

CALCULUS
PROBLEMS LIST 2

7.10.09

- (1) Find the natural powers of the number $\mathbf{i}$, that is write out the complex numbers of the form $\mathbf{i}^n$ for all natural n .
- (2) For given complex numbers $z = a + b\mathbf{i}$ oraz $w = c + d\mathbf{i}$ express $\Re(z + w)$, $\Im(z + w)$, $\Re(zw)$, $\Im(zw)$, in terms of a, b, c, d .
- (3) Prove the following properties of the complex conjugation:
 - (a) $\overline{\overline{z}} = z$,
 - (b) $\overline{z + w} = \overline{z} + \overline{w}$,
 - (c) $\overline{zw} = \overline{z}\overline{w}$,
 - (d) $\Re(z) = (z + \overline{z})/2$, $\Im(z) = (z - \overline{z})/2\mathbf{i}$.
- (4) Find the moduli of the complex numbers $z = -2 - 3\mathbf{i}$ and $z = 1 - \mathbf{i}$.
- (5) Prove that arbitrary numbers $z, w \in \mathbf{C}$ have the properties:
 - (a) $|z| \geq 0$ and $|z| = 0$ if and only if $z = 0$,
 - (b) $|zw| = |z||w|$,
 - (c) $|z - w| \geq ||z| - |w||$.
- (6) Describe geometrically (sketch on the plane) the set $\{z \in \mathbf{C} : |z - 1 - \mathbf{i}| = 1\}$.
- (7) Sketch on the plane the sets of numbers $z \in \mathbf{C}$ satisfying given inequalities:
 - (a) $|z| < 2$,
 - (b) $|z + 3\mathbf{i}| < 1$,
 - (c) $|z + 4 - 2\mathbf{i}| \leq 3$.
- (8) Find the trigonometric form of the following complex numbers:
 - (a) $-6 + 6\mathbf{i}$,
 - (b) $2\mathbf{i}$,
 - (c) $1 + \mathbf{i}$,
 - (d) $\sqrt{6} + \mathbf{i}$.
- (9) Compute:
 - (a) $\frac{1 + \mathbf{i}}{1 - \mathbf{i}}$,
 - (b) $\frac{2\mathbf{i}}{1 + \mathbf{i}}$,
 - (c) $\frac{4 - 3\mathbf{i}}{4 + 3\mathbf{i}}$.
- (10) Prove that for $z = r(\cos \varphi + \mathbf{i} \sin \varphi)$ and $z = s(\cos \psi + \mathbf{i} \sin \psi)$ we have

$$z \cdot w = r \cdot s (\cos(\varphi + \psi) + \mathbf{i} \sin(\varphi + \psi)).$$

Conclude, that for any number $k \in \mathbf{Z}$ the formula holds:

$$z^k = r^k (\cos(k\varphi) + \mathbf{i} \sin(k\varphi)).$$

- (11) Prove that for $z \in \mathbf{C}$, $z \neq 0$ there exists $w \in \mathbf{C}$, its inverse, that is such that $z \cdot w = 1$.
Hint: Write and then solve an appropriate system of equations.
- (12) Determine all values of the roots:
 - (a) $\sqrt[4]{1}$,
 - (b) $\sqrt[3]{-1}$,
 - (c) $\sqrt[4]{1 + \mathbf{i}}$,
 - (d) $\sqrt[3]{2 - 2\mathbf{i}}$,
 - (e) $\sqrt[6]{-27}$,
 - (f) $\sqrt{3 + 4\mathbf{i}}$,
 - (g) $\sqrt[3]{1}$,
 - (h) $\sqrt[3]{\mathbf{i}}$.Show their positions on the plane.
- (13) Prove that $|z + w|^2 + |z - w|^2 = 2|z|^2 + 2|w|^2$.
- (14) Let $a, b, c \in \mathbf{C}$ be arbitrary, and let $d \in \mathbf{C}$ be one of the roots $\sqrt{b^2 - 4ac}$. Prove that the roots of the equation $az^2 + bz + c = 0$ have the form

$$z = \frac{-b \pm d}{2a}.$$