

MATHEMATICAL ANALYSIS

PROBLEMS LIST 12

4.01.10

(1) Find the formula for $C_n = \sum_{i=1}^n \frac{b-a}{n} f(a + i \frac{b-a}{n})$, and the compute $\lim_{n \rightarrow \infty} C_n$:

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| (a) $f(x) = 1, a = 5, b = 8,$ | (b) $f(x) = x, a = 0, b = 1,$ |
| (c) $f(x) = x, a = 1, b = 5,$ | (d) $f(x) = x^2, a = 0, b = 5,$ |
| (e) $f(x) = x^3, a = 0, b = 1,$ | (f) $f(x) = 2x + 5, a = -3, b = 4,$ |
| (g) $f(x) = x^2 + 1, a = -1, b = 2,$ | (h) $f(x) = x^3 + x, a = 0, b = 4,$ |
| (i) $f(x) = e^x, a = 0, b = 1.$ | |

(2) Compute the following definite integrals by constructing a sequence of partitions of the interval, corresponding Riemann sums, and their limits:

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| (a) $\int_2^4 x^{10} dx, (x_i = 2 \cdot 2^{i/n}),$ | (b) $\int_1^e \frac{\log(x)}{x} dx, (x_i = e^{i/n}),$ |
| (c) $\int_0^{20} x dx,$ | (d) $\int_1^{10} e^{2x} dx,$ |
| (e) $\int_0^1 \sqrt[3]{x} dx, (x_i = \frac{i^3}{n^3}),$ | (f) $\int_{-1}^1 x dx,$ |
| (g) $\int_1^2 \frac{dx}{x}, (x_i = 2^{i/n}),$ | (h) $\int_0^4 \sqrt{x} dx, (x_i = \frac{4i^2}{n^2}).$ |

(3) Compute the definite integrals:

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| (a) $\int_{-\pi}^{\pi} \sin(x^{2007}) dx,$ | (b) $\int_0^2 \arctan([x]) dx,$ |
| (c) $\int_0^2 [\cos(x^2)] dx,$ | (d) $\int_0^1 \sqrt{1+x} dx,$ |
| (e) $\int_{-2}^{-1} \frac{1}{(11+5x)^3} dx,$ | (f) $\int_{-13}^2 \frac{1}{\sqrt[5]{(3-x)^4}} dx,$ |
| (g) $\int_0^1 \frac{x}{(x^2+1)^2} dx,$ | (h) $\int_0^3 \operatorname{sgn}(x^3 - x) dx,$ |
| (i) $\int_0^1 x e^{-x} dx,$ | (j) $\int_0^{\pi/2} x \cos(x) dx,$ |
| (k) $\int_0^{e-1} \log(x+1) dx,$ | (l) $\int_0^{\pi} x^3 \sin(x) dx,$ |
| (m) $\int_4^9 \frac{\sqrt{x}}{\sqrt{x}-1} dx,$ | (n) $\int_1^{e^3} \frac{1}{x \sqrt{1+\log(x)}} dx,$ |
| (o) $\int_1^2 \frac{1}{x+x^3} dx,$ | (p) $\int_0^2 \frac{1}{\sqrt{x+1} + \sqrt{(x+1)^3}} dx,$ |
| (q) $\int_0^5 x^2 - 5x + 6 dx,$ | (r) $\int_0^1 \frac{e^x}{e^x - e^{-x}} dx,$ |

- (s) $\int_1^2 x \log_2(x) dx,$ (t) $\int_0^{\sqrt{7}} \frac{x^3}{\sqrt[3]{1+x^2}} dx,$
(u) $\int_0^{6\pi} |\sin(x)| dx,$ (w) $\int_0^{\pi/2} \cos(x) \sin^{11}(x) dx,$
(x) $\int_0^{\log 5} \frac{e^x \sqrt{e^x - 1}}{e^x + 5} dx,$ (y) $\int_{-\pi}^{\pi} x^{2007} \cos(x) dx,$
(z) $\int_0^{2\pi} (x - \pi)^{2007} \cos(x) dx.$

(4) Prove the following estimates:

- (a) $\int_0^{\pi/2} \frac{\sin(x)}{x} dx < 2,$ (b) $\frac{1}{5} < \int_1^2 \frac{1}{x^2 + 1} dx < \frac{1}{2},$
(c) $\frac{1}{11} < \int_9^{10} \frac{1}{x + \sin(x)} dx < \frac{1}{8},$ (d) $\int_{-1}^2 \frac{|x|}{x^2 + 1} dx < \frac{3}{2},$
(e) $\int_0^1 x(1 - x^{99+x}) dx < \frac{1}{2},$ (f) $2\sqrt{2} < \int_2^4 x^{1/x} dx,$
(g) $5 < \int_1^3 x^x dx < 31,$ (h) $\int_1^2 \frac{1}{x} dx < \frac{3}{4}.$

(5) Compute the following limits:

- (a) $\lim_{n \rightarrow \infty} \left(\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{2n} \right),$
(b) $\lim_{n \rightarrow \infty} \left(\frac{1^{20} + 2^{20} + 3^{20} + \cdots + n^{20}}{n^{21}} \right),$
(c) $\lim_{n \rightarrow \infty} \left(\frac{1}{n^2} + \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \frac{1}{(n+3)^2} + \cdots + \frac{1}{(2n)^2} \right) \cdot n,$
(d) $\lim_{n \rightarrow \infty} \left(\frac{1}{\sqrt{n}\sqrt{2n}} + \frac{1}{\sqrt{n}\sqrt{2n+1}} + \frac{1}{\sqrt{n}\sqrt{2n+2}} + \frac{1}{\sqrt{n}\sqrt{2n+3}} + \cdots + \frac{1}{\sqrt{n}\sqrt{3n}} \right),$
(e) $\lim_{n \rightarrow \infty} \left(\sin\left(\frac{1}{n}\right) + \sin\left(\frac{2}{n}\right) + \sin\left(\frac{3}{n}\right) + \cdots + \sin\left(\frac{n}{n}\right) \right) \cdot \frac{1}{n},$
(f) $\lim_{n \rightarrow \infty} \left(\sqrt{4n} + \sqrt{4n+1} + \sqrt{4n+2} + \cdots + \sqrt{5n} \right) \cdot \frac{1}{n\sqrt{n}},$
(g) $\lim_{n \rightarrow \infty} \left(\frac{1}{\sqrt[3]{n}} + \frac{1}{\sqrt[3]{n+1}} + \frac{1}{\sqrt[3]{n+2}} + \cdots + \frac{1}{\sqrt[3]{8n}} \right) \cdot \frac{1}{\sqrt[3]{n^2}},$
(h) $\lim_{n \rightarrow \infty} \left(\frac{\sqrt[6]{n} \cdot (\sqrt[3]{n} + \sqrt[3]{n+1} + \sqrt[3]{n+2} + \cdots + \sqrt[3]{2n})}{\sqrt{n} + \sqrt{n+1} + \sqrt{n+2} + \cdots + \sqrt{2n}} \right),$
(i) $\lim_{n \rightarrow \infty} \left(\frac{n}{n^2} + \frac{n}{n^2+1} + \frac{n}{n^2+4} + \frac{n}{n^2+9} + \frac{n}{n^2+16} + \cdots + \frac{n}{n^2+n^2} \right),$
(j) $\lim_{n \rightarrow \infty} \left(\frac{4}{5n} + \frac{4}{5n+3} + \frac{4}{5n+6} + \frac{4}{5n+9} + \cdots + \frac{4}{26n} \right),$
(k) $\lim_{n \rightarrow \infty} \left(\frac{1}{7n} + \frac{1}{7n+2} + \frac{1}{7n+4} + \frac{1}{7n+6} + \cdots + \frac{1}{9n} \right),$
(l) $\lim_{n \rightarrow \infty} \left(\frac{1}{7n^2} + \frac{1}{7n^2+1} + \frac{1}{7n^2+2} + \frac{1}{7n^2+3} + \cdots + \frac{1}{8n^2} \right),$
(m) $\lim_{n \rightarrow \infty} \frac{1}{n} \left(e^{\sqrt{\frac{1}{n}}} + e^{\sqrt{\frac{2}{n}}} + e^{\sqrt{\frac{3}{n}}} + \cdots + e^{\sqrt{\frac{n}{n}}} \right),$
(n) $\lim_{n \rightarrow \infty} \left(\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+3}} + \frac{1}{\sqrt{n+6}} + \frac{1}{\sqrt{n+9}} + \cdots + \frac{1}{\sqrt{7n}} \right) \frac{1}{\sqrt{n}},$
(o) $\lim_{n \rightarrow \infty} \left(\frac{n^2+0}{(3n)^3} + \frac{n^2+1}{(3n+1)^3} + \frac{n^2+2}{(3n+2)^3} + \frac{n^2+3}{(3n+3)^3} + \cdots + \frac{n^2+n}{(4n)^3} \right),$
(p) $\lim_{n \rightarrow \infty} \left(\frac{n}{2n^2} + \frac{n}{2(n+1)^2} + \frac{n}{2(n+2)^2} + \frac{n}{2(n+3)^2} + \cdots + \frac{n}{50n^2} \right),$
(r) $\lim_{n \rightarrow \infty} \left(\frac{n}{2n^2} + \frac{n}{n^2+(n+1)^2} + \frac{n}{n^2+(n+2)^2} + \frac{n}{n^2+(n+3)^2} + \cdots + \frac{n}{50n^2} \right).$