beta_t - \beta_s)(\beta_t - \beta_{s+1})}, \qquad s = 0, 1, \ldots, t-2,$$

is a chain sequence.

(ii) *The numbers $F(r,s,t)$ are all nonnegative.*

(iii) *The matrices $A(t)$ are all positive definite.*

(iv) *The determinants* $\det A(t)$ *are all nonnegative.*

(v) *The finite continued fractions*

$$\beta_{t+1} - \beta_t - \frac{c_{t-1}|}{|\beta_{t+1} - \beta_{t-1}} - \frac{c_{t-2}|}{|\beta_{t+1} - \beta_{t-2}} - \cdots - \frac{c_0|}{|\beta_{t+1} - \beta_0}$$

are all nonnegative.

Before proving the proposition let us consider the following auxiliary continued fractions:

$$B(r,s,t) = \beta_{t+1} - \beta_s - \frac{c_{s-1}|}{|\beta_{t+1} - \beta_{s-1}} - \frac{c_{s-2}|}{|\beta_{t+1} - \beta_{s-2}} - \cdots - \frac{c_r|}{|\beta_{t+1} - \beta_r},$$

$0 \le r \le s \le t$. They satisfy the following recurrence:

$$\begin{aligned} B(s,s,t) &= \beta_{t+1} - \beta_s, && 0 \le s \le t,\\ B(r,s,t) &= \beta_{t+1} - \beta_s - \frac{c_{s-1}}{B(r,s-1,t)}, && 0 \le r < s \le t. \end{aligned}$$

Note that the continued fraction which appears in Proposition 1(v) is equal to $B(0,t,t)$ and that the consecutive denominators of $B(r,s,t)$ are $B(r,r,t)$, $B(r,r+1,t), \ldots,$ $B(r,s-1,t)$.

Lemma 2. *Suppose that* $B(0,k,k)$ *are well defined and nonnegative for all* $k < s$. *Then* $B(j,k,l)$ *are well defined for* $0 \le j \le k \le l$, $k \le s$, *and nonnegative if* $k < s$. *Moreover,*

(i) $B(r,s,t) + \beta_{t+2} - \beta_{t+1} \le B(r,s,t+1)$ *for* $0 \le r \le s \le t$;
(ii) $B(r-1,s,t) < B(r,s,t)$ *for* $1 \le r \le s \le t$.

In particular, if $B(0,s,s) \ge 0$, *then* $B(r,s,t)$ *are positive for all* $0 \le r \le s \le t$, $(r,t) \ne (0,s)$.

Proof. For $s = 0$ the statement is obvious. Suppose it holds for some $s \ge 0$ and that $B(0,k,k) \ge 0$ for $k \le s$. Then, by induction, $B(r,s+1,t)$ are all well defined and

$$\begin{aligned} B(r,s+1,t) + \beta_{t+2} - \beta_{t+1} &= \beta_{t+2} - \beta_{s+1} - \frac{c_s}{B(r,s,t)} \\ &< \beta_{t+2} - \beta_{s+1} - \frac{c_s}{B(r,s,t+1)} = B(r,s+1,t+1) \end{aligned}$$

if $r < s+1 \le t$ and

$$B(s+1,s+1,t) + \beta_{t+2} - \beta_{t+1} = \beta_{t+2} - \beta_{s+1} = B(s+1,s+1,t+1),$$

if $s+1 \le t$. This proves (i) for $s+1$.

Now we turn to proving (ii). We have

$$B(s,s+1,t) = \beta_{t+1} - \beta_{s+1} - \frac{c_s}{\beta_{t+1} - \beta_s} < \beta_{t+1} - \beta_{s+1} = B(s+1,s+1,t)$$

and

$$\begin{aligned} B(r-1,s+1,t) &= \beta_{t+1} - \beta_{s+1} - \frac{c_s}{B(r-1,s,t)} \\ &< \beta_{t+1} - \beta_{s+1} - \frac{c_s}{B(r,s,t)} = B(r,s+1,t), \end{aligned}$$

if $r \le s+1 \le t$. This completes the proof of the lemma. ■

Proof of Proposition 1. Put $f(r,r,t) = 1$ and for $0 \le r < s \le t$:

$$f(r,s,t) = \frac{F(r,s,t)}{(\beta_t - \beta_r)(\beta_t - \beta_{r+1})\ldots(\beta_t - \beta_{s-1})}.$$

Then we have $f(r,r+1,t) = 1$ for $0 \le r < t$, and

$$f(r,s+1,t) = f(r,s,t) - f(r,s-1,t)\frac{c_{s-1}}{(\beta_t - \beta_{s-1})(\beta_t - \beta_s)},$$

for $0 \le r < s < t$.

Next observe that the positive definiteness of $A(t)$ is equivalent to that of $B(t) = (b(t)_{i,j})_{i,j=1}^t$, where

$$\begin{aligned} b(t)_{i,i} &= 1, \\ b(t)_{i,i+1} = b(t)_{i+1,i} &= \sqrt{\frac{c_{i-1}}{(\beta_t - \beta_{i-1})(\beta_t - \beta_i)}}, \\ b(t)_{i,j} &= 0 \qquad \text{for} \quad |i-j| > 1. \end{aligned}$$

In view of Lemma 1, this proves that conditions (i), (ii), and (iii) are equivalent. Moreover, they imply (iv) because $\det A(t) = F(0, t, t)$.

Assume that

$$\frac{c_s}{(\beta_{t+1} - \beta_s)(\beta_{t+1} - \beta_{s+1})} = (1 - g_s)g_{s+1}, \qquad s = 0, 1, \ldots, t-1,$$

where $g_0 = 0$, $0 < g_1, g_2, \ldots, g_{t-1} < 1$, $0 < g_t \le 1$. Then by induction on s, we get

$$\beta_{t+1} - \beta_s - \frac{c_{s-1}|}{|\beta_{t+1} - \beta_{s-1}} - \frac{c_{s-2}|}{|\beta_{t+1} - \beta_{s-2}} - \cdots - \frac{c_0|}{|\beta_{t+1} - \beta_0} = (\beta_{t+1} - \beta_s)(1 - g_s).$$

Therefore the continued fraction in (v) is well defined and equal to $(\beta_{t+1} - \beta_t)(1 - g_t) \ge 0$.

Assume now that (v) holds. By Lemma 2, the quantity $B(r, s, t)$ is well defined and nonnegative for every $0 \le r \le s \le t$. One can check that

$$F(r, s, t) = B(r, r, t-1)B(r, r+1, t-1) \cdots B(r, s-1, t-1),$$

hence all $F(r, s, t)$ are nonnegative.

Finally, assume that $\det A(t) = F(0, t, t)$ is nonnegative for every $t \ge 0$. We will show the nonnegativity of $F(r, s, t)$ by induction on t. Fix t and suppose that $F(j, k, l)$ is nonnegative whenever $j \le k \le l \le t$. As we noticed in the proof of Lemma 1, the numbers $F(j, k, l)$ are strictly positive for $j \le k < l \le t$. Thus the numbers

$$B(0, k, k) = \frac{F(0, k+1, k+1)}{F(0, k, k+1)}, \qquad 0 \le k < t,$$

are well defined and nonnegative. Therefore, by Lemma 2, the quantities $B(j, k, l)$ are also well defined and positive whenever $0 \le j \le k \le l$, $k < t$, $(j, l) \ne (0, k)$. Since

$$F(0, t+1, t+1) = B(0, 0, t)\, B(0, 1, t) \ldots B(0, t-1, t) B(0, t, t) \ge 0,$$

we also have $B(0, t, t) \ge 0$, and hence $B(r, s, t) \ge 0$ for $r \le s \le t$. The latter implies $F(r, s, t+1) \ge 0$ for every $r \le s \le t+1$. ■

The next corollary provides three sufficient and one necessary condition for simultaneous nonnegativity of $F(r, s, t)$.

Corollary 1. *Suppose that one of the following conditions holds*:

(i) *For every* $s \ge 0$:

$$c_s \le (\beta_{s+2} - \beta_{s+1}) \left(\beta_{s+2} - \beta_s - \frac{c_{s-1}}{\beta_{s+2} - \beta_s} \right), \qquad \textit{with} \quad c_{-1} = 0.$$

(ii) $c_s \le (\beta_{s+2} - \beta_{s+1})^2$ *for every* $s \ge 0$.

(iii) *The infinite sequence*

$$\frac{c_s}{(\beta_{s+1}-\beta_s)(\beta_{s+2}-\beta_{s+1})}$$

is a chain sequence.

Then $F(r,s,t)$ *are nonnegative for all* $0 \le r \le s \le t$.

On the other hand, if $F(r,s,t)$ *are all nonnegative, then*

$$c_s \le (\beta_{s+2}-\beta_{s+1})\,(\beta_{s+2}-\beta_s)\,, \qquad s = 0,1,2,\ldots. \tag{3}$$

Proof. (i) We will proceed by induction. Assume that $B(0,k,k)$ are nonnegative for $k \le s$. Then we have to prove that so is

$$B(0,s+1,s+1) = \beta_{s+2}-\beta_{s+1} - \frac{c_s}{B(0,s,s+1)},$$

i.e., that

$$c_s \le (\beta_{s+2}-\beta_{s+1})B(0,s,s+1).$$

By assumption we have

$$\begin{aligned}\frac{c_s}{(\beta_{s+2}-\beta_{s+1})} &\le \beta_{s+2}-\beta_s - \frac{c_{s-1}}{\beta_{s+2}-\beta_s}\\ &\le \beta_{s+2}-\beta_s - \frac{c_{s-1}}{\beta_{s+2}-\beta_s+B(0,s-1,s-1)}\\ &\le \beta_{s+2}-\beta_s - \frac{c_{s-1}}{B(0,s-1,s+1)} = B(0,s,s+1).\end{aligned}$$

This shows that (i) yields the conclusion. Next observe that if condition (ii) holds, then so does (i).

Finally, assume

$$c_s = (\beta_{s+1}-\beta_s)(\beta_{s+2}-\beta_{s+1})(1-g_s)g_{s+1},$$

where $0 \le g_0 < 1$ and $0 < g_i < 1$ for $i \ge 1$. We will prove by induction that $B(0,s,s) \ge (\beta_{s+1}-\beta_s)(1-g_s)$. This is true for $s=0$, and if it holds for every $k \le s$, then

$$\begin{aligned}B(0,s+1,s+1) &= (\beta_{s+2}-\beta_{s+1})\left(1-\frac{(\beta_{s+1}-\beta_s)(1-g_s)g_{s+1}}{B(0,s,s+1)}\right)\\ &\ge (\beta_{s+2}-\beta_{s+1})\left(1-\frac{(\beta_{s+1}-\beta_s)(1-g_s)g_{s+1}}{B(0,s,s)+\beta_{s+2}-\beta_{s+1}}\right)\\ &\ge (\beta_{s+2}-\beta_{s+1})\left(1-\frac{(\beta_{s+1}-\beta_s)(1-g_s)g_{s+1}}{(\beta_{s+1}-\beta_s)(1-g_s)+\beta_{s+2}-\beta_{s+1}}\right)\\ &\ge (\beta_{s+2}-\beta_{s+1})(1-g_{s+1}).\end{aligned}$$

On the other hand, if $F(r,s,t)$ are all nonnegative, then

$$c_s \le (\beta_{s+2}-\beta_{s+1})B(0,s,s+1) \le (\beta_{s+2}-\beta_{s+1})(\beta_{s+2}-\beta_s).$$ ■

Remark. Note that if β_n is increasing and bounded then (3) implies that c_s tends to 0.

3. Motzkin Paths Method

This part contains the first proof of Theorem 1, which uses ideas of de Médicis and Stanton [4] and Viennot [7]. Assume that P_n and $g(n,m,k)$ satisfy (2) and (1), respectively.

Define L to be the linear functional on $\mathbb{R}[x]$ satisfying $L(P_0) = 1$ and $L(P_n) = 0$ for $n \geq 1$. Note that by (2) we have $L(x^m P_n(x)) = 0$ and, consequently, $L(P_m(x)P_n(x)) = 0$ if $m < n$. We also have

$$L(P_m(x)P_m(x)) = \frac{\gamma_0\gamma_1\cdots\gamma_{m-1}}{\alpha_0\alpha_1\cdots\alpha_{m-1}}. \tag{4}$$

Indeed,

$$\alpha_m L(P_{m+1}P_{m+1}) = L(xP_m P_{m+1}) = \gamma_m L(P_m P_m).$$

Now multiplying both sides of (1) by P_k and applying L yields

$$L(P_k P_m P_n) = g(n,m,k)L(P_k^2).$$

Thus, we can examine the nonnegativity of $L(P_k P_m P_n)$ instead of $g(n,m,k)$. We have

$$\begin{aligned} L(xP_kP_mP_n) &= \alpha_k L(P_{k+1}P_mP_n) + \beta_k L(P_kP_mP_n) + \gamma_{k-1}L(P_{k-1}P_mP_n), \\ L(xP_kP_mP_n) &= \alpha_n L(P_kP_mP_{n+1}) + \beta_n L(P_kP_mP_n) + \gamma_{n-1}L(P_kP_mP_{n-1}). \end{aligned}$$

This implies the recurrence relation

$$\begin{aligned} \alpha_k L(P_{k+1}P_mP_n) &= \alpha_n L(P_kP_mP_{n+1}) + \gamma_{n-1}L(P_kP_mP_{n-1}) \\ &\quad + (\beta_n - \beta_k)L(P_kP_mP_n) - \gamma_{k-1}L(P_{k-1}P_mP_n). \end{aligned} \tag{5}$$

In order to evaluate $L(P_kP_mP_n)$ we need to introduce two sets of so-called Motzkin paths. Define the following classes of directed edges (steps):

$$\begin{aligned} \mathcal{U} &= \{((i,j),(i+1,j+1)) : i,j \geq 0\}, \\ \mathcal{D} &= \{((i,j),(i+1,j-1)) : i \geq 0,\ j \geq 1\}, \\ \mathcal{H}_d &= \{((i,j),(i+d,j)) : i,j \geq 0\}, \end{aligned}$$

where $d \geq 1$. For a path $\mathbf{f} = (S_0, S_1, \ldots, S_p)$, with $(S_{i-1}, S_i) \in \mathcal{A}_i \in \{\mathcal{U}, \mathcal{D}, \mathcal{H}_1, \mathcal{H}_2, \ldots\}$, we define its *type* by $t(\mathbf{f}) = (\mathcal{A}_1, \ldots, \mathcal{A}_p)$. Note that the pair $(S_0, t(\mathbf{f}))$ determines $\mathbf{f}$.

Now fix $k, m, n \geq 0$ and denote by $\mathbf{M}(k,m,n)$ the family of paths

$$\mathbf{f} = (S_0, S_1, \ldots, S_p)$$

with $S_0 = (0,m)$, $S_p = (k,n)$ and satisfying $(S_{i-1}, S_i) \in \mathcal{U} \cup \mathcal{D} \cup \mathcal{H}_1 \cup \mathcal{H}_2$. Note that by the definition of the classes $\mathcal{U}$, $\mathcal{D}$, and $\mathcal{H}_d$, $\mathbf{f}$ lies at or above the x-axis. For such a path we define its *weight* to be

$$w(\mathbf{f}) = w(S_0, S_1)w(S_1, S_2)\cdots w(S_{p-1}, S_p),$$

where for an edge $e = (S', S'')$ with $S' = (i, j)$ we set

$$w(e) = \begin{cases} \gamma_j & \text{if } e \in \mathcal{U}, \\ \alpha_{j-1} & \text{if } e \in \mathcal{D}, \\ \beta_j - \beta_i & \text{if } e \in \mathcal{H}_1, \\ -\alpha_i \gamma_i & \text{if } e \in \mathcal{H}_2. \end{cases}$$

Put

(6)
$$w(k, m, n) = \sum_{\mathbf{f} \in \mathbf{M}(k,m,n)} w(\mathbf{f}) .$$

Formulas (4) and (5) yield

(7)
$$L(P_k P_m P_n) = \frac{\gamma_0 \gamma_1 \cdots \gamma_{m-1}}{\alpha_0 \alpha_1 \cdots \alpha_{m-1} \alpha_0 \alpha_1 \cdots \alpha_{k-1}} w(k, m, n).$$

By definition, the class $\mathbf{M}(k, m, n)$ contains paths in which the only horizontal steps allowed are of length 1 or 2. Collecting all the consecutive horizontal steps together, into single long horizontal steps in each path of $\mathbf{M}(k, m, n)$, gives rise to a new class of paths denoted by $\widetilde{\mathbf{M}}(k, m, n)$. Precisely, this is the class of paths $\mathbf{g} = (T_0, T_1, \ldots, T_q)$ such that

$$\begin{aligned} T_0 &= (0, m), \\ T_q &= (k, n), \\ (T_{r-1}, T_r) &\in \mathcal{U} \cup \mathcal{D} \cup \bigcup_{d=1}^{\infty} \mathcal{H}_d, \end{aligned}$$

and with no two consecutive horizontal steps allowed, i.e., if $(T_{i-1}, T_i) \in \mathcal{H}_d$, then either $i = q$ or $(T_i, T_{i+1}) \notin \bigcup_{d=1}^{\infty} \mathcal{H}_d$.

There is a natural mapping Λ from $\mathbf{M}(k, m, n)$ onto $\widetilde{\mathbf{M}}(k, m, n)$ given as follows. If $\mathbf{f} \in \mathbf{M}(k, m, n)$ and $t(\mathbf{f}) = (\mathcal{A}_1, \ldots, \mathcal{A}_p)$, then $t(\Lambda(\mathbf{f}))$ is defined by replacing in $t(\mathbf{f})$ every maximal block $(\mathcal{A}_u, \mathcal{A}_{u+1}, \ldots, \mathcal{A}_v)$ of type $(\mathcal{H}_{\varepsilon_u}, \mathcal{H}_{\varepsilon_{u+1}}, \ldots, \mathcal{H}_{\varepsilon_v})$, where $\varepsilon_j \in \{1, 2\}$, by $\mathcal{H}_\varepsilon$, $\varepsilon = \varepsilon_u + \varepsilon_{u+1} + \cdots + \varepsilon_v$.

We can endow $\widetilde{\mathbf{M}}(k, m, n)$ with the weight

$$\widetilde{w}(\mathbf{g}) = \sum_{\substack{\mathbf{f} \in \mathbf{M}(k,m,n) \\ \Lambda(\mathbf{f}) = \mathbf{g}}} w(\mathbf{f}).$$

In this way we get (see (6)):

(8)
$$w(k, m, n) = \sum_{\mathbf{g} \in \widetilde{\mathbf{M}}(k,m,n)} \sum_{\substack{\mathbf{f} \in \mathbf{M}(k,m,n) \\ \Lambda(\mathbf{f}) = \mathbf{g}}} w(\mathbf{f}) = \sum_{\mathbf{g} \in \widetilde{\mathbf{M}}(k,m,n)} \widetilde{w}(\mathbf{g}).$$

Moreover,

$$\widetilde{w}(T_0, T_1, \ldots, T_q) = \widetilde{w}(T_0, T_1) \widetilde{w}(T_1, T_2) \cdots \widetilde{w}(T_{q-1}, T_q),$$

where $\widetilde{w}(e) = w(e)$ if $e \in \mathcal{U} \cup \mathcal{D}$. For $e = ((r, t), (s, t))$ denote $\widetilde{h}(r, s, t) = \widetilde{w}(e)$. Observe that

$$\widetilde{h}(r, s, t) = \sum w(e_1) w(e_2) \cdots w(e_p),$$

where the sum is taken over all paths $(e_1, e_2, \ldots, e_p)$ from (r, t) to (s, t) with $e_i \in \mathcal{H}_1 \cup \mathcal{H}_2$. Now by decomposing this sum into $\Sigma^{(1)} + \Sigma^{(2)}$, according to whether the last step e_p belongs to $\mathcal{H}_1$ or $\mathcal{H}_2$, we get, for $s - r \geq 2$:

$$\begin{aligned}\widetilde{h}(r, s, t) &= \sum\nolimits^{(1)} w(e_1) \cdots w(e_p) + \sum\nolimits^{(2)} w(e_1) \cdots w(e_p) \\ &= \widetilde{h}(r, s-1, t)(\beta_t - \beta_{s-1}) - \widetilde{h}(r, s-2, t)\alpha_{s-2}\gamma_{s-2},\end{aligned}$$

with the initial values $\widetilde{h}(r, r, t) = 1$ and $\widetilde{h}(r, r+1, t) = \beta_t - \beta_r$.

Assume that $k \leq m \leq n$. Then we have $T_0, T_1, \ldots, T_q \in \{(i, j) : 0 \leq i \leq j\}$ for every $(T_0, T_1, \ldots, T_q) \in \widetilde{\mathbf{M}}(k, m, n)$. Putting $c_s = \alpha_s \gamma_s$ leads to $\widetilde{h}(r, s, t) = F(r, s, t)$. Therefore, if for $0 \leq r \leq s \leq t$ all the numbers $F(r, s, t)$ are nonnegative, then so are all the weights $\widetilde{w}(e)$ and, by (8), so is $w(k, m, n)$. This, in view of Proposition 1, concludes the proof of Theorem 1. ■

4. Maximum Principle Method

In this section we prove Theorem 1 using a discrete version of the boundary value problem associated with the recurrence relation (2).

The nonnegativity of the linearization coefficients does not depend on the normalization of the polynomials P_n. Therefore we may replace the polynomials P_n with their positive multiples Q_n and the new polynomials satisfy

$$x Q_n(x) = \alpha_n' Q_{n+1}(x) + \beta_n' Q_n(x) + \gamma_{n-1}' Q_{n-1}(x). \tag{9}$$

The fact that the polynomials Q_n and P_n are the positive multiples of each other is equivalent to

$$\beta_n' = \beta_n, \qquad \alpha_n' \gamma_n' = \alpha_n \gamma_n, \tag{10}$$

for $n \geq 0$. In particular, when Q_n are orthonormal they satisfy

$$x Q_n(x) = \lambda_n Q_{n+1}(x) + \beta_n Q_n(x) + \lambda_{n-1} Q_{n-1}(x), \tag{11}$$

where $\lambda_n^2 = \alpha_n \gamma_n$. As had been observed in [5], nonnegative linearization is equivalent to the following maximum principle (note the slight difference between (2) and the notation in [5]).

Let $u = u(n, m)$ denote a matrix defined for $n, m \geq 0$. Let L_1 and L_2 denote operators acting on u according to the rule

$$\begin{aligned}L_1 u(n, m) &= \alpha_n u(n+1, m) + \beta_n u(n, m) + \gamma_{n-1} u(n-1, m), \\ L_2 u(n, m) &= \alpha_m u(n, m+1) + \beta_m u(n, m) + \gamma_{m-1} u(n, m-1).\end{aligned}$$

Let $H = L_1 - L_2$. Assume the matrix $u = u(n, m)$ satisfies

$$\begin{cases} Hu\,(n, m) = 0 & \text{for} \quad n, m \geq 0, \\ u(n, 0) \geq 0 & \text{for} \quad n \geq 0. \end{cases} \tag{12}$$

The polynomials P_n admit nonnegative linearization if and only if every solution of the boundary value problem (12) satisfies $u(n, m) \geq 0$ for $n \geq m \geq 0$. This leads to another proof of Theorem 1.

Proof. Instead of showing the nonnegative linearization of Q_n we will show that property for their positive multiples P_n satisfying (2). The coefficients α_n, γ_n will be specified later in the proof. Assume u is a solution of (12). We will proceed by induction on m. Assume $u(t,s) \geq 0$ for $t \geq s \geq 0$ and $s \leq m-1$. We will show that $u(n,m) \geq 0$. By assumption, we have

$$\frac{\lambda_j^2}{(\beta_n - \beta_j)(\beta_n - \beta_{j+1})} = (1-g_j)g_{j+1},$$

where $0 \leq g_j \leq 1$ for $j = 0, 1, \ldots, n-2$. Set

$$\begin{aligned} \alpha_j &= (\beta_n - \beta_{j+1})g_{j+1}, & j &= 0, 1, \ldots, n-2, \\ \alpha_j &= \lambda_j, & j &\geq n-1, \\ \gamma_j &= (\beta_n - \beta_j)(1-g_j), & j &= 0, 1, \ldots, n-2, \\ \gamma_j &= \lambda_j, & j &\geq n-1. \end{aligned}$$

Hence the coefficients α_j and γ_j are nonnegative and $\alpha_j\gamma_j = \lambda_j^2$. This means the polynomials satisfying (2) are positive multiples of Q_n. Therefore it suffices to show nonnegative linearization for P_n. This in turn amounts to showing that the problem (12) admits nonnegative solutions. By (12) and by the definition of H we have (see Figure 1):

$$\begin{aligned} 0 &= \sum_{s=0}^{m-1} Hu(n,s) \\ &= -\alpha_{m-1}u(n,m) + \sum_{s=0}^{m-1} \gamma_{n-1}u(n-1,s) \\ &\quad + \sum_{s=0}^{m-1} \alpha_n u(n+1,s) + \sum_{s=0}^{m-1} c_s u(n,s) \\ &\geq -\alpha_{m-1}u(n,m) + \sum_{s=0}^{m-1} c_s u(n,s). \end{aligned}$$

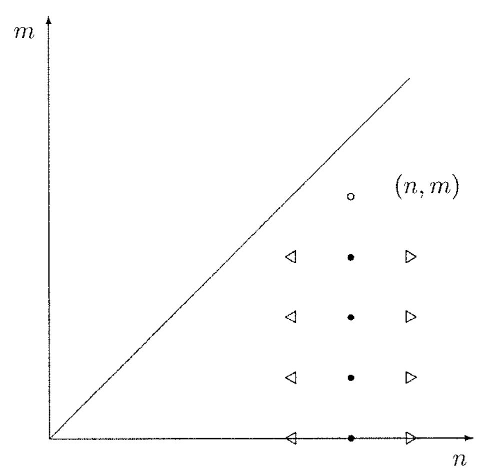

Fig. 1.

It can be easily computed that

$$\begin{aligned} c_{m-1} &= \beta_n - \beta_{m-1} - \alpha_{m-2}, \\ c_s &= \beta_n - \beta_s - \alpha_{s-1} - \gamma_s, \qquad s = 0, 1, \ldots, m-2. \end{aligned}$$

By the definition of α_j and γ_j, we get

$$\begin{aligned} c_{m-1} &= \beta_n - \beta_{m-1} - (\beta_n - \beta_{m-1})g_{m-1} = (\beta_n - \beta_{m-1})(1 - g_{m-1}) \geq 0, \\ c_s &= \beta_n - \beta_s - (\beta_n - \beta_s)g_s - (\beta_n - \beta_s)(1 - g_s) = 0. \end{aligned}$$

Thus $u(n, m) \geq 0$. ■

5. The Symmetric Case

The orthogonal polynomials P_n are called *symmetric* if they satisfy

$$P_n(-x) = (-1)^n P_n(x).$$

This is equivalent to the fact that the orthogonality measure is symmetric about the origin (in the determinate case). Another equivalent condition is that the general recurrence relation for the symmetric polynomials is

$$x P_n(x) = \alpha_n P_{n+1}(x) + \gamma_{n-1} P_{n-1}(x), \tag{13}$$

where $P_{-1} = 0$, $P_0 = 1$, and

$$x Q_n(x) = \lambda_n Q_{n+1}(x) + \lambda_{n-1} Q_{n-1}(x), \tag{14}$$

$\lambda_n > 0$ for orthonormalized symmetric polynomials. As we have seen in the previous section, the polynomials P_n and Q_n are the multiples of each other if and only if

$$\lambda_n^2 = \alpha_n \gamma_n.$$

By [6] nonnegative linearization of the polynomials P_n satisfying (13) is equivalent to the following boundary problem

$$\begin{cases} Hu\,(n, m) = 0 & \text{for} \quad n, m \geq 0, \\ u(2n+1, 0) = 0 & \text{for} \quad n \geq 0, \\ u(2n, 0) \geq 0 & \text{for} \quad n \geq 0, \end{cases} \tag{15}$$

having only nonnegative solutions $u(n, m)$.

Theorem 2. *Let the orthogonal polynomials Q_n satisfy* (14). *Assume that either*:

(i) *For every $N = 0, 1, 2, \ldots,$ the sequence $\lambda_n^2/\lambda_{2N+1}^2$ for $n = 0, 1, \ldots, 2N$, is a chain sequence.*
(ii) *For every $N = 0, 1, 2, \ldots,$ the sequence $\lambda_n^2/\lambda_{2N}^2$ for $n = 0, 1, \ldots, 2N-1$, is a chain sequence.*

Then the polynomials Q_n admit nonnegative product linearization.

Proof. We will show part (i) only, since part (ii) can be shown similarly.

Again as in the proof of Theorem 1 we consider the polynomials P_n satisfying (13) and being positive multiples of Q_n.

Assume u is a solution of (15). First observe that this implies

$$u(n,m) = 0 \qquad \text{for} \quad n+m \text{ odd}.$$

We will proceed by induction on m. Assume $u(t,s) \geq 0$ for $t \geq s \geq 0$ and $s \leq m-1$. We will show that $u(n,m) \geq 0$.

We will consider two cases depending on the parity of n. First let $n = 2N$. If m is odd $u(n,m) = 0$. Let m be an even number, i.e., $m = 2M$. By assumption we have

$$\frac{\lambda_j^2}{\lambda_{2N-1}^2} = (1-g_j)g_{j+1},$$

where $0 \leq g_j \leq 1$ for $j = 0, 1, \ldots, 2N-2$. Set

$$\alpha_j = \lambda_{2N-1} g_{j+1}, \qquad j = 0, 1, \ldots, 2N-2, \tag{16}$$

$$\alpha_j = \lambda_j, \qquad j \geq 2N-1, \tag{17}$$

$$\gamma_j = \lambda_{2N-1}(1-g_j), \qquad j = 0, 1, \ldots, 2N-2, \tag{18}$$

$$\gamma_j = \lambda_j, \qquad j \geq 2N-1. \tag{19}$$

Hence the coefficients α_j and γ_j are nonnegative and $\alpha_j \gamma_j = \lambda_j^2$. Thus the polynomials satisfying (13) are positive multiples of Q_n. By (15) we have (see Figure 2):

$$\begin{aligned} 0 &= \sum_{s=0}^{M-1} Hu(2N-1, 2s) + \sum_{s=1}^{M} Hu(2N, 2s-1) \\ &= -\alpha_{2M-1} u(2N, 2M) + \sum_{s=1}^{M} c_{2s-1} u(2N-1, 2s-1) + \sum_{s=0}^{M-1} c_{2s} u(2N, 2s) \end{aligned} \tag{20}$$

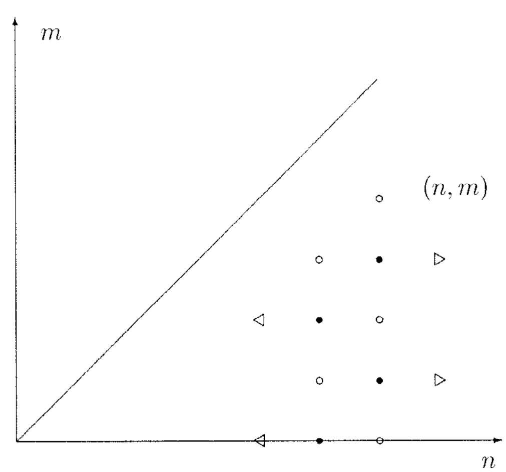

Fig. 2.

$$+\sum_{s=0}^{M-1}\gamma_{2N-2}u(2N-2,2s)+\sum_{s=1}^{M}\alpha_{2N}u(2N+1,2s-1).$$

The coefficients c_s are given by

$$\begin{aligned}c_{2M-1} &= \gamma_{2N-1}-\alpha_{2M-2},\\ c_{2s-1} &= \gamma_{2N-1}-\alpha_{2s-2}-\gamma_{2s-1},\\ c_{2s} &= \alpha_{2N-1}-\alpha_{2s-1}-\gamma_{2s}.\end{aligned}$$

By the definition of α_j and γ_j we obtain

$$\begin{aligned}c_{2M-1} &= \lambda_{2N-1}-\lambda_{2N-1}g_{2M-1}\ge 0,\\ c_{2s-1} &= \lambda_{2N-1}-\lambda_{2N-1}g_{2s-1}-\lambda_{2N-1}(1-g_{2s-1})=0,\\ c_{2s} &= \lambda_{2N-1}-\lambda_{2N-1}g_{2s}-\lambda_{2N-1}(1-g_{2s})=0.\end{aligned}$$

Thus, by the inductive hypothesis, all the terms in (20), except $-\alpha_{2M-1}u(2N,2M)$, are nonnegative. Hence $u(n,m)=u(2N,2M)\ge 0$.

The case when both $n=2N-1$ and $m=2M-1$ are odd numbers can be dealt with similarly, by analyzing the expression

$$0 = \sum_{s=1}^{M-1}Hu(2N-1,2s-1)+\sum_{s=0}^{M-1}Hu(2N,2s). \qquad \blacksquare$$

By Wall's characterization of chain sequences we immediately get the following:

Theorem 3. *Let the orthogonal polynomials* Q_n *satisfy* (14). *Assume that either*:

(i) *For every* $N=1,2,\dots,$ *the matrix*

$$\begin{pmatrix}\lambda_{2N-1} & \lambda_0 & 0 & \cdots & 0 & 0\\ \lambda_0 & \lambda_{2N-1} & \lambda_1 & \cdots & 0 & 0\\ 0 & \lambda_1 & \lambda_{2N-1} & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & \lambda_{2N-1} & \lambda_{2N-2}\\ 0 & 0 & 0 & \cdots & \lambda_{2N-2} & \lambda_{2N-1}\end{pmatrix} \tag{21}$$

is positive definite.

(ii) *For every* $N=1,2,\dots,$ *the matrix*

$$\begin{pmatrix}\lambda_{2N} & \lambda_0 & 0 & \cdots & 0 & 0\\ \lambda_0 & \lambda_{2N} & \lambda_1 & \cdots & 0 & 0\\ 0 & \lambda_1 & \lambda_{2N} & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & \lambda_{2N-1} & \lambda_{2N-1}\\ 0 & 0 & 0 & \cdots & \lambda_{2N-1} & \lambda_{2N}\end{pmatrix}$$

is positive definite.

Then the polynomials Q_n *admit nonnegative product linearization*.

Corollary 2. *Assume that either*:

(i) *For every* $n \geq 0$ *there holds*

$$\lambda_{2n-2} + \lambda_{2n-1} \leq \lambda_{2n+1},$$
$$\lambda_{2n-1} + \lambda_{2n} \leq \lambda_{2n+1}.$$

(ii) *For every* $n \geq 0$ *there holds*

$$\lambda_{2n-1} + \lambda_{2n} \leq \lambda_{2n+2},$$
$$\lambda_{2n} + \lambda_{2n+1} \leq \lambda_{2n+2}.$$

Then the polynomials Q_n *satisfying* (14) *admit nonnegative product linearization.*

Proof. We will show (i) only, since the proof of (ii) is similar. By assumption, the sequence λ_{2n-1} is increasing. Hence

$$\lambda_{2n-2} + \lambda_{2n-1} \leq \lambda_{2n+1} \leq \lambda_{2N-1},$$
$$\lambda_{2n-1} + \lambda_{2n} \leq \lambda_{2n+1} \leq \lambda_{2N-1},$$

for $n < N$. Thus, in matrix (21), the sum of absolute values of the entries off the main diagonal in the nth row is less than the entry on the main diagonal of this row. This implies that the matrix (21) is positive definite. ■

Remark. 1. If λ_{2n} is chosen to be decreasing then we can drop the second inequality of assumption (i). A typical example of a sequence λ_n satisfying assumption (i) can be obtained by picking up an increasing sequence λ_{2n-1} and then choosing a sequence λ_{2n} so that (i) is satisfied. For example, the following choice of λ_n satisfies (i) if $0 < a \leq q < 1$:

$$\lambda_{2n-1} = 1 - q^n,$$
$$\lambda_{2n} = aq^n(1-q).$$

2. Assume λ_n is bounded and satisfies assumption (i). Since λ_{2n-1} is increasing it is convergent, say to λ, and therefore the sequence λ_{2n} tends to 0. The polynomials satisfying (14) are orthonormal with respect to the measure μ. The support of μ is symmetric about 0 by (14) and it coincides with the spectrum of the following difference operator on ℓ^2:

$$(La)_n = \lambda_n a_{n+1} + \lambda_{n-1} a_{n-1}.$$

We have

$$\lambda_{n-1} + \lambda_n \leq \lambda \qquad \text{and} \qquad \lambda_{n-1} + \lambda_n \to \lambda.$$

Thus the norm of L is equal to λ. Observe that

$$(L^2 a)_n = \lambda_n \lambda_{n+1} a_{n+2} + (\lambda_n^2 + \lambda_{n-1}^2) a_n + \lambda_{n-1} \lambda_{n-2} a_{n-2}.$$

Thus we have

$$\lambda_n \lambda_{n+1} \to 0, \qquad n \to \infty,$$
$$\lambda_n^2 + \lambda_{n-1}^2 \to \lambda^2, \qquad n \to \infty.$$

Therefore the operator $L^2 - \lambda^2 I$ is compact. We can conclude that the spectrum of the operator L consists of $\pm\lambda$ and the eigenvalues $\pm x_n$ where x_n is a sequence convergent to λ. Since the norm of L is equal to λ, we may assume that $x_n \nearrow \lambda$.

Thus also the support of the orthogonality measure consists of the sequence $\pm x_n$ where $x_n \nearrow \lambda$. The same conclusion holds if λ_n satisfies (ii).

6. Relation Between Symmetric and Nonsymmetric Cases

Let polynomials Q_n satisfy (14). Then they satisfy

$$Q_n(-x) = (-1)^n Q_n(x).$$

This implies that the polynomials Q_{2n} involve even powers of x only. Therefore the functions defined as

$$R_n(y) = Q_{2n}(\sqrt{y})$$

are polynomials of degree n. By iterating (14) twice and substituting $y = x^2$ we get

$$yR_n(y) = \lambda_{2n}\lambda_{2n+1}R_{n+1}(y) + (\lambda_{2n}^2 + \lambda_{2n-1}^2)R_n(y) + \lambda_{2n-2}\lambda_{2n-1}R_{n-1}(y).$$

By Theorem 2 we get the following:

Corollary 3. *Assume the polynomials R_n satisfy*

$$yR_n(y) = \Lambda_n R_{n+1}(y) + \beta_n R_n(y) + \Lambda_{n-1}R_{n-1}(y)$$

and that there are coefficients λ_n such that

$$\begin{aligned} \Lambda_n &= \lambda_{2n}\lambda_{2n+1}, \\ \beta_n &= \lambda_{2n}^2 + \lambda_{2n-1}^2. \end{aligned}$$

If the coefficients λ_n satisfy the assumptions of Theorem 2 *the polynomials R_n admit nonnegative product linearization.*

Remark. In examples that can be constructed by using Corollary 3 combined with Corollary 2 the sequence β_n is always increasing. Indeed, by the assumptions of Corollary 2, we have

$$\begin{aligned} \beta_{n+1} &= \lambda_{2n+1}^2 + \lambda_{2n+2}^2 \\ &> (\lambda_{2n-1} + \lambda_{2n})^2 > \lambda_{2n-1}^2 + \lambda_{2n}^2 = \beta_n. \end{aligned}$$

It would be interesting to determine if the condition $\beta_{n+1} > \beta_n$, for every $n \geq 0$, is a necessity for nonnegative linearization in the case of $\lambda_n \to 0$.

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