MATHEMATICAL ANALYSIS

PROBLEMS LIST 12

18.12.08

(1) Give the formula for $C_n = \sum_{i=1}^{n} \frac{b-a}{n} f\big(a + i\,\frac{b-a}{n}\big)$, and the compute $\lim_{n\to\infty} C_n$

(a) $f(x) = 1,\ a = 5,\ b = 8;$ (b) $f(x) = x,\ a = 0,\ b = 1,$

(c) $f(x) = x,\ a = 1,\ b = 5;$ (d) $f(x) = x^2,\ a = 0,\ b = 5,$

(e) $f(x) = x^3,\ a = 0,\ b = 1;$ (f) $f(x) = 2x+5,\ a = -3,\ b = 4,$

(g) $f(x) = x^2 + 1,\ a = -1,\ b = 2,$

(h) $f(x) = x^3+x,\ a = 0,\ b = 4;$ (i) $f(x) = e^x,\ a = 0,\ b = 1.$

(2) Compute the following definite integrals by constructing a sequence of partitions of the interval, corresponding Riemann sums, and their limit:

(a) $\int_2^4 x^{10}\,dx,\ (t_i = 2\cdot 2^{i/n}),$ (b) $\int_1^e \frac{\log x}{x}\,dx,\ (t_i = e^{i/n}),$

(c) $\int_0^{20} x\,dx,$ (d) $\int_1^{10} e^{2x}\,dx,$

(e) $\int_0^1 \sqrt[3]{x}\,dx,\ (t_i = \frac{i^3}{n^3}),$ (f) $\int_{-1}^1 |x|\,dx,$

(g) $\int_1^2 \frac{dx}{x}\,dx,\ (t_i = 2^{i/n}),$ (h) $\int_0^4 \sqrt{x}\,dx,\ (t_i = \frac{4i^2}{n^2}).$

(3) Compute the definite integrals:

(a) $\int_{-\pi}^{\pi} \sin x^{2007}\,dx,$ (b) $\int_0^2 \arctan([x])\,dx,$

(c) $\int_0^2 [\cos(x^2)]\,dx,$ (d) $\int_0^1 \sqrt{1+x}\,dx,$

(e) $\int_{-2}^{-1} \frac{1}{(11+5x)^3}\,dx,$ (f) $\int_{-13}^{2} \frac{1}{\sqrt[5]{(3-x)^4}}\,dx,$

(g) $\int_0^1 \frac{x}{(x^2+1)^2}\,dx,$ (h) $\int_0^3 \operatorname{sgn}(x^3 - x)\,dx,$

(i) $\int_0^1 x\,e^{-x}\,dx,$ (j) $\int_0^{\pi/2} x\cos x\,dx,$

(k) $\displaystyle\int_0^{e-1} \log(x+1)\,dx$, (l) $\displaystyle\int_0^{\pi} x^3 \sin x\,dx$,

(m) $\displaystyle\int_4^9 \frac{\sqrt{x}}{\sqrt{x}-1}\,dx$, (n) $\displaystyle\int_1^{e^3} \frac{1}{x\sqrt{1+\log x}}\,dx$,

(o) $\displaystyle\int_1^2 \frac{1}{x+x^3}\,dx$, (p) $\displaystyle\int_0^2 \frac{1}{\sqrt{x+1}+\sqrt{(x+1)^3}}\,dx$,

(q) $\displaystyle\int_0^5 |x^2-5x+6|\,dx$, (r) $\displaystyle\int_0^1 \frac{e^x}{e^x-e^{-x}}\,dx$,

(s) $\displaystyle\int_1^2 x\,\log_2 x\,dx$, (t) $\displaystyle\int_0^{\sqrt{7}} \frac{x^3}{\sqrt[3]{1+x^2}}\,dx$,

(u) $\displaystyle\int_0^{6\pi} |\sin x|\,dx$, (w) $\displaystyle\int_0^{\pi/2} \cos x \sin^{11} x\,dx$,

(x) $\displaystyle\int_0^{\log 5} \frac{e^x\sqrt{e^x-1}}{e^x+5}\,dx$, (y) $\displaystyle\int_{-\pi}^{\pi} x^{2007}\cos x\,dx$,

(z) $\displaystyle\int_0^{2\pi} (x-\pi)^{2007}\cos x\,dx$.

(4) Prove the following estimates:

(a) $\displaystyle\int_0^{\pi/2} \frac{\sin x}{x}\,dx < 2$, (b) $\displaystyle\frac{1}{5} < \int_1^2 \frac{1}{x^2+1}\,dx < \frac{1}{2}$,

(c) $\displaystyle\frac{1}{11} < \int_9^{10} \frac{1}{x+\sin x}\,dx < \frac{1}{8}$, (d) $\displaystyle\int_{-1}^2 \frac{|x|}{x^2+1}\,dx < \frac{3}{2}$,

(e) $\displaystyle\int_0^1 x(1-x^{99+x})\,dx < \frac{1}{2}$, (f) $\displaystyle 2\sqrt{2} < \int_2^4 x^{1/x}\,dx$,

(g) $\displaystyle 5 < \int_1^3 x^x\,dx < 31$, (h) $\displaystyle\int_1^2 \frac{1}{x}\,dx < \frac{3}{4}$.

(5) Compute the following limits:

(a) $\lim\limits_{n\to\infty} \frac{1}{n}+\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots+\frac{1}{2n}$,

(b) $\lim\limits_{n\to\infty} \frac{1^{20}+2^{20}+3^{20}+\cdots+n^{20}}{n^{21}}$,

(c) $\lim\limits_{n\to\infty} \left(\frac{1}{n^2}+\frac{1}{(n+1)^2}+\frac{1}{(n+1)^2}+\frac{1}{(n+3)^2}+\cdots+\frac{1}{(2n)^2}\right)\cdot n$,

(d) $\lim\limits_{n\to\infty} \frac{1}{\sqrt{n}\sqrt{2n}}+\frac{1}{\sqrt{n}\sqrt{2n+1}}+\frac{1}{\sqrt{n}\sqrt{2n+2}}+\frac{1}{\sqrt{n}\sqrt{2n+3}}+\cdots+\frac{1}{\sqrt{n}\sqrt{3n}}$,

(e) $\lim\limits_{n\to\infty} \left(\sin\frac{1}{n}+\sin\frac{2}{n}+\sin\frac{3}{n}+\cdots+\sin\frac{n}{n}\right)\cdot\frac{1}{n}$,

(f) $\lim\limits_{n\to\infty} \left(\sqrt{4n}+\sqrt{4n+1}+\sqrt{4n+2}+\cdots+\sqrt{5n}\right)\cdot\frac{1}{n\sqrt{n}}$,

(g) $\lim\limits_{n\to\infty} \left(\frac{1}{\sqrt[3]{n}}+\frac{1}{\sqrt[3]{n+1}}+\frac{1}{\sqrt[3]{n+2}}+\cdots+\frac{1}{\sqrt[3]{8n}}\right)\cdot\frac{1}{\sqrt[3]{n^2}}$,

(h) $\lim\limits_{n\to\infty} \frac{\sqrt[6]{n}\cdot(\sqrt[3]{n}+\sqrt[3]{n+1}+\sqrt[3]{n+2}+\cdots+\sqrt[3]{2n})}{\sqrt{n}+\sqrt{n+1}+\sqrt{n+2}+\cdots+\sqrt{2n}}$,

(i) $\lim\limits_{n\to\infty} \frac{n}{n^2}+\frac{n}{n^2+1}+\frac{n}{n^2+4}+\frac{n}{n^2+9}+\frac{n}{n^2+16}+\cdots+\frac{n}{n^2+n^2}$,

(j) $\lim\limits_{n\to\infty} \frac{4}{5n}+\frac{4}{5n+3}+\frac{4}{5n+6}+\frac{4}{5n+9}+\cdots+\frac{4}{26n}$,

(k) $\lim\limits_{n\to\infty} \frac{1}{7n} + \frac{1}{7n+2} + \frac{1}{7n+4} + \frac{1}{7n+6} + \cdots + \frac{1}{9n}$,

(l) $\lim\limits_{n\to\infty} \frac{1}{7n^2} + \frac{1}{7n^2+1} + \frac{1}{7n^2+2} + \frac{1}{7n^2+3} + \cdots + \frac{1}{8n^2}$,

(m) $\lim\limits_{n\to\infty} \frac{1}{n} \left(e^{\sqrt{\frac{1}{n}}} + e^{\sqrt{\frac{2}{n}}} + e^{\sqrt{\frac{3}{n}}} + \cdots + e^{\sqrt{\frac{n}{n}}} \right)$,

(n) $\lim\limits_{n\to\infty} \left(\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+3}} + \frac{1}{\sqrt{n+6}} + \frac{1}{\sqrt{n+9}} + \cdots + \frac{1}{\sqrt{7n}} \right) \frac{1}{\sqrt{n}}$,

(o) $\lim\limits_{n\to\infty} \frac{n^2+0}{(3n)^3} + \frac{n^2+1}{(3n+1)^3} + \frac{n^2+2}{(3n+2)^3} + \frac{n^2+3}{(3n+3)^3} + \cdots + \frac{n^2+n}{(4n)^3}$,

(p) $\lim\limits_{n\to\infty} \frac{n}{2n^2} + \frac{n}{2(n+1)^2} + \frac{n}{2(n+2)^2} + \frac{n}{2(n+3)^2} + \cdots + \frac{n}{50n^2}$,

(r) $\lim\limits_{n\to\infty} \frac{n}{2n^2} + \frac{n}{n^2+(n+1)^2} + \frac{n}{n^2+(n+2)^2} + \frac{n}{n^2+(n+3)^2} + \cdots + \frac{n}{50n^2}$.