MULTIPLICATIVE FREE SQUARE OF THE FREE POISSON MEASURE AND EXAMPLES OF FREE SYMMETRIZATION

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ABSTRACT. We compute moments and free cumulants of the measure $\rho_t := \pi_t \boxtimes \pi_t$, where π_t denotes the free Poisson law with parameter $t > 0$. We also compute free cumulants of the symmetrization of ρ_t. Finally we introduce free symmetrization of a probability measure on $\mathbb{R}$ and provide some examples.

1. Introduction

Free convolution is a binary operation on the class $\mathcal{M}$ of probability measures on $\mathbb{R}$, which corresponds to the notion of free independence in noncommutative probability (see [2, 7, 5]). Namely, if X, Y are free noncommuting random variables, with distributions $\mu, \nu \in \mathcal{M}$ respectively, then the *(additive) free convolution* $\mu \boxplus \nu$ is the distribution of the sum $X + Y$. Similarly, if moreover $X \geq 0$ then the *multiplicative free convolution* $\mu \boxtimes \nu$ can be defined as the distribution of the product $\sqrt{X} Y \sqrt{X}$.

For the sake of this paper we can confine ourselves to the class $\mathcal{M}^{\mathbf{c}}$ of compactly supported measures in $\mathcal{M}$. Then these operations can be described in the following way. For $\mu \in \mathcal{M}^{\mathbf{c}}$ we define its *moment generating function*

$$M_\mu(z) := \sum_{m=0}^{\infty} s_m(\mu) z^m, \tag{1}$$

defined in some neighborhood of 0, where

$$s_m(\mu) := \int_{\mathbb{R}} x^m \, d\mu(x) \tag{2}$$

is the mth moment of μ. Then we define its *R-transform* $R_\mu(z)$ by the equation:

$$M_\mu(z) = R_\mu(z M_\mu(z)) + 1. \tag{3}$$

If $R_\mu(z) = \sum_{m=1}^{\infty} r_m(\mu) z^m$ then the numbers $r_m(\mu)$ are called *free cumulants* of μ. For $\mu, \nu \in \mathcal{M}^{\mathbf{c}}$ their free convolution $\mu \boxplus \nu$ can be defined as the unique measure in $\mathcal{M}^{\mathbf{c}}$ satisfying

$$R_{\mu \boxplus \nu}(z) = R_\mu(z) + R_\nu(z). \tag{4}$$

The *free S-transform* (see [6]) of $\mu \in \mathcal{M}^{\mathbf{c}}$ is defined by the relation

$$R_\mu(z S_\mu(z)) = z \qquad \text{or} \qquad M_\mu\big(z(1+z)^{-1} S_\mu(z)\big) = 1 + z. \tag{5}$$

If $\mu, \nu \in \mathcal{M}^{\mathbf{c}}$ and at least one of them has support contained in $[0, \infty)$ then the *multiplicative free convolution* $\mu \boxtimes \nu$ is defined by

$$S_{\mu \boxtimes \nu}(z) := S_\mu(z) S_\nu(z). \tag{6}$$

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For $\mu \in \mathcal{M}$, concentrated on $[0,\infty)$, we define its *symmetrization* by $\int_{\mathbb{R}} f(x^2)\,d\mu^{\mathbf{s}}(x) = \int_{\mathbb{R}} f(x)\,d\mu(x)$ for every compactly supported continuous function $f:\mathbb{R}\to\mathbb{R}$. If $M_\mu(z)$ is the moment generating function of μ then the moment generating function of $\mu^{\mathbf{s}}$ is $M_{\mu^{\mathbf{s}}}(z) = M_\mu(z^2)$ (which means that $s_{2m}(\mu^{\mathbf{s}}) = s_m(\mu)$ and $s_n(\mu^{\mathbf{s}}) = 0$ if n is odd).

The aim of this paper is to compute moments and free cumulants of the measure $\rho_t := \pi_t \boxtimes \pi_t$, where π_t denotes the free Poisson measure. We also compute the free cumulants of the symmetric measures $\rho_t^{\mathbf{s}}$. Finally we introduce and study notion of free symmetrization, which can be considered as a free analog of the map $\mu \mapsto \mu^{\mathbf{s}}$, and provide a one-parameter family of examples.

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2. A family of sequences

For real parameters t, r we define a sequence $\{c_m(t,r)\}_{m=0}^{\infty}$ putting $c_0(t,r) := 1$ and for $m \geq 1$

$$c_m(t,r) := \sum_{k=1}^{m} \binom{2m}{m+k}\binom{m+r-1}{k-1}\frac{rt^k}{m}, \tag{7}$$

where $\binom{a}{m}$ denotes the generalized binomial coefficient: $\binom{a}{m} := \frac{a(a-1)(a-2)\ldots(a-m+1)}{m!}$. By convention we also put $c_{-1}(t,r) := 0$. For example, using the Cauchy-Vandermonde convolution formula (see formula (5.22) in [3]) one can see that for $m \geq 1$

$$c_m(1,r) = \binom{3m-1+r}{m-1}\frac{r}{m}. \tag{8}$$

Proposition 2.1. *For $m \geq 0$*

$$\begin{aligned} t\cdot c_m(t,r) = c_{m-1}(t,r+2) + 2(t-1)c_{m-1}(t,r+1) \\ + (t-1)^2 c_{m-1}(t,r) + t\cdot c_m(t,r-1). \end{aligned} \tag{9}$$

Proof. First we note that

$$c_m(t,r) - c_m(t,r-1) = \sum_{k=1}^{m}\binom{2m}{m+k}\left[\binom{m+r-2}{k-2}r + \binom{m+r-2}{k-1}\right]\frac{t^k}{m}. \tag{10}$$

Now we observe that (9) can be written as

$$\begin{aligned} & t\,[c_m(t,r) - c_m(t,r-1)] \\ &= [c_{m-1}(t,r+2) - c_{m-1}(t,r+1)] - [c_{m-1}(t,r+1) - c_{m-1}(t,r)] \\ &\quad + 2t\,[c_{m-1}(t,r+1) - c_{m-1}(t,r)] + t^2 c_{m-1}(t,r). \end{aligned} \tag{11}$$

Applying (10) and the binomial identity: $\binom{a-1}{b-1}+\binom{a-1}{b}=\binom{a}{b}$ to the right hand side of (11) we obtain

$$\sum_{k=1}^{m-1}\binom{2m-2}{m+k-1}\left[\binom{m+r-1}{k-2}(r+2)+\binom{m+r-1}{k-1}\right]\frac{t^k}{m-1}$$
$$-\sum_{k=1}^{m-1}\binom{2m-2}{m+k-1}\left[\binom{m+r-2}{k-2}(r+1)+\binom{m+r-2}{k-1}\right]\frac{t^k}{m-1}$$
$$+2\sum_{k=1}^{m-1}\binom{2m-2}{m+k-1}\left[\binom{m+r-2}{k-2}(r+1)+\binom{m+r-2}{k-1}\right]\frac{t^{k+1}}{m-1}$$
$$+\sum_{k=1}^{m-1}\binom{2m-2}{m+k-1}\binom{m+r-2}{k-1}r\frac{t^{k+2}}{m-1}$$

$$=\sum_{k=1}^{m-1}\binom{2m-2}{m+k-1}\left[\binom{m+r-2}{k-3}r+2\binom{m+r-1}{k-2}\right]\frac{t^k}{m-1}$$
$$+2\sum_{k=1}^{m-1}\binom{2m-2}{m+k-1}\left[\binom{m+r-2}{k-2}r+\binom{m+r-1}{k-1}\right]\frac{t^{k+1}}{m-1}$$
$$+\sum_{k=1}^{m-1}\binom{2m-2}{m+k-1}\binom{m+r-2}{k-1}\frac{rt^{k+2}}{m-1}.$$

Now we substitute $k':=k-1$ in the first sum and $k'':=k+1$ in the last one, obtaining

$$\sum_{k=0}^{m-2}\binom{2m-2}{m+k}\left[\binom{m+r-2}{k-2}r+2\binom{m+r-1}{k-1}\right]\frac{t^{k+1}}{m-1}$$
$$+2\sum_{k=1}^{m-1}\binom{2m-2}{m+k-1}\left[\binom{m+r-2}{k-2}r+\binom{m+r-1}{k-1}\right]\frac{t^{k+1}}{m-1}$$
$$+\sum_{k=2}^{m}\binom{2m-2}{m+k-2}\binom{m+r-2}{k-2}\frac{rt^{k+1}}{m-1}.$$

Note that each sum can be taken from $k=1$ to $k=m$. Applying the binomial identity we finally get

$$\sum_{k=1}^{m}\left[\binom{2m}{m+k}\binom{m+r-2}{k-2}r+2\binom{2m-1}{m+k}\binom{m+r-1}{k-1}\right]\frac{t^{k+1}}{m-1}. \tag{12}$$

To see that this is equal to the left hand side of (11) we use the identity: $\binom{2m-1}{m+k}=\binom{2m}{m+k}\frac{m-k}{2m}$ so it remains to check that

$$\binom{m+r-2}{k-2}\frac{r}{m-1}+\binom{m+r-1}{k-1}\frac{m-k}{m(m-1)}=\binom{m+r-2}{k-2}\frac{r}{m}+\binom{m+r-2}{k-1}\frac{1}{m}.\ \square$$

Proposition 2.2. *For every $r,s,t\in\mathbb{R}$ and every $m\geq 0$*

$$\sum_{k=0}^{m}c_{m-k}(t,r)c_k(t,s)=c_m(t,r+s). \tag{13}$$

Proof. It is easy to check that (13) is true for $m = 0, 1$. Assume this holds for $m-1$ and for all $r, s, t \in \mathbb{R}$. To prove that it holds for m we use induction on r. For $r = 0$ it is clear. Assume it holds for $r-1$. Then using (9), the induction and (9) again, we get

$$\begin{aligned} t \cdot \sum_{k=0}^{m} & c_{m-k}(t,r)c_k(t,s) \\ &= \sum_{k=0}^{m} \big[c_{m-k-1}(t,r+2) + 2(t-1)c_{m-k-1}(t,r+1) \\ &\qquad + (t-1)^2 c_{m-k-1}(t,r) + t \cdot c_{m-k}(t,r-1)\big] c_k(t,s) \\ &= c_{m-1}(t,r+s+2) + 2(t-1)c_{m-1}(t,r+s+1) \\ &\qquad + (t-1)^2 c_{m-1}(t,r+s) + t \cdot c_m(t,r+s-1) \\ &= t \cdot c_m(t,r+s). \end{aligned}$$

In this way we prove that (13) holds for all natural r. Since each side of (13) is a polynomial on r, the equality holds for all $r \in \mathbb{R}$. □

Denote by $C_t(z)$ the generating function for the sequence $\{c_m(t,1)\}_{m=0}^{\infty}$:

$$C_t(z) := \sum_{m=0}^{\infty} c_m(t,1) z^m. \tag{14}$$

Since $\binom{2m}{m+k} \le 4^m$, we have

$$|c_m(t,1)| \le \frac{4^m}{m} \sum_{k=1}^{m} \binom{m-1}{k-1} |t|^k = \frac{|t| 4^m (1+|t|)^{m-1}}{m},$$

so $C_t(z)$ is defined in some neighborhood of 0. Moreover, since $C_t(0) = 1$, the powers $C_t(z)^r$, $r \in \mathbb{R}$, are well defined on a (possibly smaller) neighborhood of 0. Then (13) implies that

$$C_t(z)^r := \sum_{m=0}^{\infty} c_m(t,r) z^m. \tag{15}$$

Proposition 2.3. *For fixed $t \in \mathbb{R}$ the function C_t satisfies equation*

$$t\left(C_t(z) - 1\right) = z C_t(z) \left(C_t(z) - 1 + t\right)^2, \tag{16}$$

for z belonging to some neighborhood of 0.

Proof. It is sufficient to multiply both sides of (9) by z^m and to take sum $\sum_{m=0}^{\infty}$, putting $r = 1$, and then to apply (15). □

3. Multiplicative square of the free Poisson measure

For $t > 0$ let π_t denote the free Poisson measure with parameter t:

$$\pi_t = \max\{1-t, 0\}\delta_0 + \frac{\sqrt{4t - (x-1-t)^2}}{2\pi x}\, dx, \tag{17}$$

with the absolutely continuous part supported on $\big[(1-\sqrt{t})^2,(1+\sqrt{t})^2\big]$. Then

$$M_{\pi_t}(z) = \frac{2}{1+(1-t)z+\sqrt{\big(1-(1+t)z\big)^2-4tz^2}} \tag{18}$$

$$= 1+\sum_{m=1}^{\infty} z^m \sum_{k=1}^{m} \binom{m}{k}\binom{m}{k-1}\frac{t^k}{m}, \tag{19}$$

$$R_{\pi_t}(z) = \frac{tz}{1-z}, \qquad S_{\pi_t}(z) = \frac{1}{t+z}. \tag{20}$$

From now on we are going to study the multiplicative free square $\rho_t := \pi_t \boxtimes \pi_t$. One can note that ρ_1 corresponds to $\pi_{2,1}$ in [1].

Theorem 3.1. *For the moment generating function and the free R-transform of ρ_t we have:*

$$M_{\rho_t}(z) = 1+\sum_{m=1}^{\infty} z^m \sum_{k=1}^{m} \binom{2m}{m+k}\binom{m}{k-1}\frac{t^{m+k}}{m}, \tag{21}$$

$$R_{\rho_t}(z) = \frac{1-2tz-\sqrt{1-4tz}}{2z} = t\sum_{m=1}^{\infty}\binom{2m+1}{m}\frac{(tz)^m}{2m+1}. \tag{22}$$

Proof. Since $S_{\rho_t}(z) = (t+z)^{-2}$, the function $M_{\rho_t}(z)$ satisfies equation

$$M_{\rho_t}\left(\frac{z}{(1+z)(t+z)^2}\right) = 1+z, \tag{23}$$

which means that $M_{\rho_t}(z)-1$ is the composition inverse of the function $z \mapsto \frac{z}{(1+z)(t+z)^2}$. This leads to the equation

$$\frac{M_{\rho_t}(z)-1}{M_{\rho_t}(z)\big(M_{\rho_t}(z)-1+t\big)^2} = z, \tag{24}$$

or equivalently

$$M_{\rho_t}(z)-1 = zM_{\rho_t}(z)\left(M_{\rho_t}(z)-1+t\right)^2. \tag{25}$$

Comparing (25) with (16) we see that $M_{\rho_t}(z) = C_t(tz)$.

For the R-transform we have $R_{\rho_t}\left(\frac{z}{(t+z)^2}\right) = z$ which is equivalent to

$$\frac{R_{\rho_t}(z)}{\big(t+R_{\rho_t}(z)\big)^2} = z. \tag{26}$$

Solving this equation we get

$$R_{\rho_t}(z) = \frac{1-2tz-\sqrt{1-4tz}}{2z} = \frac{2t^2z}{1-2tz+\sqrt{1-4tz}}. \tag{27}$$

□

For $c \in \mathbb{R}\setminus\{0\}$ and $\mu \in \mathcal{M}$ we define *dilation* $\mathrm{D}_c\mu \in \mathcal{M}$ by $\mathrm{D}_c\mu(X) := \mu(\frac{1}{c}X)$ for a borel subset of $\mathbb{R}$. Then we have $M_{\mathrm{D}_c\mu}(z) = M_\mu(cz)$ and $R_{\mathrm{D}_c\mu}(z) = R_\mu(cz)$.

Corollary 3.1. *Put $\widetilde{\rho}_t := \mathrm{D}_{\frac{1}{t}}\rho_t$. Then $\{\widetilde{\rho}_t\}_{t>0}$ is a $\boxplus$-semigroup, i.e. $\widetilde{\rho}_s \boxplus \widetilde{\rho}_t = \widetilde{\rho}_{s+t}$ whenever $s,t>0$.*

Proof. This is a direct consequence of (22). □

4. Free symmetrization

Let μ be a probability measure on $\mathbb{R}$ with support contained in $[0,\infty)$. Then its *symmetrization* $\mu^{\mathbf{s}}$ is defined by

$$\int_{\mathbb{R}} f(x^2)\,d\mu^{\mathbf{s}}(x) = \int_{\mathbb{R}} f(x)\,d\mu(x) \tag{28}$$

for every compactly supported continuous function on $\mathbb{R}$. If $M_\mu(z)$ is the moment generating function of μ then the moment generating function of $\mu^{\mathbf{s}}$ is $M_{\mu^{\mathbf{s}}}(z) = M_\mu(z^2)$. For example

$$\pi_t^{\mathbf{s}} = \max\{1-t,0\}\delta_0 + \frac{\sqrt{4t-(x^2-1-t)^2}}{\pi|x|}\,dx, \tag{29}$$

where the absolutely continuous part is supported on

$$\big[-1-\sqrt{t},-|1-\sqrt{t}|\big] \cup \big[|1-\sqrt{t}|,1+\sqrt{t}\big].$$

It is known (see Corollary 3.2 together with the remark in [4]) that $\pi_t^{\mathbf{s}}$ is not $\boxplus$-infinitely divisible, except of the Wigner measure $\pi_1^{\mathbf{s}} = \frac{1}{\pi}\sqrt{4-x^2}\cdot\chi_{[-2,2]}\,dx$.

Let us now consider the symmetrization $\rho_t^{\mathbf{s}}$ of the measure ρ_t. Therefore for the R-transform we have

$$R_{\rho_t^{\mathbf{s}}}(zM_{\rho_t}(z^2)) + 1 = M_{\rho_t}(z^2). \tag{30}$$

Proposition 4.1. *For the R-transform of $\rho_t^{\mathbf{s}}$ we have:*

$$R_{\rho_t^{\mathbf{s}}}(z) = \frac{2tz^2-1+\sqrt{1+4tz^2(t-1)}}{2(1-z^2)} \tag{31}$$

$$= \sum_{m=1}^{\infty} z^{2m} \sum_{k=1}^{m} \binom{2m}{m+k}\binom{m+k-1}{m}\frac{(-1)^{k-1}t^{m+k}}{m+k-1}. \tag{32}$$

Proof. Put $R_t := R_{\rho_t^{\mathbf{s}}}$. Then

$$R_t(zM_{\rho_t}(z^2)) + 1 = M_{\rho_t}(z^2). \tag{33}$$

To prove (31) we note that R_t satisfies quadratic equation:

$$R_t(z)\big(1+R_t(z)\big) = z^2\big(R_t(z)+t\big)^2. \tag{34}$$

Indeed, it is sufficient to substitute $z \mapsto zM_{\rho_t}(z^2)$ and use (33) and (25).

For (32) we apply the Taylor expansion to (31):

$$\begin{aligned} R_t(z) &= \frac{1}{2}\left[2tz^2 - 1 + \sum_{k=0}^{\infty}\binom{1/2}{k}\left(4tz^2(t-1)\right)^k\right]\sum_{l=0}^{\infty} z^{2l} \\ &= \frac{1}{2}\left[2t^2z^2 + \sum_{k=2}^{\infty}\binom{1/2}{k}\left(4tz^2(t-1)\right)^k\right]\sum_{l=0}^{\infty} z^{2l}. \end{aligned}$$

Now we note that

$$\frac{1}{2}4^k\binom{1/2}{k} = -\frac{(-1)^k}{2k-1}\binom{2k-1}{k-1}, \tag{35}$$

so that for the coefficient r_{2m} at z^{2m} we have

$$r_{2m} = t^2 - \sum_{k=2}^{m} \binom{2k-1}{k-1} \frac{\big(t(1-t)\big)^k}{2k-1}, \tag{36}$$

$m \geq 2$. Now it remains to prove that

$$t^2 - \sum_{k=2}^{m} \binom{2k-1}{k-1} \frac{\big(t(1-t)\big)^k}{2k-1} = \sum_{k=1}^{m} \binom{2m}{m+k} \binom{m+k-1}{m} \frac{(-1)^{k-1} t^{m+k}}{m+k-1}. \tag{37}$$

Denoting the left (resp. right) hand side of (37) by $LHS(m)$ (resp. $RHS(m)$) we have $LHS(1) = RHS(1) = t^2$ and for $m \geq 1$

$$\begin{aligned}
&RHS(m-1) - RHS(m) \\
&\quad = \sum_{k=1}^{m-1} \binom{2m-2}{m+k-1} \binom{m+k-2}{m-1} \frac{(-1)^{k-1} t^{m+k-1}}{m+k-2} \\
&\qquad\qquad - \sum_{k=1}^{m} \binom{2m}{m+k} \binom{m+k-1}{m} \frac{(-1)^{k-1} t^{m+k}}{m+k-1} \\
&\quad = \sum_{k=0}^{m-2} \binom{2m-2}{m+k} \binom{m+k-1}{m-1} \frac{(-1)^{k} t^{m+k}}{m+k-1} \\
&\qquad\qquad + \sum_{k=1}^{m} \binom{2m}{m+k} \binom{m+k-1}{m} \frac{(-1)^{k} t^{m+k}}{m+k-1} \\
&\quad = \sum_{k=0}^{m} \binom{2m-1}{m-1} \binom{m}{k} \frac{(-t)^k}{2m-1} t^m \\
&\quad = \binom{2m-1}{m-1} \frac{t^m (1-t)^m}{2m-1} = LHS(m-1) - LHS(m).
\end{aligned}$$

Now we can conclude by induction. □

One can check that if X, Y are independent random variables with distributions μ and $\frac{1}{2}(\delta_{-1} + \delta_1)$ respectively and with $X \geq 0$ then $\mu^{\mathbf{s}}$ is the distribution of the product $Y\sqrt{X}$. Let $\sqrt{\mu}$ denote the distribution of $\sqrt{X}$, so that

$$\int_{\mathbb{R}} f(x)\, d\sqrt{\mu}(x) := \int_{\mathbb{R}} f\big(\sqrt{x}\big)\, d\mu(x) \tag{38}$$

for every continuous compactly supported function $f : \mathbb{R} \to \mathbb{R}$. Similarly, we define *free symmetrization* of a probability measure μ, with $\operatorname{supp}\mu \subseteq [0, +\infty)$, by putting $\mu^{\mathbf{fs}} := \nu_0 \boxtimes \sqrt{\mu}$, where $\nu_0 := \frac{1}{2}(\delta_{-1} + \delta_1)$. It is easy to check that $S_{\nu_0}(z) = \sqrt{\frac{1+z}{z}}$, so that $S_{\mu^{\mathbf{fs}}}(z) = \sqrt{\frac{1+z}{z}} S_\mu(z)$.

Proposition 4.2. *If μ is a probability measure with support contained in $[0, \infty)$ then*

$$\mu^{\mathbf{fs}} = (\sqrt{\mu} \boxtimes \sqrt{\mu})^{\mathbf{s}}. \tag{39}$$

Moreover, if $\mu^{\boxtimes \frac{1}{2}}$ exists then

$$\mu^{\mathbf{s}} = \nu_0 \boxtimes \mu^{\boxtimes \frac{1}{2}}. \tag{40}$$

Proof. We have

$$1+z=M_{\mu^{\mathbf{fs}}}\left(\frac{z}{1+z}S_{\mu^{\mathbf{fs}}}(z)\right)=M_{\mu^{\mathbf{fs}}}\left(\sqrt{\frac{z}{1+z}}S_{\sqrt{\mu}}(z)\right)$$

and, on the other hand,

$$M_{(\sqrt{\mu}\boxtimes\sqrt{\mu})^{\mathbf{s}}}\left(\sqrt{\frac{z}{1+z}}S_{\sqrt{\mu}}(z)\right)=M_{\sqrt{\mu}\boxtimes\sqrt{\mu}}\left(\frac{z}{1+z}S_{\sqrt{\mu}}(z)^2\right)=1+z,$$

which means that $M_{\mu^{\mathbf{fs}}}=M_{(\sqrt{\mu}\boxtimes\sqrt{\mu})^{\mathbf{s}}}$ and consequently $\mu^{\mathbf{fs}}=(\sqrt{\mu}\boxtimes\sqrt{\mu})^{\mathbf{s}}$.

For the second statement we note that

$$M_\mu\big(z(1+z)^{-1}S_\mu(z)\big)=1+z=M_{\mu^{\mathbf{s}}}\big(z(1+z)^{-1}S_{\mu^{\mathbf{s}}}(z)\big)=M_\mu\big(z^2(1+z)^{-2}S_{\mu^{\mathbf{s}}}(z)^2\big),$$

which implies that

$$S_{\mu^{\mathbf{s}}}(z)=\sqrt{\frac{1+z}{z}}\cdot\sqrt{S_\mu(z)}. \tag{41}$$

□

Example. For $t>0$ define

$$\mu_t:=\max\{1-t,0\}\delta_0+\frac{\sqrt{4t-(\sqrt{x}-1-t)^2}}{4\pi x}\,dx, \tag{42}$$

with the absolutely continuous part supported on $\left[\left|1-\sqrt{t}\right|,1+\sqrt{t}\right]$. Then we have $\pi_t=\sqrt{\mu_t}$ and therefore $\mu_t^{\mathbf{fs}}=(\pi_t\boxtimes\pi_t)^{\mathbf{s}}=\rho_t^{\mathbf{s}}$.

Final remarks. Denote by $\mathcal{M}_{\mathbf{s}}$ (resp. $\mathcal{M}_+$) the class of probability measures on $\mathbb{R}$ which are symmetric (resp. have support in the positive half-line $[0,\infty)$). Then it is easy to see from (28) that the symmetrization $\mathcal{M}_+\ni\mu\mapsto\mu^{\mathbf{s}}\in\mathcal{M}_{\mathbf{s}}$ is a bijection. On the other hand, in view of (39) the free symmetrization is a well defined map $\mathcal{M}_+\to\mathcal{M}_{\mathbf{s}}$ which is one-to-one but not onto. Indeed, if $\nu\in\mathcal{M}_{\mathbf{s}}$ is free symmetrization of some measure $\mu\in\mathcal{M}_+$ then ν is of the form $\eta^{\mathbf{s}}$ for such $\eta\in\mathcal{M}_+$ for which there exists the multiplicative free power $\eta^{\frac{1}{2}\boxtimes}$.

Let us finally mention that it is also possible to investigate other free versions of classical symmetrization, namely $\mathcal{M}\ni\mu\mapsto\mu\boxplus\widetilde{\mu}\in\mathcal{M}_{\mathbf{s}}$, where $\widetilde{\mu}:=\mathrm{D}_{-1}(\mu)$ denotes the reflection of μ, or $\mathcal{M}_+\ni\mu\mapsto\frac{1}{2}(\delta_{-1}+\delta_1)\boxtimes\mu=(\mu\boxtimes\mu)^{\mathbf{s}}\in\mathcal{M}_{\mathbf{s}}$, where the last equality can be proved in the same way as Proposition 4.2.

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References

[1] T. Banica, S. T. Belinschi, M. Capitaine, B. Collins, *Free Bessel laws,* preprint.
[2] H. Bercovici, D. V. Voiculescu, *Free convolutions of measures with unbounded support,* Indiana Univ. Math. J. **42** (1993). 733–773.
[3] R. L. Graham, D. E. Knuth, O. Patashnik, *Concrete mathematics,* Addison-Wesley 1994.
[4] W. Młotkowski, *Combinatorial relation between free cumulants and Jacobi parameters*, to appear in Inf. Dim. Anal. Quantum Probab. Rel. Topics.
[5] A. Nica, R. Speicher, *Lectures on the Combinatorics of Free Probability*, Cambridge University Press, 2006.
[6] N. R. Rao, R. Speicher, *Multiplication of free random variables and the S-transform for vanishing mean,* Elect. Comm. in Probab. 12 (2007), 248–258.
[7] D. V. Voiculescu, K. J. Dykema, A. Nica, *Free random variables*, CRM 1992.

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