Exercise 1. For what values of parameters a and b is the following function continuous?

$$f(x) = \begin{cases} x & : \quad x < 1 \\ x^2 + ax + b & : \quad 1 \leq x < 2 \\ x + 3 & : \quad 2 \leq x \end{cases} .$$

Solution: Each of functions involved is continuous everywhere, so the only issue is the points in which different functions are spliced, that is 1 and 2. We compute one-sided limits

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x = 1,$$
$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} (x^2 + ax + b) = 1 + a + b,$$
$$\lim_{x \to 2^-} f(x) = \lim_{x \to 2^-} (x^2 + ax + b) = 4 + 2a + b,$$
$$\lim_{x \to 2^+} f(x) = \lim_{x \to 2^+} (x + 3) = 5$$

We thus have two equations with two unknowns:

$$1 = 1 + a + b \Rightarrow a + b = 0, \quad \text{and} \quad 4 + 2a + b = 5 \Rightarrow 2a + b = 1,$$

and so $a = 1$ and $b = -1$.

Exercise 2. Is the following series convergent?

$$\sum_{n=1}^{\infty} \frac{3^n}{2^{2^n}}.$$

Solution: We can apply the d'Alembert's criterion:

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{3^{n+1}}{2^{2^{n+1}}} \cdot \frac{2^{2^n}}{3^n} = \frac{3}{2^{(2^{n+1}-2^n)}} = \frac{3}{2^{(2\cdot 2^n-2^n)}} = \frac{3}{2^{2^n}} \to 0.$$

Thus the series is convergent.

Exercise 3. Find the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$$

Solution: We compute the inverse of the radius:

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \frac{(n+1)}{(n+1)^{n+1}} \cdot \frac{n^n}{1} = \frac{1}{(n+1)^n} \cdot \frac{n^n}{1} =$$
$$= \frac{n^n}{(n+1)^n} = \left(\frac{n}{n+1}\right)^n = \frac{1}{(\frac{n+1}{n})^n} = \frac{1}{(1+\frac{1}{n})^n} \to \frac{1}{e}.$$

The radius of convergence is the inverse of the above limit, so $R = e$.

Exercise 4. Analyse the following function. Determine the extrema, intervals of monotonicity, find the asymptotes, sketch the graph.

$$f(x) = \frac{x^2 - 3x + 2}{x^2 + 3x + 2}.$$

Solution: The function's domain is all real numbers except for -2 and -2, where the denominator is zero. Most important is to compute the derivative.

$$\begin{aligned} f'(x) &= \frac{(2x-3)(x^2+3x+2) - (2x+3)(x^2-3x+2)}{(x^2+3x+2)^2} \\ &= \frac{2x^3+6x^2+4x-3x^2-9x-6-2x^3+6x^2-4x-3x^2+9x-6}{(x^2+3x+2)^2} \\ &= \frac{6x^2-12}{(x^2+3x+2)^2} \end{aligned}$$

The sign of the derivative is determined by the sign of the numerator, so $f'(x) > 0 \Leftrightarrow x^2 > 2$ and so $f(x)$ increases on $(-\infty, -2)$, $(-2, -\sqrt{2})$ and $(\sqrt{2}, \infty)$, and decreases on $(-\sqrt{2}, -1)$ and $(-1, \sqrt{2})$. $f'(x) = 0 \Leftrightarrow x^2 = 2$, so $x = \pm\sqrt{2}$. These are the only possibilities for local extrema. Clearly, at $-\sqrt{2}$ we have a maximum, since $f(x) \nearrow$ to the left, and $\searrow$ to the right of $-\sqrt{2}$, while at $\sqrt{2}$ we have a minimum, since $f(x) \searrow$ to the left, and $\nearrow$ to the right of $\sqrt{2}$. The function "explodes" at -2 and -1, since the denominator is zero there, while the numerator is non-zero. Thus the function has two vertical asymptotes $x = -2$ and $x = -1$. Moreover,

$$\lim_{x \to \pm\infty} f(x) = \lim_{x \to \pm\infty} \frac{1 - \frac{3}{x} + \frac{2}{x^2}}{1 + \frac{3}{x} + \frac{2}{x^2}} = 1,$$

so the function has horizontal asymptotes $y = 1$ at both $+\infty$ and $-\infty$

Exercise 5. Compute the derivative of order 3 of the function

$$f(x) = \log(x^2).$$

Solution: Couldn't be any simpler:

$$\begin{aligned} f'(x) &= \frac{1}{x^2} \cdot 2x = \frac{2}{x}, \\ f''(x) &= \frac{-2}{x^2}, \\ f'''(x) &= \frac{-2 \cdot -2}{x^3} = \frac{4}{x^3}. \end{aligned}$$

Exercise 6. Compute the limit

$$\lim_{x\to 0} \frac{2\cos x - x^2 - 2}{x\sin x - x^2}.$$

Solution: It is an indeterminate expression of the type $\frac{0}{0}$ at zero, so we use de l'Hôspital.

$$\lim_{x\to 0} \frac{2\cos x - x^2 - 2}{x\sin x - x^2} = \lim_{x\to 0} \frac{-2\sin x - 2x}{\sin x + x\cos x - 2x}.$$

This again is an indeterminate expression of the type $\frac{0}{0}$, but we can factor out x from both numerator and the denominator, and get

$$\lim_{x\to 0} \frac{-2\frac{\sin x}{x} - 2}{\frac{\sin x}{x} + \cos x - 2}.$$

This is no longer indeterminate: the numerator has finite limit -4, while the denominator goes to zero, through negatives. Thus this last limit (and hence the original limit) is improper $+\infty$.

Exercise 7. Find the maximal and minimal values of the function

$$f(x) = |x^2 - 1| + 3x$$

on the interval $[-2, 2]$.

Solution: The maximal and minimal values are attained at the end-points, points of non-differentiability or at points where the derivative is zero. The function might be non-differentiable at ± 1, since the absolute value is non-differentiable at zero. So, we need to compare the values of the function at points ± 2, ± 1, and at any possible zeros of the derivative. We thus look for these zeros. We consider two cases:

$x^2 > 1 \Rightarrow f(x) = x^2 - 1 + 3x \Rightarrow f'(x) = 2x + 3 \Rightarrow f'(x) = 0 \Leftrightarrow x = -\frac{3}{2}$.

This point indeed falls into the considered case $x^2 > 1$, so there we have a zero of the derivative. Now consider the other case:

$x^2 < 1 \Rightarrow f(x) = -x^2 + 1 + 3x \Rightarrow f'(x) = -2x + 3 \Rightarrow f'(x) = 0 \Leftrightarrow x = \frac{3}{2}$.

This point is, however, outside the considered range $x^2 < 1$, so we do not have any new zeros of the derivative. So, the maximal and minimal values have to be chosen from:

$$\begin{aligned} f(-2) &= 3 - 6 = -3, \\ f(-\tfrac{3}{2}) &= \tfrac{5}{4} - \tfrac{9}{2} = -\tfrac{13}{4}, \\ f(-1) &= -3, \\ f(1) &= 3, \\ f(2) &= 3 + 6 = 9. \end{aligned}$$

Thus the minimal value is $-\frac{13}{4}$ (because it is less than -3), and the maximal value is 9.