MATHEMATICAL ANALYSIS

PROBLEMS LIST 14

18.01.10

(1) Verify the convergence of the given function sequences, and uniform convergence on given sets:

(a) $f_n(x) = \sqrt{x^4 + \dfrac{x^2}{n}}$, $(-\infty, \infty)$, (b) $f_n(x) = \sqrt[n]{1 + x^{2n}}$, $(-\infty, \infty)$,

(c) $f_n(x) = x^n - x^{2n}$, $[0, 1]$, (d) $f_n(x) = \sin\left(\dfrac{x}{n}\right)$, $[0, \pi]$,

(e) $f_n(x) = \sin^n(x)$, $(-\infty, \infty)$, (f) $f_n(x) = \dfrac{1}{1 + x + n}$, $[0, \infty)$,

(g) $f_n(x) = \dfrac{1}{1 + (x + n)^2}$, $(-\infty, \infty)$, (h) $f_n(x) = \dfrac{1}{nx}$, $(0, 1]$,

(i) $f_n(x) = \dfrac{nx}{1 + nx^2}$, $[-1, 1]$, (j) $f_n(x) = \dfrac{nx}{1 + n^2x^2}$, $[-1, 1]$,

(k) $f_n(x) = n \sin\left(\dfrac{x}{n}\right)$, $[-1, 1]$, (l) $f_n(x) = nx^{-nx^2}$, $[-1, 1]$.

(2) Establish the set on which the function series is convergent, and check whether the convergence is uniform:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}\, e^{-nx^2}$, (b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n\sqrt{1 + nx}}$, (c) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(n\,x)}{10^n}$,

(d) $\displaystyle\sum_{n=1}^{\infty} n\, e^{-nx}$, (e) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n!\, x^n}$, (f) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3 + x^2}}$,

(g) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n\, x^n}{n^2}$, (h) $\displaystyle\sum_{n=1}^{\infty} 2^n\, x^n$, (i) $\displaystyle\sum_{n=1}^{\infty} \frac{5^n\, x^n}{n}$,

(j) $\displaystyle\sum_{n=1}^{\infty} n\big(\sqrt{x\,(1 - x)}\big)^n$, (k) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^x}$, (l) $\displaystyle\sum_{n=1}^{\infty} \sin\left(\frac{x}{n^2}\right)$,

(m) $\displaystyle\sum_{n=1}^{\infty} \frac{x}{x^2 + n^2}$, (n) $\displaystyle\sum_{n=1}^{\infty} \sin(nx)$.

(3) Prove that the following series are uniformly convergent on the entire real line $(-\infty, \infty)$:

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{\sin(nx)}{n!}$, (b) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos(nx)}{10^n}$, (c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{x^2 + n^2}$.

(4) Prove that the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n\sqrt{1 + nx}}$ is uniformly convergent on the set $[0, \infty)$.

(5) Prove that the series $\sum_{n=1}^{\infty} \frac{\log(1+nx)}{n\,x^n}$ converges pointwise, but not uniformly on the set $[1,\infty)$, and that it is convergent uniformly on the set $[2,\infty)$.

(6) Find the derivative $f'(x)$ and the indefinite integral $\int f(x)\,dx$ of the following functions:

(a) $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}\,x^n$, (b) $f(x) = \sum_{n=0}^{\infty} \frac{1}{n^2+1}\,x^n$,

(c) $f(x) = \sum_{n=1}^{\infty} (n+1)\,x^n$, (d) $f(x) = \sum_{n=1}^{\infty} x^n$.

(7) "Compact" the following power series, that is find the formula for their sum, and then determine the domain of so obtained function:

(a) $\sum_{n=0}^{\infty} x^{2n}$, (b) $\sum_{n=1}^{\infty} n\,x^{2n}$, (c) $\sum_{n=1}^{\infty} n^2\,x^{2n}$,

(d) $\sum_{n=1}^{\infty} (-1)^n\,n\,x^n$, (e) $\sum_{n=1}^{\infty} n\,(n+1)\,x^n$, (f) $\sum_{n=1}^{\infty} n\,(n+1)\,(n+2)\,x^n$.