MATHEMATICAL ANALYSIS

PROBLEMS LIST 2

9.10.08

(1) Find the natural powers f the number i, that is write out the complex numbers of the form i^n for all natural n.

(2) For given complex numbers $z, w \in \mathbf{C}$ compute: $\Re(z+w), \Im(z+w), \Re(zw), \Im(zw)$, in terms of $\Re(z), \Im(z), \Re(w)$ and $\Im(w)$.

(3) Compute $\Re(1/z)$ in terms of $\Re(z)$ and $\Im(z)$.

(4) Prove the following properties of the complex conjugation:

(i) $\overline{(\overline{z})} = z$;

(ii) $\overline{z+w} = \overline{z} + \overline{w}$;

(iii) $\overline{(z\,w)} = \overline{z}\,\overline{w}$;

(iv) $\Re(z) = (z+\overline{z})/2,\ \Im(z) = (z-\overline{z})/2i$.

(5) Find the moduli of the complex numbers $z = -2 - 3\,i$ and $z = 1 - i$.

(6) Prove that arbitrary numbers $z, w \in \mathbf{C}$ have the properties:

(i) $|z| \geq 0$ and $|z| = 0$ if and only if $z = 0$;

(ii) $|z\,w| = |z|\,|w|$;

(iii) $|z - w| \geq ||z| - |w||$.

(7) Describe geometrically (sketch on the plane) the set $\{z \in \mathbf{C} : |z-1| = 1\}$.

(8) Sketch on the plane the set of numbers $z \in \mathbf{C}$ satisfying the inequality $|z + 4 - 2\,i| \leq 3$.

(9) Find the trigonometric form of the following complex numbers: $-6 + 6\,i$, $2\,i$, $1 + i$, $\sqrt{6} + i$.

(10) Find the trigonometric form of the complex numbers of modulus 1.

(11) Prove that for $z = r\,(\cos\varphi + i\,\sin\varphi)$ and $z = s\,(\cos\psi + i\,\sin\psi)$ we have

$$z \cdot w = r \cdot s\,(\cos(\varphi+\psi) + i\,\sin(\varphi+\psi)).$$

(12) Prove that for $z \in \mathbf{C}$, $z \neq 0$ there exists $w \in \mathbf{C}$, its inverse, that is such that $z \cdot w = 1$.

(13) Prove that for any $n \in \mathbf{N}$ and any $z \in \mathbf{C}$ with the trigonometric form $z = r\,(\cos\varphi + i\,\sin\varphi)$ we have the formula

$$z^n = r^n\,(\cos(n\,\varphi) + i\,\sin(n\,\varphi)).$$

(14) Find all different roots of order 3 and 4 of numbers 1, -1, $1+i$ and $2-2\,i$ (in the trigonometric and in the normal forms). Show their location on the plane.

(15) Let $\epsilon_1, \ldots, \epsilon_n$ be different roots of order n of the number 1. What is the sum $\epsilon_1 + \cdots + \epsilon_n$? And what is the sum of all n different rootos of order n of the number i?

(16) Prove the equality $|z+w|^2 + |z-w|^2 = 2|z|^2 + 2|w|^2$.

(17) Let $a, b, c \in \mathbf{C}$ be arbitrary, and let $d \in \mathbf{C}$ be one of the roots $\sqrt{b^2 - 4\,a\,c}$. Prove that the roots of the equation $a\,z^2 + b\,z + c = 0$ have the form

$$z = \frac{-b \pm d}{2\,a}.$$