CALCULUS

PROBLEMS LIST 2

8.10.2010

(1) Find the natural powers of the number $\mathbf{i}$, that is write out the complex numbers of the form $\mathbf{i}^n$ for all natural n.

(2) For given complex numbers $z = a + b\,\mathbf{i}$ and $w = c + d\,\mathbf{i}$ express $\Re(z+w), \Im(z+w), \Re(zw), \Im(zw)$, in terms of a, b, c, d.

(3) Prove the following properties of the complex conjugation:
   (a) $\overline{(\overline{z})} = z$,
   (b) $\overline{z+w} = \overline{z} + \overline{w}$,
   (c) $\overline{(z\,w)} = \overline{z}\,\overline{w}$,
   (d) $\Re(z) = (z+\overline{z})/2,\ \Im(z) = (z-\overline{z})/2\,\mathbf{i}$.

(4) Find the moduli of the complex numbers $z = -2 - 3\,\mathbf{i}$ and $z = 1 - \mathbf{i}$.

(5) Prove that arbitrary numbers $z, w \in \mathbf{C}$ have the properties:
   (a) $|z| \ge 0$ and $|z| = 0$ if and only if $z = 0$,
   (b) $|z\,w| = |z|\,|w|$,
   (c) $|z - w| \ge ||z| - |w||$.

(6) Describe geometrically (sketch on the plane) the set $\{z \in \mathbf{C} : |z - 1 - \mathbf{i}| = 1\}$.

(7) Sketch on the plane the sets of numbers $z \in \mathbf{C}$ satisfying given inequalities:
   (a) $|z| < 2$, (b) $|z + 3\,\mathbf{i}| < 1$, (c) $|z + 4 - 2\,\mathbf{i}| \le 3$.

(8) Find the trigonometric form of the following complex numbers:
   (a) $-6 + 6\,\mathbf{i}$, (b) $2\,\mathbf{i}$, (c) $1 + \mathbf{i}$, (d) $2\sqrt{2} + \mathbf{i}$.

(9) Compute:
   (a) $\dfrac{1+\mathbf{i}}{1-\mathbf{i}}$, (b) $\dfrac{2\,\mathbf{i}}{1+\mathbf{i}}$, (c) $\dfrac{4-3\,\mathbf{i}}{4+3\,\mathbf{i}}$.

(10) Prove that for $z = r\,(\cos\varphi + \mathbf{i}\,\sin\varphi)$ and $z = s\,(\cos\psi + \mathbf{i}\,\sin\psi)$ we have

$$z \cdot w = r \cdot s\,(\cos(\varphi + \psi) + \mathbf{i}\,\sin(\varphi + \psi)).$$

Conclude, that for any number $k \in \mathbf{Z}$ the formula holds:

$$z^k = r^k\,(\cos(k\,\varphi) + \mathbf{i}\,\sin(k\,\varphi)).$$

(11) Prove that for $z \in \mathbf{C},\ z \ne 0$ there exists $w \in \mathbf{C}$, its inverse, that is such that $z \cdot w = 1$.
Hint: Write and then solve an appropriate system of equations.

(12) Determine all values of the roots:
   (a) $\sqrt[4]{1}$, (b) $\sqrt[3]{-1}$, (c) $\sqrt[4]{1+\mathbf{i}}$, (d) $\sqrt[3]{2-2\,\mathbf{i}}$,
   (e) $\sqrt[6]{-27}$, (f) $\sqrt{3+4\,\mathbf{i}}$, (g) $\sqrt[3]{1}$, (h) $\sqrt[3]{\mathbf{i}}$.
Show their positions on the plane.

(13) Prove that $|z+w|^2 + |z-w|^2 = 2|z|^2 + 2|w|^2$.

(14) Let $a, b, c \in \mathbf{C}$ be arbitrary, $a \ne 0$ and let $d \in \mathbf{C}$ be one of the roots $\sqrt{b^2 - 4\,a\,c}$. Prove that the roots of the equation $a\,z^2 + b\,z + c = 0$ have the form

$$z = \frac{-b \pm d}{2\,a}.$$