MATHEMATICAL ANALYSIS

PROBLEMS LIST 4

23.10.08

(1) Prove the inequality: $2^k < (k+1)!$ for each natural $k \geq 2$.

(2) Prove, that for any natural numbers $k \leq n$ the following inequality holds

$$\binom{n}{k}\frac{1}{n^k} = \frac{1}{k!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\dots\left(1-\frac{k-1}{n}\right).$$

(3) Prove the Bernoulli's inequality: for $x > -1$ and arbitrary natural $n > 1$ we have $(1+x)^n > 1+nx$. Also, show that for $x > 0$ and $n \in \mathbf{N}$, $n > 1$ we have the following inequality

$$(1+x)^n > 1 + \frac{n(n-1)}{2}x^2.$$

(4) Prove, that for any $n \in \mathbf{N}$ the following inequalities hold

(a) $$\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} = 2^n,$$

(b) $$\sum_{\substack{k=1 \\ \text{k-odd}}}^{n} \binom{n}{k} = \sum_{\substack{k=0 \\ k\text{-even}}}^{n} \binom{n}{k}.$$

(5) Show, that for every natural number $n \geq 2$ we have the inequality $\binom{2n}{n} < 4^n$.

(6) Prove, that for any number $a \in \mathbf{R}$ or $a \in \mathbf{C}$ satisfying the condition $|a| < 1$ we have $\lim_{n\to\infty} a^n = 0$.

(7) Find the limits:

(a) $\lim_{n\to\infty} \left(1+\frac{1}{n^2}\right)^n$, (b) $\lim_{n\to\infty} \left(1-\frac{1}{n}\right)^n$.

(8) Find the limits of sequences:

(a) $a_n = \sqrt[n]{2^n+3^n}$, (b) $a_n = \sqrt[n]{2^n+3^n+5^n}$.

(9) For which real α exists the limit

$$\lim_{n\to\infty} \sqrt[3]{n+n^\alpha} - \sqrt[3]{n}.$$

Find the limits for those α for which they exist.

(10) Compute the limits:

(a) $\displaystyle\lim_{n\to\infty} \frac{1+2+3+\cdots+n}{n^2}$, (b) $\displaystyle\lim_{n\to\infty} \frac{1^2+2^2+3^2+\cdots+n^2}{n^3}$.

(11) Compute the limits of sequences:

(a) $a_n = \dfrac{\sin^2 n}{n}$, (b) $a_n = \sqrt[n]{\ln n}$,

(c) $a_n = \dfrac{1}{n^2}\ln\big(1+\dfrac{(-1)^n}{n}\big)$.

(12) Prove, that if

$$\lim_{n\to\infty} a_n = a$$

then the sequence of absolute values $\{|a_n|\}$ is also convergent, and

$$\lim_{n\to\infty} |a_n| = |a|.$$

Show, that the above theorem does not hold the other way around, that is find a sequence $\{a_n\}$ which is not convergent, despite the fact that $\{|a_n|\}$ does converge. On the other hand, prove, that if

$$\lim_{n\to\infty} |a_n| = 0$$

then $\{a_n\}$ also converges to 0.

(13) Prove, that if sequences $\{a_n\}$ and $\{b_n\}$ satisfy $a_n \le b_n$ and are convergent, then

$$\lim_{n\to\infty} a_n \le \lim_{n\to\infty} b_n.$$

(14) The sequence a_n is given in the following way: $a_1 = 0$, $a_2 = 1$, and

$$a_{n+2} = \frac{a_n + a_{n+1}}{2}, \quad \text{for } n = 1, 2, \ldots.$$

Show that

$$\lim_{n\to\infty} a_n = \frac{2}{3}.$$

(15) Show that if $\lim_{n\to\infty} a_n = 0$ and the sequence $\{b_n\}$ is bounded, then

$$\lim_{n\to\infty} (a_n \cdot b_n) = 0.$$

(16) Show that if $a_n > 0$, and $\lim_{n\to\infty} a_n = 0$ then

$$\lim_{n\to\infty} \frac{1}{a_n} = \infty$$

(improper limit).

(17) Given is a sequence $\{b_n\}$, about which it is known, that

$$\forall\ \epsilon > 0\ \forall\ n \geq 10/\epsilon \quad |b_n + 2| < \epsilon.$$

Exhibit M such that

$$\forall\ n \in \mathbf{N} \quad |b_n| < M,$$

N_1 such that

$$\forall\ n \geq N_1 \quad b_n < 0,$$

N_2 such that

$$\forall\ n \geq N_2 \quad b_n > -3,$$

and N_3 such that

$$\forall\ n \geq N_3 \quad |b_n - 2| > \frac{1}{10}.$$

(18) Let $a_n = \dfrac{\sqrt{n^2+n}}{n}$ and $\epsilon = \dfrac{1}{100}$. Find $n_0 \in \mathbf{N}$ such, that for $n \geq n_0$ we have $|a_n - 1| < \epsilon$.